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Original Research Paper

A Finite Difference Approximation for the Solution of the Space Fractional Diffusion Equation

H. Hajinezhad*
Payame Noor Univesity

Ali R. Soheili
Ferdowsi University of Mashhad

Abstract. The objective of this paper is to present a finite difference scheme that estimates the solution of space fractional diffusion equation with the Caputo fractional derivative. The proposed scheme's stability, and convergence are proved. To assess the efficiency of this program, a set of tests is carried out. The results of these tests demonstrate the reliability and accuracy of the proposed scheme.

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Keywords and Phrases: Space fractional diffusion equation, Convergence, Stability, Finite difference method

1 Introduction

In this paper, the solution of space fractional diffusion equation

$$\frac{\partial u(x,t)}{\partial t} = \alpha(x)\frac{\partial^{\beta} u(x,t)}{\partial x^{\beta}} + f(x,t),\ 0 < x < L,\ 0 < t \leqslant T,\ 1 < \beta < 2 \tag{1}$$

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*Corresponding Author

with initial condition

$$u(x,0) = \Psi_{t_0}(x), \qquad 0 \leqslant x \leqslant L, \tag{2}$$

and boundary conditions

$$\frac{\partial u(0,t)}{\partial x} = \Psi_{x0}(t), \quad 0 \leqslant t \leqslant T, \tag{3}$$

$$u(L,t) = \Psi_{x_1}(t), \qquad 0 \leqslant t \leqslant T, \tag{4}$$

is approximated, where $u(x,t)$ is an unknown function and the space fractional derivative is assumed to be based on the Caputo fractional derivative [9] as follows.

$$\frac{\partial^\beta u(x,t)}{\partial x^\beta} = \frac{1}{\Gamma(2-\beta)} \int_0^x \frac{\partial^2 u(s,t)}{\partial s^2}(x-s)^{1-\beta} ds, \quad 1 < \beta < 2. \tag{5}$$

The topic of fractional derivatives has garnered increased attention in recent years within the scientific community [12], [2]. The space fractional diffusion equation with the Riemann fractional derivative was approximated by a shifted *Grünwald* finite difference formula in [7]. In [14], for equation (1) with the Riemann fractional derivative, the Crank–Nicolson method was applied based on a *Grünwald* formula then an extrapolation was used to obtain a second-order approximation.

Equation (1) with fractional derivative (5) was solved numerically using orthogonal polynomials by some authors. The Legendre polynomials with the tau method were used to approximate this equation in [11]. Ren et al. [10] applied the shifted Chebyshev polynomials with the tau method to obtain an approximation for this equation. Some other authors used Chebyshev polynomials to estimate the solution of equation (1) with fractional derivative (5) [4], [13], [5], [1]. Khader [4] applied the Chebyshev polynomials to reduce this equation into a system of ordinary differential equations and then a finite difference approximation was used to obtain the numerical solution of this system. Safdari et al. [13] approximated this equation by using the compact finite difference to obtain the semi-discretization in the time derivative and then used the

Chebyshev collocation method to estimate the space fractional derivative.

In this paper, we propose a novel finite difference method for approximating equation (1) in Caputo sense (5) subject to conditions (2)-(4). Our method has a distinct advantage over other methods used for space fractional diffusion with the Caputo derivative. Specifically, we demonstrate that our proposed scheme is unconditionally stable and convergent through rigorous proof. To evaluate the accuracy of our method, we conduct several numerical tests.

The structure of our paper is as follows: The discretization of equation (1) is explained in the next section. Section 3 is dedicated to proving the stability and convergence of our proposed scheme. In Section 4, the numerical tests are provided. The final section presents the conclusion.

2 Finite Difference Method for the Problem

The discretization of equation (1) using a proposed finite difference method is explained in this section.

Let Δt and Δx represent the grid sizes in time and space, respectively, for the finite difference scheme. Then, $x_j = j\Delta x$ $(j = 0, 1, ..., J)$ and $t^n = n\Delta t$ $(n = 0, 1, ..., N)$, where $J\Delta x = L$ and $N\Delta t = T$. Assume u_j^n is the value of $u(x_j, t^n)$ for $j = 0, 1, ..., J$ and $n = 0, 1, ..., N$.

The following lemma provides the essential tools for the discretization of equation (1).

Lemma 2.1. *Assuming* $a_s = (s+1)^{2-\beta} - s^{2-\beta}$, $(s = 0, 1, ..., 1 < \beta < 2)$. *Then, the discretization of* $\frac{\partial^\beta u(x,t)}{\partial x^\beta}$ *at* (x_j, t^n) *for* $1 \leqslant j \leqslant J-1$ *and*

$0 \leqslant n \leqslant N$ *is as follows.*

$$\begin{aligned}
&\frac{\partial^\beta u(x,t)}{\partial x^\beta}|_1^n \\
&= \frac{(\Delta x)^{1-\beta}}{\Gamma(3-\beta)}[-a_0 \frac{\partial u}{\partial x}|_0^n - \frac{a_0}{\Delta x}u_1^n + \frac{a_0}{\Delta x}u_2^n] + O(\Delta x)^{2-\beta}, \ (0 \leqslant n \leqslant N), \\
&\frac{\partial^\beta u(x,t)}{\partial x^\beta}|_j^n \\
&= \frac{(\Delta x)^{1-\beta}}{\Gamma(3-\beta)}[-a_{j-1}\frac{\partial u}{\partial x}|_0^n + \frac{(a_{j-2}-a_{j-1})}{\Delta x}u_1^n \\
&\qquad + \Sigma_{k=2}^{j-1}\frac{(a_{j-k+1}-2a_{j-k}+a_{j-k-1})}{\Delta x}u_k^n + \frac{a_1-2a_0}{\Delta x}u_j^n + \frac{a_0}{\Delta x}u_{j+1}^n] \\
&\qquad + O(\Delta x)^{2-\beta}, \quad (2 \leqslant j \leqslant J-1, \ 0 \leqslant n \leqslant N).
\end{aligned}$$

Proof.

$$\begin{aligned}
&\frac{\partial^\beta u(x,t)}{\partial x^\beta}|_j^n \\
&= \frac{1}{\Gamma(2-\beta)}\int_0^{x_j}\frac{\partial^2 u(s,t^n)}{\partial s^2}(x_j-s)^{1-\beta}ds \\
&= \frac{1}{\Gamma(2-\beta)}\Sigma_{k=1}^{j}[(\frac{\frac{\partial u(x,t)}{\partial x}|_k^n - \frac{\partial u(x,t)}{\partial x}|_{k-1}^n}{\Delta x} \\
&\qquad + O(\Delta x))\int_{(k-1)\Delta x}^{k\Delta x}(x_j-s)^{1-\beta}ds] \\
&= \frac{(\Delta x)^{1-\beta}}{\Gamma(3-\beta)}\left\{-a_{j-1}\frac{\partial u}{\partial x}|_0^n + \Sigma_{k=1}^{j-1}(a_{j-k}-a_{j-k-1})\frac{\partial u}{\partial x}|_k^n + a_0\frac{\partial u}{\partial x}|_j^n\right\} \\
&\qquad + O(\Delta x)^{3-\beta}, \qquad (1 \leqslant j \leqslant J-1, \quad 0 \leqslant n \leqslant N).
\end{aligned} \tag{6}$$

Consider

$$\frac{\partial u}{\partial x}|_k^n = \frac{u|_{k+1}^n - u|_k^n}{\Delta x} + O(\Delta x), \quad (1 \leqslant k \leqslant J-1, \quad 0 \leqslant n \leqslant N). \tag{7}$$

Then, the relations (6) and (7) complete the proof. □

Assume

$$\frac{\partial u}{\partial t}|_j^{n+\frac{1}{2}} = \frac{u_j^{n+1}-u_j^n}{\Delta t} + O(\Delta t)^2, \ (0 \leqslant n \leqslant N-1, \ 1 \leqslant j \leqslant J-1), \tag{8}$$

$$\frac{\partial^\beta u}{\partial x^\beta}|_j^{n+\frac{1}{2}} = \frac{1}{2}\left[\frac{\partial^\beta u}{\partial x^\beta}|_j^{n+1} + \frac{\partial^\beta u}{\partial x^\beta}|_j^{n}\right] + O(\Delta t)^2, \qquad (0 \leqslant n \leqslant N-1,\ 1 \leqslant j \leqslant J-1). \tag{9}$$

By disregarding the truncation errors, the discretization of equation (1) with conditions (2)-(4) at the grid point x_j $(j = 1, 2, ..., J-1)$ and time step $(n+\frac{1}{2})$ for $0 \leqslant n \leqslant N-1$ using lemma 2.1 and relations (8) and (9) is as follows.

$$u_1^{n+1} - \gamma_1\{\frac{-a_0}{\Delta x}u_1^{n+1} + \frac{a_0}{\Delta x}u_2^{n+1}\} = u_1^n + \gamma_1\{\frac{-a_0}{\Delta x}u_1^n + \frac{a_0}{\Delta x}u_2^n\} + (\Delta t)f_1^{n+\frac{1}{2}} - \gamma_1 a_0(\frac{\partial u}{\partial x}|_0^n + \frac{\partial u}{\partial x}|_0^{n+1}),\ 0 \leqslant n \leqslant N-1, \tag{10}$$

$$\begin{aligned} u_j^{n+1} &- \gamma_j[\frac{a_{j-2}-a_{j-1}}{\Delta x}u_1^{n+1} + \frac{\Sigma_{k=2}^{j-1}(a_{j-k+1} - 2a_{j-k} + a_{j-k-1})}{\Delta x}u_k^{n+1} \\ &+ \frac{a_1 - 2a_0}{\Delta x}u_j^{n+1} + \frac{a_0}{\Delta x}u_{j+1}^{n+1}] \\ = u_j^n &+ \gamma_j[\frac{a_{j-2}-a_{j-1}}{\Delta x}u_1^n + \frac{\Sigma_{k=2}^{j-1}(a_{j-k+1} - 2a_{j-k} + a_{j-k-1})}{\Delta x}u_k^n \\ &+ \frac{a_1 - 2a_0}{\Delta x}u_j^n + \frac{a_0}{\Delta x}u_{j+1}^n] + (\Delta t)f_j^{n+\frac{1}{2}} \\ &- \gamma_j a_{j-1}(\frac{\partial u}{\partial x}|_0^n + \frac{\partial u}{\partial x}|_0^{n+1}), \quad 2 \leqslant j \leqslant J-2,\ 0 \leqslant n \leqslant N-1, \end{aligned} \tag{11}$$

$$\begin{aligned} u_{J-1}^{n+1} &- \gamma_{J-1}[\frac{a_{J-3}-a_{J-2}}{\Delta x}u_1^{n+1} + \frac{\Sigma_{k=2}^{J-2}[(a_{J-k} - 2a_{J-k-1} + a_{J-k-2})]}{\Delta x}u_k^{n+1} \\ &+ \frac{a_1 - 2a_0}{\Delta x}u_{J-1}^{n+1}] \\ = u_{J-1}^n &+ \gamma_{J-1}[\frac{a_{J-3}-a_{J-2}}{\Delta x}u_1^n + \frac{\Sigma_{k=2}^{J-2}[(a_{J-k} - 2a_{J-k-1} + a_{J-k-2})]}{\Delta x}u_k^n \\ &+ \frac{a_1 - 2a_0}{\Delta x}u_{J-1}^n] + (\Delta t)f_{J-1}^{n+\frac{1}{2}} + \gamma_{J-1}\frac{a_0}{\Delta x}(u_J^n + u_J^{n+1}) \\ &- \gamma_{J-1}a_{J-2}(\frac{\partial u}{\partial x}|_0^n + \frac{\partial u}{\partial x}|_0^{n+1}), \quad 0 \leqslant n \leqslant N-1, \end{aligned} \tag{12}$$

where $\gamma_j = \frac{\alpha(x_j)(\Delta t)(\Delta x)^{1-\beta}}{2\Gamma(3-\beta)}$ for $1 \leqslant j \leqslant J-1$, $f_j^{n+\frac{1}{2}} = f(x_j, t^{n+\frac{1}{2}})$ for $1 \leqslant j \leqslant J-1$, and $0 \leqslant n \leqslant N-1$. Now the following theorem is easy to prove.

Theorem 2.2. *The discretization of equation (1) with conditions (2)-(4) using lemma 2.1 and relations (8)-(9) is consistent with accuracy* $(O(\Delta x)^{2-\beta} + O(\Delta t)^2)$.

3 Stability and Convergence

The stability and convergence of schemes (10)-(12) are presented in this section. A stable finite difference scheme can approximate the solution of complex equations accurately [8]. The idea of showcasing stability is based on reference [14]. Equations (10)-(12) can be considered as follows:

$$(I-B)U^{n+1} = (I+B)U^n + F^{n+\frac{1}{2}}, \quad 0 \leqslant n \leqslant N-1, \tag{13}$$

where

$$U^n = [u_1^n, u_2^n, ..., u_{J-1}^n]^T, \quad 0 \leqslant n \leqslant N-1,$$

$$F^{n+\frac{1}{2}} =$$

$$[(\Delta t)f_1^{n+\frac{1}{2}}, (\Delta t)f_2^{n+\frac{1}{2}}, ..., (\Delta t)f_{J-2}^{n+\frac{1}{2}}, (\Delta t)f_{J-1}^{n+\frac{1}{2}} + \frac{\gamma_{J-1}a_0}{\Delta x}(u_J^n + u_J^{n+1})]^T$$

$$-\left(\frac{\partial u}{\partial x}|_0^n + \frac{\partial u}{\partial x}|_0^{n+1}\right)[\gamma_1 a_0, \gamma_2 a_1, ..., \gamma_{J-1} a_{J-2}]^T, \quad 0 \leqslant n \leqslant N-1,$$

I is a $(J-1)\times(J-1)$ identity matrix, and in matrix B, the elements B_{jk} $(j,k = 1,2,...,J-1)$ are as follows.

$$B_{jk} = \begin{cases} -\gamma_1 \frac{a_0}{\Delta x} & if \quad k=j=1, \\ \gamma_j \frac{a_1 - 2a_0}{\Delta x} & if \quad k=j\neq 1, \\ \gamma_j \frac{a_{j-2}-a_{j-1}}{\Delta x} & if \quad k=1, 2\leqslant j, \\ \gamma_j \frac{a_{j-k+1}-2a_{j-k}+a_{j-k-1}}{\Delta x} & if \quad 2\leqslant k \leqslant j-1,\ 2\leqslant j, \\ \gamma_j \frac{a_0}{\Delta x} & if \quad k=j+1, \\ 0 & if \quad j+1<k. \end{cases}$$

The following theorem establishes that the finite difference discretization of equation (1) with conditions (2)-(4), as defined by (10)-(12), is unconditionally stable. This stability property is crucial for ensuring reliable numerical solutions.

Theorem 3.1. *The finite difference discretization of equation* (1) *with conditions* (2)-(4) *defined by* (10)-(12) *is unconditionally stable.*

Proof. Equations (10)-(12) are equivalent to (13). First, it is argued that matrix B has the eigenvalues with a non–positive real–part. According to the Gershgorin theorem ([3] p. 294), for matrix B, we have

$$\begin{cases} |\lambda_1 + \gamma_1 \frac{a_0}{\Delta x}| \leqslant \gamma_1 \frac{a_0}{\Delta x}, \\ |\lambda_j + \gamma_j \frac{2a_0 - a_1}{\Delta x}| \\ \qquad \leqslant \gamma_j [\frac{a_{j-2} - a_{j-1}}{\Delta x} + \Sigma_{k=2}^{j-1} |\frac{(a_{j-k+1} - 2a_{j-k} + a_{j-k-1})}{\Delta x}| + \frac{a_0}{\Delta x}], \\ \qquad\qquad for\ 2 \leqslant j \leqslant J-2, \\ |\lambda_{J-1} + \gamma_{J-1} \frac{2a_0 - a_1}{\Delta x}| \\ \qquad \leqslant \gamma_{J-1} [\frac{a_{J-3} - a_{J-2}}{\Delta x} + \Sigma_{k=2}^{J-2} |\frac{(a_{J-k} - 2a_{J-k-1} + a_{J-k-2})}{\Delta x}|], \end{cases} \tag{14}$$

where λ_j $(1 \leqslant j \leqslant J-1)$ is the eigenvalue of the matrix B. It is easy to show that $a_0 > a_1 > ... > a_n$ and $lim_{n\to\infty} a_n = 0$. Also, we can demonstrate that

$$(a_n - a_{n+1}) > (a_{n+1} - a_{n+2}), \qquad n = 1, 2, ...$$

Therefore, for $2 \leqslant j \leqslant J-1$ and $k = 2, 3, ..., j-1$, we have

$$(a_{j-k+1} - 2a_{j-k} + a_{j-k-1}) > 0.$$

So,

$$\Sigma_{k=2}^{j-1} |(a_{j-k+1} - 2a_{j-k} + a_{j-k-1})|$$
$$= \Sigma_{k=2}^{j-1} ((a_{j-k+1} - a_{j-k}) - (a_{j-k} - a_{j-k-1})) = (a_{j-1} - a_{j-2}) + (a_0 - a_1).$$

Now the relations (14) can be written as follows.

$$\begin{cases} |\lambda_1 + \gamma_1 \frac{a_0}{\Delta x}| \leqslant \gamma_1 \frac{a_0}{\Delta x}, \\ |\lambda_j + \gamma_j \frac{2a_0 - a_1}{\Delta x}| \leqslant \gamma_j \frac{2a_0 - a_1}{\Delta x}, \quad 2 \leqslant j \leqslant J-2, \\ |\lambda_{J-1} + \gamma_{J-1} \frac{2a_0 - a_1}{\Delta x}| \leqslant \gamma_{J-1} \frac{a_0 - a_1}{\Delta x}. \end{cases} \tag{15}$$

It is obvious that matrix B is invertible, so matrix B has the non–zero eigenvalues. Therefore according to (15), matrix B has the eigenvalues with the non–positive real–part.
Now, λ_j is an eigenvalue of the matrix B if and only if $\frac{1+\lambda_j}{1-\lambda_j}$ is an eigenvalue of the matrix $(I-B)^{-1}(I+B)$. Since the real part of λ_j is not positive, $|\frac{1+\lambda_j}{1-\lambda_j}| < 1$. Thus, the system of equation (13) is unconditionally stable. $\square$

By using Lax's equivalence theorem [6], theorem 2.2 and theorem 3.1 indicate that our proposed scheme (13) is convergent.

4 Numerical Tests

Some numerical tests are presented in this section, to check the validity of our proposed scheme. We measure the accuracy of the proposed method by assuming Δx and Δt, using the following maximum absolute error

$$L_\infty(\Delta x, \Delta t) = max_{1\leqslant j\leqslant J-1, 1\leqslant n\leqslant N} |\widehat{u}_j^n - u_j^n|,$$

where $\widehat{u}_j^n$, and u_j^n are the approximation and the exact solutions of equation (1) with conditions (2)–(4) at x_j and time t^n, respectively. To test our proposed method, we consider the following three examples in which the exact solutions are available.

Example 4.1. Assume the equation [4]

$$\frac{\partial u(x,t)}{\partial t} = \alpha(x)\frac{\partial^{1.8} u(x,t)}{\partial x^{1.8}} + f(x,t), \quad 0 < x < 1, \quad 0 < t \leqslant 1,$$

where $\alpha(x) = \Gamma(1.2)x^{1.8}$, $f(x,t) = (6x^3 - 3x^2)e^{-t}$. The exact solution $u(x,t) = (x^2 - x^3)e^{-t}$ is used to consider conditions (2)–(4).

Example 4.2. Assuming the equation [11]

$$\frac{\partial u(x,t)}{\partial t} = \alpha(x)\frac{\partial^{1.5} u(x,t)}{\partial x^{1.5}} + f(x,t), \quad 0 < x < 1, \quad 0 < t \leqslant 1,$$

where $\alpha(x) = \Gamma(1.5)x^{0.5}$, $f(x,t) = (x^2+1)cos(t+1) - 2xsin(t+1)$. The exact solution $u(x,t) = (x^2+1)sin(t+1)$ is used to consider conditions (2)–(4).

Table 1: The maximum absolute errors and Convergence rates with different values Δt and Δx in Example 1.

$\Delta t, \Delta x$	$L_\infty(\Delta x, \Delta t)$	Convergence rate
$\frac{1}{10}$	$7.7904e-3$	$-$
$\frac{1}{20}$	$4.6223e-3$	$1.68 \approx \frac{20}{10}$
$\frac{1}{50}$	$2.1667e-3$	$2.13 \approx \frac{50}{20}$
$\frac{1}{100}$	$1.1842e-3$	$1.83 \approx \frac{100}{50}$
$\frac{1}{200}$	$6.3542e-4$	$1.86 \approx \frac{200}{100}$
$\frac{1}{500}$	$2.7344e-4$	$2.32 \approx \frac{500}{200}$
$\frac{1}{1000}$	$1.4287e-4$	$1.91 \approx \frac{1000}{500}$

Example 4.3. Assume the equation [14]

$$\frac{\partial u(x,t)}{\partial t} = \alpha(x)\frac{\partial^{1.8}u(x,t)}{\partial x^{1.8}} + f(x,t), \quad 0 < x < 1, \quad 0 < t \leqslant 1,$$

where $\alpha(x) = \Gamma(2.2)\frac{x^{2.8}}{6}$, $f(x,t) = -(1+x)e^{-t}x^3$. The exact solution $u(x,t) = e^{-t}x^3$ is used to consider conditions (2)–(4).

It is essential to note that the implementation of the proposed method for each equation in the examples 4.1, 4.2, and 4.3, with their initial and boundary conditions, involves discretizing the equations at the grid point x_j $(j = 1, 2, ..., J-1)$ and time step $(n + \frac{1}{2})$ for $0 \leqslant n \leqslant N-1$ using Lemma 2.1 and relations (8) and (9) to form a system of equations (as shown in Equation (13)). By solving this system at each time step, the value of unknown function u in the next time step is obtained. It is obvious that the value of u_j^n is known for $n = 0$ at each space point, initially. By solving the system, the value of u_j^n at time step $n = 1$ for

Table 2: The maximum absolute errors and Convergence rates with different values Δt and Δx in Example 2.

$\Delta t, \Delta x$	$L_\infty(\Delta x, \Delta t)$	Convergence rate
$\frac{1}{10}$	$5.4187e-2$	$-$
$\frac{1}{20}$	$2.6464e-2$	$2.05 \approx \frac{20}{10}$
$\frac{1}{50}$	$1.0139e-2$	$2.61 \approx \frac{50}{20}$
$\frac{1}{100}$	$4.9237e-3$	$2.06 \approx \frac{100}{50}$
$\frac{1}{200}$	$2.4043e-3$	$2.05 \approx \frac{200}{100}$
$\frac{1}{500}$	$9.3982e-4$	$2.56 \approx \frac{500}{200}$
$\frac{1}{1000}$	$4.6414e-4$	$2.04 \approx \frac{1000}{500}$

each space point is determined, and this process continues iteratively for subsequent time steps.

The results of our proposed method for Examples 4.1, 4.2, and 4.3 are presented in Tables 1, 2, and 3, respectively. The second columns of these Tables show that the maximum absolute error, with different values Δt and Δx, is small enough and reduces as the grids are refined.

To test the rate of convergence of our proposed scheme, we started with $\Delta x = \Delta t = \frac{1}{10}$ and obtained numerical solutions for Examples 4.1, 4.2, and 4.3. We then repeated the computations using finer grids. Here, the Convergence rate is defined by the ratio of the errors as refining the grids, as follows.

$$\mathit{Convergence\ rate} = \frac{L_\infty((\Delta x)_1, (\Delta t)_1)}{L_\infty((\Delta x)_2, (\Delta t)_2)},$$

where$(\Delta x)_2 = (\Delta t)_2 < (\Delta x)_1 = (\Delta t)_1$. According to the third columns of Tables 1, 2, and 3, the behavior of errors is (almost) linear. It means

Table 3: The maximum absolute errors and Convergence rates with different values Δt and Δx in Example 3.

$\Delta t, \Delta x$	$L_\infty(\Delta x, \Delta t)$	Convergence rate
$\frac{1}{10}$	$4.3914e-3$	$-$
$\frac{1}{20}$	$2.4129e-3$	$1.82 \approx \frac{20}{10}$
$\frac{1}{50}$	$1.1076e-3$	$2.18 \approx \frac{50}{20}$
$\frac{1}{100}$	$5.7498e-4$	$1.93 \approx \frac{100}{50}$
$\frac{1}{200}$	$3.0367e-4$	$1.89 \approx \frac{200}{100}$
$\frac{1}{500}$	$1.2873e-4$	$2.36 \approx \frac{500}{200}$
$\frac{1}{1000}$	$6.6693e-5$	$1.93 \approx \frac{1000}{500}$

that the ratio of $L_\infty((\Delta x)_1, (\Delta t)_1)$ to $L_\infty((\Delta x)_2, (\Delta t)_2)$ is approximately equal to the ratio of $(\Delta x)_1$ or $(\Delta t)_1$ to $(\Delta x)_2$ or $(\Delta t)_2$, where $(\Delta x)_2 = (\Delta t)_2 < (\Delta x)_1 = (\Delta t)_1$. Therefore, in our method, when the grid sizes in space and time are divided by n, the maximum absolute error is also divided by n approximately.

5 Conclusion

This paper introduces a novel finite difference scheme for approximating the solution of the space fractional diffusion equation with the Caputo fractional derivative. The proposed scheme has been rigorously proven to be stable and convergent. Through numerical tests and comparisons with exact solutions, the reliability and accuracy of the method have been demonstrated. Additionally, the numerical tests show that the error behavior of the proposed scheme is nearly linear. It means when the grid sizes in space and time are divided by n, the maximum absolute error is also divided by n approximately. This study contributes to the advancement of numerical methods for solving fractional differential equations and underscores the importance of rigorous analysis and testing in computational mathematics research.

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Haniye Hajinezhad
Assistant Professor of Mathematics
Department of Mathematics
Payame Noor University
Tehran, Iran
E-mail: H.Hajinezhad@pnu.ac.ir

Ali Reza Soheili
Professor of Applied Mathematics
Department of Applied Mathematics
Faculty of Mathematical Sciences
Ferdowsi University of Mashhad
Mashhad, Iran

E-mail: soheili@um.ac.ir