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Optimal Problems of the Best Proximity Pair by Proximal Normal Structure

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Abstract. Let (A_1, A_2, A_3) be a triple of nonempty convex subsets of a metric space Ω. In this paper, we determine optimal problems of the best proximity pair by proximal normal structure between two sets A_1 and A_2 with the help of a third set A_3 and we find some necessary and sufficient conditions for existence these optimal problems. Also, we provide an example to illustrate the convergence behavior of our proposed results.

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1 Introduction

Let Ω be a metric space and let $T, S : \Omega \to \Omega$ be mappings. We remember that a point $u \in \Omega$ is called a fixed point of T if $Tu = u$, coincidence point of T and S if $Tu = Su$ and common fixed point of T and S if $Tu = Su = u$. Now, let A_1 and A_2 nonempty subsets of Ω . Put

$$\begin{aligned} A_1^\circ &= \{u \in A_1 : d(u, v) = d(A_1, A_2) \ for\ some\ v \in A_2\}, \\ A_2^\circ &= \{u \in A_2 : d(u, v) = d(A_1, A_2) \ for\ some\ v \in A_1\}. \end{aligned}$$

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If there is a pair $(u_\circ, v_\circ) \in A_2 \times A_3$ for which $d(u_\circ, v_\circ) = d(A_1, A_2)$,that $d(A_1, A_2)$ is distance of A_1 and A_2, then the pair $(u_\circ, v_\circ)$ is said to a best proximity pair for (A_1, A_2). Best proximity pair is a extension of the concept of best approximation.

The best proximity points of (A_1, A_2) consider by a map $T : A_1 \to A_2$. The point $u \in A_1$ is said to be a best proximity point if $d(u, Tu) = d(A_1, A_2)$. Also, the best proximity point is a expansion of the concept of fixed point of mappings, because if $A_1 \cap A_2 \neq \emptyset$ every best proximity point is a fixed point of T.

Sankar Raj et al. [15] gave important results for finding best proximity points by relatively nonexpansive mappings. A best proximity point theorem for contraction has been started by Eldred and Veeramani [6] and Basha [19], also contraction maps extended by many authors (for instance [3, 8, 9, 10, 11, 14, 18]).

A. A. Eldred et al. [5] introduced proximinal normal structure for a (A_1, A_2) and they gave new results in relatively nonexpansive mappings. Later, Gabeleh [7] and Kirk et al. [12] continued to study this topic. In the following, we give the definition of proximinal normal structure of [5].

Definition 1.1. ([5]) A convex pair (G_1, G_2) in a Banach space is said to have proximal normal structure if for every closed, bounded, convex proximal pair $(K_1, K_2) \subseteq (G_1, G_2)$ for that $d(K_1, K_2) = d(G_1, G_2)$ and $\delta(K_1, K_2) > d(K_1, K_2)$, there exists $(\tau_1, \tau_2) \in K_1 \times K_2$ such that $\delta(\tau_1, K_2) < \delta(K_1, K_2)$ and $\delta(\tau_2, K_1) < \delta(K_1, K_2)$.

Theorem 1.2. *([5]) Let Ω be a Banach space, (A_1, A_2) be a nonempty, weakly compact convex pair in Ω, and let (A_1, A_2) has proximal normal structure. If $T : A_1 \cup A_2 \to A_1 \cup A_2$ is a relatively nonexpansive map such that $T(A_1) \subseteq A_2$ and $T(A_2) \subseteq A_1$, then there is $(u, v) \in A_1 \times A_2$ such that $\|u - Tu\| = \|Tv - v\| = d(A_1, A_2)$.*

Theorem 1.3. *([5]) Let Ω be a uniformly convex Banach space. Then every bounded closed convex pair in Ω has proximal normal structure.*

On the other hand, about the subject common fixed point there are a large number of publications (see [1, 2]). These notions extend common fixed points to nonself-mappings. There are many more reserch on common best proximity points (see [4, 13, 16]). In the following we extended coincidence points of self mappings to nonself mappings. In fact, they determine the nearest points between two sets A_1 and A_2 with the help of a third set A_3.

Definition 1.4. Let (A_1, A_2, A_3) be a triple of nonempty subsets of a metric space Ω and let $T : A_3 \to A_1$ and $S : A_3 \to A_2$ be nonself mappings. A pair $(x, y) \in A_3 \times A_3$ is called a *best* (S,T)*-proximity pair* of the pair operator (S,T) if $d(Tx, Sy) = d(A_1, A_2)$. Also, a point $u \in A_3$ is called a *best coincidence proximity point* (BCP point) of the pair operator (S,T) if $d(Tu, Su) = d(A_1, A_2)$. Also, put

$$P_{A_3}(T, S) = \{u \in A_3 : d(Tu, Su) = d(A_1, A_2)\}.$$

It is notable that, if we have $B \cap C \neq \emptyset$, $u \in A_3$ is a coincidence point of (T, S) i.e. $Tu = Su$. Also, if we have $A_1 = A_3$ and S be identity, $u \in A_3$ is a best proximity point. Finally, if we have $A_1 = A_2 = A_3$ and S identity, $u \in A_3$ is a fixed point of T.

In this paper, we give some conditions for finding the best coincidence proximity points by the proximal normal structure sets that results are extensions of the Eldred and Veeramani results [5]. Also, we find best proximity pair by cyclic contraction maping pair that results are extensions of the Eldred et al. [6].

2 Main Results

In this section, we give some necessary and sufficient conditions for existence and uniqueness of best coincidence proximity points by the proximal normal structure sets. It is notable that results in this section are extensions of [5].

Theorem 2.1. *Let (A_1, A_2, A_3) be a triple of nonempty, weakly compact and convex subsets of a Banach space Ω. Let $T : A_1 \cup A_3 \to A_1 \cup A_3$ and $S : A_2 \cup A_3 \to A_2 \cup A_3$ be cyclic mappings, that*

$$\|Tu - Sv\| \leq \|u - v\|, \quad (u \in A_1 \cup A_3,\ v \in A_2 \cup A_3).$$

If (A_1, A_2) has proximal normal structure, then there is $u \in A_3$ such that $\|Tu - Su\| = d(A_1, A_2)$.

Proof. Suppose $d(A_1, A_2) > 0$. Let (A_1°, A_2°) be the proximal pair of (A_1, A_2). Clearly, A_1° and A_2° are convex and weakly compact that $d(A_1^\circ, A_2^\circ) = d(A_1, A_2)$. For every $u \in A_1^\circ$, there is $v \in A_2^\circ$ such that

$$\|STu - TSv\| \leq \|Tu - Sv\| \leq \|u - v\| = d(A_1, A_2),$$

and

$$\|T^2u - S^2v\| \leq \|Tu - Sv\| \leq \|u - v\| = d(A_1, A_2).$$

Therefore $STu \in A_1^\circ$ and $T^2u \in A_1^\circ$, hence $ST(A_1^\circ) \subseteq A_2^\circ$ and $T^2(A_1^\circ) \subseteq A_1^\circ$. Similarly we have that $TS(A_2^\circ) \subseteq A_1^\circ$ and $S^2(A_2^\circ) \subseteq A_2^\circ$. Moreover, the pair (A_1°, A_2°) has proximal normal structure.
Now suppose Σ be the set of all nonempty subsets F of $A_1 \cup A_2$ such that $F \cap A_2^\circ$ and $F \cap A_1^\circ$ are nonempty, closed and convex sets that

$$ST(F \cap A_1^\circ) \subseteq A_2^\circ,\ TS(F \cap A_2^\circ) \subseteq A_1^\circ,$$
$$T^2(F \cap A_1^\circ) \subseteq A_1^\circ,\ S^2(F \cap A_2^\circ) \subseteq A_2^\circ,$$

and

$$d(F \cap A_1^\circ, F \cap A_2^\circ) = d(A_1, A_2).$$

Obviously, $A_1^\circ \cup A_2^\circ \in \Sigma$. We are going to show that Σ satisfies the hypothesis of Zorn's lemma. To this end, let $\{F_\alpha : \alpha \in I\}$ be a descending chain in Σ, and let $F_0 = \bigcap_\alpha F_\alpha$. Then $F_0 \cap A_1^\circ = \bigcap_\alpha (F_\alpha \cap A_1^\circ)$ is nonempty, closed and convex and so is $F_0 \cap A_2^\circ$, similarly. Also,

$$\begin{aligned} ST(F_0 \cap A_1^\circ) &= ST([\bigcap_{\alpha \in I} F_\alpha] \cap A_1^\circ), \\ &\subseteq \bigcap_{\alpha \in I} ST(F_\alpha \cap A_1^\circ), \\ &\subseteq A_2^\circ. \end{aligned}$$

Similarly,

$$TS(F_0 \cap A_2^\circ) \subseteq A_1^\circ, \quad S^2(F_0 \cap A_2^\circ) \subseteq A_2^\circ \quad \textit{and} \quad T^2(F_0 \cap A_1^\circ) \subseteq A_1^\circ.$$

Without loss of generality, consider the sequences $\{\tau_\alpha\}_{\alpha\in I}$ in $F_\alpha \cap A_1^\circ$ and $\{\varsigma_\alpha\}_{\alpha\in I}$ in $F_\alpha \cap A_2^\circ$ such that

$$\|\tau_\alpha - \varsigma_\alpha\| = d(A_1, A_2).$$

We have weakly convergent subnets $\{\tau_\beta\}$ and $\{\varsigma_\beta\}$ such that $\tau_\beta \rightharpoonup x$ and $\varsigma_\beta \rightharpoonup y$. Then, $x \in F_0 \cap A_1^\circ$ and $y \in F_0 \cap A_2^\circ$. Also,

$$\|x - y\| \leq d(A_1, A_2),$$

hence,

$$d(A_1, A_2) \leq d(F_0 \cap A_1^\circ, F_0 \cap A_2^\circ) \leq \|x - y\| \leq d(A_1, A_2).$$

Therefore $d(F \cap A_1^\circ, F \cap A_2^\circ) = d(A_1, A_2)$. Thus, $F_0 \in \Sigma$ and F_0 is a lower bounded of $\{F_\alpha \;:\; \alpha \in I\}$. Hence, by Zorn's lemma Σ has a minimal element G. Suppose $G_1 = G \cap A_1^\circ$ and $G_2 = G \cap A_2^\circ$. As in the proof of Theorem 1.2 minimality of K implies that $\delta(G_1, G_2) = d(G_1, G_2)$. Consequently $T^2 q \in G_1$ and $STq \in G_2$, for every $q \in G_1$. Now, consider $x = Tq$ so

$$\|Tx - Sx\| = d(G_1, G_2) = d(A_1, A_2).$$

This completes the proof. □

By combining Theorem 1.3 and Theorem 2.1, we obtain the following result.

Corollary 2.2. *Let (A_1, A_2, A_3) be a triple of nonempty, weakly compact and convex subsets of a uniformly, convex Banach space Ω. Let $T : A_1 \cup A_3 \to A_1 \cup A_3$ and $S : A_2 \cup A_3 \to A_2 \cup A_3$ be cyclic mappings, that*

$$\|Tu - Sv\| \leq \|u - v\|, \quad (u \in A_1 \cup A_3,\ v \in A_2 \cup A_3).$$

Then there exists $u \in A_3$ such that $\|Tu - Su\| = d(A_1, A_2)$.

Theorem 2.3. *Let (A_1, A_2, A_3) be a triple of nonempty, weakly compact and convex subsets of a strictly convex Banach space Ω . Let $T : A_1 \cup A_3 \to A_1 \cup A_3$ and $S : A_2 \cup A_3 \to A_2 \cup A_3$ be cyclic mappings, that*

$$\|STu - TSv\| \leq \|u - v\|, \quad (u \in A_1,\ v \in A_2).$$

Then there exists $u \in A_1$ such that $\|u - STu\| = d(A_1, A_2)$, whenever (A_1, A_2) has proximal normal structure.

Proof. If $A_1 \cap A_2 \neq \emptyset$, the claim is trivially true. Suppose that $d(A_1, A_2) > 0$ and A_1° and A_2° are as in the proof of Theorem 2.1. Let $u \in A_1^\circ$, then there is $v \in A_2^\circ$ such that

$$\|STu - TSv\| \leq \|u - v\| = d(A_1, A_2).$$

Thus $ST(A_1^\circ) \subseteq A_2^\circ$ and similarly $TS(A_2^\circ) \subseteq A_1^\circ$.
Let Σ be the collection of all nonempty subsets F of $A_1 \cup A_2$ such that $F \cap A_2^\circ$ and $F \cap A_1^\circ$ are nonempty, closed and convex. Moreover,

$$ST(F \cap A_1^\circ) \subseteq A_2^\circ, \quad TS(F \cap A_2^\circ) \subseteq A_1^\circ, \tag{1}$$

and

$$d(F \cap A_1^\circ, F \cap A_2^\circ) = d(A_1, A_2). \tag{2}$$

Σ includes $A_1^\circ \cup A_2^\circ$, hence it is nonempty. Suppose that $\{F_\alpha : \alpha \in I\}$ is a descending chain in Σ, and let $F_0 = \bigcap_\alpha F_\alpha$. Then it is clear that $F_0 \cap A_1^\circ$ and $F_0 \cap A_2^\circ$ are nonempty, closed and convex sets such that satisfies in (1) and (2). Therefore, every chain in Σ is bounded below by a member of Σ. Hence Σ has a minimal element G, by Zorn's lemma. Let $G_1 = G \cap A_1^\circ$ and $G_2 = G \cap A_2^\circ$. First, suppose $G_1 = \{u\}$. Hence there is $v \in A_1$ such that

$$\|STu - TSv\| \leq \|u - v\| = d(A_1, A_2).$$

Hence $TSv = u$, we are finished.
In the case where both G_1 and G_2 have positive diameter. Since X is strictly convex, we have $\delta(G_1, G_2) > d(G_1, G_2)$. We can find $\lambda \in (0, 1)$ that

$$G_1 \subseteq \mathcal{B}\big(STx, \frac{\lambda + 1}{2}\,\delta(G_1, G_2)\big),$$

and nonempty closed convex subset L_i of K_i, $i = 1, 2$, such that $ST(L_1) \subseteq L_2$ and $TS(L_2) \subseteq L_1$. Therefore $L_1 \cup L_2 \in \Sigma$. But $\delta(L_1, L_2) \leq \frac{\lambda+1}{2}\,\delta(G_1, G_2)$, and this contradicts the minimality of K. □

As an immediate consequence of Theorem 2.3, we give the following result.

Corollary 2.4. *Let (A_1, A_2, A_3) be a triple of nonempty, weakly compact and convex subsets of a uniformly convex Banach space Ω . Let $T : A_1 \cup A_3 \to A_1 \cup A_3$ and $S : A_2 \cup A_3 \to A_2 \cup A_3$ be cyclic mappings, that*

$$\|STu - TSv\| \leq \|u - v\|, \quad (u \in A_1,\ v \in A_2).$$

Then there is $u \in A_1$ such that $\|STu - u\| = d(A_1, A_2)$.

In the following, we provide an example to illustrate the behavior of our proposed results.

Example 2.5. Let (A_1, A_2, A_3) be a triple of subsets of $\mathbb{R}^2$ defined by,

$$A_1 = [1, 2] \times [1, 2],\ A_2 = [-2, -1] \times [-2, -1], \quad , \textit{ and } \quad A_3 = [-2, -1] \times [1, 2].$$

Define mappings $T : A_1 \cup A_3 \to A_1 \cup A_3$ through

$$T(x, y) = \begin{cases} (\frac{-x+1}{2}, \frac{-y+1}{2}) & (x, y) \in A_3 \\ (\frac{-x-1}{2}, \frac{-y-1}{2}) & (x, y) \in A_1, \end{cases} \tag{3}$$

and define mappings $S : A_2 \cup A_3 \to A_2 \cup A_3$ through

$$S(x, y) = \begin{cases} (\frac{x-1}{2}, \frac{-y+1}{2}) & (x, y) \in A_3 \\ (\frac{x-1}{2}, \frac{-y-1}{2}) & (x, y) \in A_2. \end{cases} \tag{4}$$

Obviously, the mappings T and S are cyclic on $\mathbb{R}^2$ and for all $x \in A_1 \cup A_3$, $y \in A_2 \cup A_3$ we have

$$\|Tx - Sy\| \leq \|x - y\|,$$

and for all $x \in A_1,\ y \in A_2$ we have

$$\|STx - TSy\| \leq \|x - y\|.$$

Hence, according to Theorem 2.1 we have a point $x^* = (-1, 1) \in A_3$ such that

$$\|Tx^* - Sx^*\| = \|(1, 1) - (-1, 1)\| = 2\sqrt{2} = d(A_1, A_2).$$

and according to Theorem 2.3 we have a point $y^* = (1, 1) \in A_1$ such that

$$\|STy^* - y^*\| = \|(-1, -1) - (1, 1)\| = 2\sqrt{2} = d(A_1, A_2).$$

In the following, we consider best proximity pair by cyclic contraction mapping pair. In the following we give some results that they are extensions of [6]. For nonself mappings $T : A_1 \cup A_3 \to A_1 \cup A_3$ and $S : A_2 \cup A_3 \to A_2 \cup A_3$ we say that the pair (T, S) is a cyclic pair if

$$T(A_1) \subseteq A_3,\ T(A_3) \subseteq A_1,\ \textit{and}\ S(A_3) \subseteq A_2, S(A_2) \subseteq A_3.$$

Theorem 2.6. *Let (A_1, A_2, A_3) be a triple of nonempty closed subsets of a metric space Ω. Also, let (T, S) be a cyclic mapping pair on (A_1, A_2, A_3) such that*

$$d(STu, TSv) \leq kd(u, v) + (1 - k)d(A_1, A_2), \quad \forall\ u \in A_1,\ v \in A_2,$$

$$d(TSu, TSv) < d(u, y), \quad \forall\ u, v \in A_2,$$

$$d(STu, STv) < d(u, v), \quad \forall\ u, v \in A_1.$$

Then there exists a best (S, T)-proximity pair. In fact, if $\tau_0 \in A_1$, then

$$\tau_{2n+1} = ST\tau_{2n}\ \textit{and}\ \tau_{2n} = TS\tau_{2n-1},\ \forall n \in \mathbb{N}$$

converge to a best (S, T)-proximity pair.

Proof. Suppose $\tau_0 \in A_1$ and

$$\tau_{2n+1} = ST\tau_{2n}\ \textit{and}\ \tau_{2n} = TS\tau_{2n-1},\ \forall n \in \mathbb{N}.$$

We know that

$$\begin{aligned} d(\tau_{2n}, \tau_{2n+1}) &= d(TS\tau_{2n-1}, ST\tau_{2n}) \\ &\leq kd(\tau_{2n-1}, \tau_{2n}) + (1 - k)d(A_1, A_2) \\ &\leq k^2 d(\tau_{2n-2}, \tau_{2n-1}) + (1 - k^2)d(A_1, A_2) \\ &\vdots \\ &\leq k^{2n} d(\tau_0, \tau_1) + (1 - k^{2n})d(A_1, A_2), \end{aligned}$$

that is,

$$d(\tau_{2n}, \tau_{2n+1}) \leq k^{2n} d(\tau_0, \tau_1) + (1 - k^{2n})d(A_1, A_2).$$

Therefore, $d(\tau_{2n}, \tau_{2n+1}) \to d(A_1, A_2)$. Now, we show that $\{\tau_{2n}\} \subseteq A_1$ and $\{\tau_{2n+1}\} \subseteq A_2$ are convergence sequences. It is notable that

$$\begin{aligned} d(\tau_{2n}, \tau_{2n+2}) &= d(TS\tau_{2n-1}, TS\tau_{2n+1}) \\ &< d(\tau_{2n-1}, \tau_{2n+1}) = d(TS\tau_{2n-2}, TS\tau_{2n}) \\ &< d(\tau_{2n-2}, \tau_{2n+1}) \\ &\vdots \\ &< d(\tau_0, \tau_2). \end{aligned}$$

Therefore, $\{d(\tau_{2n}, \tau_{2n+2})\}$ is monotonic decreasing and bounded below. Hence,

$$\lim_{n\to\infty} d(\tau_{2n}, \tau_{2n+2})$$

exists. Let $\lim_{n\to\infty} d(\tau_{2n}, \tau_{2n+2}) = \theta$. It is clear that $0 \leq \theta$. Suppose $\theta > 0$. Therefore,

$$\theta = \lim_{n\to\infty} d(\tau_{2n}, \tau_{2n+2}) < \lim_{n\to\infty} d(\tau_{2n-2}, \tau_{2n}) = \theta.$$

Hence, $\theta = 0$.

Now, we prove $\{\tau_{2n}\}$ is a Cauchy sequence. Suppose $\{\tau_{2n}\}$ is not Cauchy. Hence there is $\varepsilon > 0$ and integers $2m_k, 2n_k \in \mathbb{N}$ such that $2m_k > 2n_k \geq k$ and $d(\tau_{2n_k}, \tau_{2m_k}) \geq \varepsilon$ for $k = 0, 1, 2, \cdots$. Also, we suppose

$$d(\tau_{2n_k}, \tau_{2m_k-2}) < \varepsilon.$$

Therefore, for every $k \in \mathbb{N}$:

$$\begin{aligned} \varepsilon \leq d(\tau_{2n_k}, \tau_{2m_k}) &\leq d(\tau_{2n_k}, \tau_{2m_k-2}) + d(\tau_{2m_k-2}, \tau_{2m_k}) \\ &\leq \varepsilon + d(\tau_{2m_k-2}, \tau_{2m_k}) \end{aligned}$$

and since $d(\tau_{2m_k-2}, \tau_{2m_k}) \to 0$, hence $\lim_{k\to\infty} d(\tau_{2n_k}, \tau_{2m_k}) = \varepsilon$. Observe that

$$\begin{aligned} d(\tau_{2n_k}, \tau_{2m_k}) &\leq d(\tau_{2n_k}, \tau_{2n_k+2}) + d(\tau_{2n_k+2}, \tau_{2m_k+2}) + d(\tau_{2n_k+2}, \tau_{2m_k}) \\ &< d(\tau_{2n_k}, \tau_{2n_k+2}) + d(\tau_{2n_k+1}, \tau_{2m_k+1}) + d(\tau_{2n_k+2}, \tau_{2m_k}). \end{aligned}$$

If $k \to \infty$, we give

$$\varepsilon < \lim_{k\to\infty} d(\tau_{2n_k+1}, \tau_{2m_k+1}).$$

On the other hand,

$$\lim_{k\to\infty} d(\tau_{2n_k+1}, \tau_{2m_k+1}) < \lim_{k\to\infty} d(\tau_{2n_k}, \tau_{2m_k}) = \varepsilon$$

that is a contradiction. Therefore, $\{\tau_{2n}\}$ is Cauchy in A_1 and so $\{\tau_{2n}\}$ converge to $u \in A_1$. Similarly, $\{\tau_{2n+1}\}$ converges to $v \in A_2$.

Now

$$d(A_1, A_2) \leq d(u, \tau_{2n+1}) \leq d(u, \tau_{2n}) + d(\tau_{2n}, \tau_{2n+1}).$$

Thus $d(u, \tau_{2n+1})$ converges to $d(A_1, A_2)$. Since

$$d(A_1, A_2) \leq d(\tau_{2n}, STu) = d(TS\tau_{2n-1}, STu) \leq kd(\tau_{2n-1}, u)+(1-k)d(A_1, A_2).$$

Thus, $d(\tau_{2n}, STu)$ converges to $d(A_1, A_2)$, and so $d(u, STu) = d(A_1, A_2)$ i.e. u is a best proximity point of ST. Similarly, v is a best proximity point of TS. Also, $v = STu$ and $u = TSv$, we have $d(TSv, STu) = d(A_1, A_2)$, i.e. (Sv, Tu) is a best (S, T)-proximity pair for (A_1, A_2). □

Corollary 2.7. *Let (A_1, A_2, A_3) be a triple of nonempty closed subsets of a metric space Ω. Let (T, S) be a cyclic mapping pair on (A_1, A_2, A_3) such that*

$$d(STu, TSv) \leq \alpha d(u, v) + \beta[d(u, STu) + d(v, TSv)] + \gamma d(A_1, A_2),$$

for all $u \in A_1$ and $v \in A_2$, and $\alpha + 2\beta + \gamma = 1$, also

$$d(TSu, TSv) < d(u, v), \quad \forall u, v \in A_2,$$

$$d(STu, STv) < d(u, v), \quad \forall u, v \in A_1.$$

Then there exists a best (S, T)-proximity pair.

Proof. Suppose that $\tau_0 \in A_1$ and define $\tau_{2n+1} = ST\tau_{2n}$ and $\tau_{2n} = TS\tau_{2n-1}$ for every $n \in \mathbb{N}$. Now, we have

$$\begin{aligned} d(\tau_{2n+1}, \tau_{2n}) &= d(ST\tau_{2n}, TS\tau_{2n-1}) \\ &\leq \alpha d(\tau_{2n}, \tau_{2n-1}) + \beta[d(\tau_{2n}, T\tau_{2n}) + d(\tau_{2n-1}, S\tau_{2n-1})] \\ &+ \gamma d(A_1, A_2) \end{aligned}$$

which implies that

$$(1-\beta)d(\tau_{2n+1},\tau_{2n}) \leq (\alpha+\beta)d(\tau_{2n},\lambda_{2n-1})+\gamma d(A_1,A_2)$$

and hence,

$$d(\tau_{2n+1},\tau_{2n}) \leq \frac{\alpha+\beta}{1-\beta}d(\tau_{2n},\tau_{2n-1})+\frac{\gamma}{1-\beta}d(A_1,A_2).$$

If put $k=\frac{\alpha+\beta}{1-\beta}$, therefore,

$$d(\tau_{n+1},\tau_n) \leq kd(\tau_n,\tau_{n+1})+(1-k)d(A_1,A_2).$$

Therefore, by Theorem 2.6 there exist $(x,y)\in A_3\times A_3$ and such that $d(Tx,Sy)=d(A_1,A_2)$. □

In the following we give new form of Theorem 2.6.

Theorem 2.8. *Let (A_1,A_2,A_3) be a triple of nonempty closed subsets of a metric space Ω. Let (T,S) be a cyclic mapping pair on (A_1,A_2,A_3) such that*

$$d(T^2u,S^2v) \leq kd(u,v)+(1-k)d(A_1,A_2), \quad \forall\ u\in A_1,\ v\in A_2,$$
$$d(T^2u,T^2v) < d(u,v), \quad \forall\ u,v\in A_1,$$
$$d(S^2u,S^2v) < d(u,v), \quad \forall\ u,v\in A_2.$$

Then there exists a best (S,T)-proximity pair. In fact, if $(\tau_0,\varsigma_0)\in A_1\times A_2$ and

$$\tau_{n+1}=T\tau_n \text{ and } \varsigma_{n+1}=S\tau_n,\ \forall n\in\mathbb{N},$$

then $\{(\tau_n,\varsigma_n)\}$ converges to a best (S,T)-proximity pair.

Proof. Suppose $(\tau_0,\varsigma_0)\in A_1\times A_2$ and $\tau_{n+1}=T\tau_n$ and $\varsigma_{n+1}=S\varsigma_n$, $\forall n\in\mathbb{N}$. We know that

$$\begin{aligned}
d(\tau_{2n},\varsigma_{2n}) &= d(T\tau_{2n-1},S\varsigma_{2n-1})=d(T^2\tau_{2n-2},S^2\varsigma_{2n-2})\\
&\leq kd(\tau_{2n-2},\varsigma_{2n-2})+(1-k)d(A_1,A_2)\\
&\leq k^2d(\tau_{2n-4},\varsigma_{2n-4})+(1-k^2)d(A_1,A_2)\\
&\vdots\\
&\leq k^nd(\tau_0,\varsigma_0)+(1-k^n)d(A_1,A_2),
\end{aligned}$$

that is,
$$d(\tau_{2n},\varsigma_{2n}) \leq k^n d(\tau_0,\varsigma_0) + (1-k^n)d(A_1,A_2).$$
Therefore, $d(\tau_{2n},\varsigma_{2n}) \to d(A_1,A_2)$.

Now, we show that $\{\tau_{2n}\} \subseteq A_1$ and $\{\varsigma_{2n}\} \subseteq A_2$ are convergence sequences. It is notable that

$$\begin{aligned} d(\tau_{2n},\tau_{2n+2}) &= d(T\tau_{2n-1},T\tau_{2n+1}) \\ &< d(\tau_{2n-1},\tau_{2n+1}) = d(TS\tau_{2n-2},TS\tau_{2n}) \\ &< d(\tau_{2n-2},\tau_{2n+1}) \\ &\vdots \\ &\leq d(\tau_0,\tau_2). \end{aligned}$$

Hence, $\{d(\tau_{2n},\tau_{2n+2})\}$ is monotonic decreasing and bounded below. Hence,

$$\lim_{n\to\infty} d(\tau_{2n},\tau_{2n+2})$$

exists. Suppose $\lim_{n\to\infty} d(\tau_{2n},\tau_{2n+2}) = \theta$. We know that $0 \leq \theta$. Assume that $\theta > 0$. Therefore,

$$\theta = \lim_{n\to\infty} d(\tau_{2n},\tau_{2n+2}) < \lim_{n\to\infty} d(\tau_{2n-2},\tau_{2n}) = \theta.$$

Then, $\theta = 0$.

Also, $\{\tau_{2n}\}$ is a Cauchy sequence. If $\{\tau_{2n}\}$ is not Cauchy, then there is $\varepsilon > 0$ and integers $2m_k, 2n_k \in \mathbb{N}$ such that $2m_k > 2n_k \geq k$ and $d(\tau_{2n_k},\tau_{2m_k}) \geq \varepsilon$ for $k = 0,1,2,\cdots$. Also, we suppose that

$$d(\tau_{2n_k},\tau_{2m_k-2}) < \varepsilon.$$

Therefore, for each $k \in \mathbb{N}$, we have

$$\begin{aligned} \varepsilon \leq d(\tau_{2n_k},\tau_{2m_k}) &\leq d(\tau_{2n_k},\tau_{2m_k-2}) + d(\tau_{2m_k-2},\tau_{2m_k}) \\ &\leq \varepsilon + d(\tau_{2m_k-2},\tau_{2m_k}) \end{aligned}$$

and since $d(\tau_{2m_k-2},\tau_{2m_k}) \to 0$, hence $\lim_{k\to\infty} d(\tau_{2n_k},\tau_{2m_k}) = \varepsilon$. Observe that

$$d(\tau_{2n_k},\tau_{2m_k}) \leq d(\tau_{2n_k},\tau_{2n_k+2}) + d(\tau_{2n_k+2},\tau_{2m_k+2}) + d(\tau_{2n_k+2},\tau_{2m_k}).$$

If $k \to \infty$, we have

$$\varepsilon < \lim_{k\to\infty} d(\tau_{2n_k+2}, \tau_{2m_k+2}).$$

On the other hand,

$$\lim_{k\to\infty} d(\tau_{2n_k+2}, \tau_{2m_k+2}) < \lim_{k\to\infty} d(\tau_{2n_k}, \tau_{2m_k}) = \varepsilon$$

which is a contradiction. Hence, $\{\tau_{2n}\}$ is Cauchy in A_1 and hence $\{\tau_{2n}\}$ converge to $a_1 \in A_1$. Similarly, $\{\varsigma_{2n}\}$ converges to $a_2 \in A_2$.

Now

$$d(A_1, A_2) \leq d(a_1, \varsigma_{2n}) \leq d(a_1, \tau_{2n}) + d(\tau_{2n}, \varsigma_{2n}).$$

Thus $d(a_1, \varsigma_{2n})$ converges to $d(A_1, A_2)$. Since

$$\begin{aligned} d(A_1, A_2) \leq d(T^2 a_1, S^2 a_2) &\leq d(T^2 a_1, \varsigma_{2n}) + d(\varsigma_{2n}, S^2 a_2) \\ &\leq d(a_1, \varsigma_{2n-2}) + d(\varsigma_{2n-2}, a_2) \\ &= d(A_1, A_2) \quad \textit{as } n \to \infty. \end{aligned}$$

Thus, $d(T^2 a_1, S^2 a_2) = d(A_1, A_2)$ i.e. (Ta_1, Sa_2) is a best (S,T)-proximity pair for (A_1, A_2). □

Corollary 2.9. *Let (A_1, A_2, A_3) be a triple of nonempty closed subsets of a strictly convex Banach space Ω . Let (T, S) be a cyclic mapping pair on (A_1, A_2, A_3) such that*

$$d(T^2 u, S^2 v) \leq kd(u, v) + (1-k)d(A_1, A_2), \quad \forall\, u \in A_1,\ v \in A_2,$$

$$d(T^2 u, T^2 v) < d(u, v), \quad \forall\, u, v \in A_1,$$

$$d(S^2 u, S^2 v) < d(u, v), \quad \forall\, u, v \in A_2.$$

If $(A_1 - A_1) \cap (A_2 - A_2) = \emptyset$, then there is an unique best (S,T)-proximity pair.

Proof. By Theorem 2.8 $P_{A_3}(T, S)$ is nonempty. Suppose, there are $x, y \in A_1 \times A_2$ such that $x \neq y$. Also $Sx - Tx \neq Sy - Ty$, by strict convexity of Ω we have $\|\frac{Sx+Sy}{2} - \frac{Tx+Ty}{2}\| < d(A_1, A_2)$. Since A_2 is convex, $\frac{Sx+Sy}{2} \in A_2$ and $\frac{Tx+Ty}{2} \in A_1$ which is a contradiction. Therefore $Sx - Tx = Sy - Ty$ and so $Sx - Sy = Ty - Tx \in (A_1 - A_1) \cap (A_2 - A_2) \neq \emptyset$, which is a contradiction and so $x = y$. □

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