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Original Research Paper

Shadowing Relations with Structural and Topological Stability in Iterated Functions Systems

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Abstract. This paper aims at formulating definitions of topological stability, structural stability, and expansiveness property for an iterated functions system (abbrev, IFS). It is going to show that the shadowing property is necessary condition for topological stability in IFSs. Then, it proves the previous weak converse demonstration with the addition of expansiveness property for IFSs. Then, by giving an example, we show that in an IFS, the shadowing property doesn't imply structural stability.

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1 Introduction

We know that topological stability and structural stability are important properties of dynamical systems, so the relation between shadowing

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these concepts in iterated functions systems is an interesting research topic.
The concept of the iterated functions system was applied in 1981 by Hutchinson[12], but this phrase was presented by Barnsley, briefly call IFS. An IFS includes a nonempty set Λ and some functions $f_\lambda, \lambda \in \Lambda$, on an arbitrary space M. As, in an IFS, the nonempty set Λ can be finite or infinite (countable) or its functions can be special, so different IFSs have been investigated. The importance of using the IFSs is their applicable attractor set that is called fractal [2].
From numerical perspective, whenever we simulate a dynamical system by a computer, since any number is represented in the computer with finite precision, there will be the small difference between the original number and the registered number in the process of the resolution. That is, the error occurs, for example, resultant error from round-off and so on. Passing the time, this error is growing and amplified. If we can find a true solution nearby the generated solution, then we say that the system has shadowing property. It's mean, shadowing property is finding of the true orbit which it remains nearby generated orbit. The approximated (or generated) orbit is called pseudo orbit. The pseudo orbits have an important role in shadowing and every shadowed pseudo orbit provide useful information about the dynamic of system [4]. Using the stable manifold theorem, Bowen in [[6], page: 335] proved the shadowing lemma for diffeomorphisms satisfying axiom A.
Petrov and Pilyugin give sufficient conditions under which a homeomorphism of a compact metric space has the shadowing property [17]. We know that in the direction of a vector field, the hyperbolic property is not satisfied. However, efforts have been done in order to explore the shadowing property in vector fields. Franke and Selgrade, for the first time in 1977, extended the shadowing lemma for the hyperbolic sets of vector fields. These attempts caused the arrival of shadowing lemma in ordinary differential equations [10]. Walters proved that the shadowing is necessary condition for topological stability of a homeomorphism on manifold of dimension≥ 2,[24]. In 1980, Yano proved that this condition is not sufficient,[26], he presented a homeomorphism on a circle that has shadowing property but is no topologically stable. Later, in 1999, Pilyugin proved the converse of Walters's demonstration by adding a

condition,[16]. In this paper, we define topological stability for an IFS. Now, the following questions arise:
Does an IFS have the topological stability if it is topologically stable whereas the functions of IFS are homeomorphisms?
Is the converse of above the demonstration true?
Robinson showed that each structurally stable diffeomorphism has the shadowing property on a closed manifold, then using this assertion, he proved the stability of a diffeomorphism nearby a hyperbolic set [18]. Then the authors developed this assertion about C^1-vector fields and showed that each C^1-vector field with no singular point belongs to C^1-interior all of vector fields with shadowing property if and only if it's Structurally stable,[13]. We know that the set of all diffeomorphisms with shadowing property isn't equivalent to the set of all structurally stable diffeomorphisms. In fact, there exist examples of diffeomorphisms with shadowing property but without strutural stability like a diffeomorphism of the circle S^1. Also this is showed that there exists an equivalence between the set of all diffeomorphisms with variational shadowing and the set of all structurally stable diffeomorphisms [19]. We can see the summary of important and new results in the theory of pseudo-orbit shadowing in the first decade of the 21st century in survey [20]. The main objective of this summary is SP, SS and some equivalent sets on these cases.
In this paper, we define the concept of structural stability for an IFS. Then we attempt to present the demonstration for IFSs similar to Robinson's assertion. We consider conditions that an IFS including some of diffeomorphisms on compact manifold M, shadowing property if it's Structurally stable and investigate reverse demonstration.
Here is a description of the sections in this paper.
In Section 2, we present the basic definitions. We also formulate definitions of the topological stability and weakly topologically stable for an IFS. Then, we prove fundamental Theorem 2.12, through proving some lemmas:
THEOREM 2.12. *Suppose that* $\mathcal{F} = \Big\{ f_\lambda,\ M\ :\lambda \in \Lambda \Big\}$ *be an IFS and* $dimM \geq 2$. *If* $\mathcal{F}$ *is topologically stable, then* $\mathcal{F}$ *has the shadowing property.* In Section 3, we define expansiveness and shadowing uniqueness properties for an IFS. In the following, we prove some lemmas to provide

a proof of the following technical theorem:
THEOREM 3.5. *Suppose that* $\mathcal{F}=\Big\{f_\lambda,\ M\ :\lambda\in\Lambda\Big\}\subset Homeo(M)$ *is an expansive IFS relative to* $\sigma=\Big\{\lambda_0,\lambda_1,\dots\Big\}$ *with expansive constant* η. *Also* $\mathcal{F}$ *has the shadowing property. Then there exist* $\epsilon>0$, $3\epsilon<\eta$, *and* $\delta>0$ *with the following property:*
If $\mathcal{G}=\Big\{g_{\overline{\lambda}},\ M\ :\overline{\lambda}\in\overline{\Lambda}\Big\}\subset Homeo(M)$ *be an IFS and* $\overline{\sigma}$ *be a sequence in* $\overline{\Lambda}^W$ *such that* $\mathcal{D}_0\Big(\mathcal{F}_\sigma,\mathcal{G}_{\overline{\sigma}}\Big)<\delta$ *then there exists a continuous function* $h:M\to M$ *such that:*

$$\begin{cases} i)\quad r\Big(G_{\overline{\sigma}_n}(x),F_{\sigma_n}(h(x))\Big)<\epsilon, & \forall x\in M \text{ and } \forall n\in W,\\ ii)\quad r\Big(x,h(x)\Big)<\epsilon, & \forall x\in M.\end{cases}$$

Moreover, if $\epsilon>0$ *is sufficiently small, then the function* h *is surjective and also* $F_{\sigma_n}oh=hoG_{\overline{\sigma}_n}$ *for every* $n\in W$.
Furthermore, we prove Corollary 3.6 that is an important conclusion of the above theorem:
COROLLARY 3.6. *If IFS* $\mathcal{F}$ *has shadowing property and moreover* $\mathcal{F}$ *is expansive relative to the sequence* σ *with small expansive constant, then* $\mathcal{F}$ *is weakly topologically stable.*
In Section 4, we present a formulation for the concept of the SS of an IFS. Then, by giving an example, we show that in an IFS, the shadowing property doesn't imply structural stability.

2 Topological Stability and Shadowing Property in an IFS

Investigations of topological and structural stability of a diffeomorphism have been simultaneously progressed. As we know examining topological stability of a diffeomorphism has been done by tools; for example, shadowing property, [15], Lyapunov functions, [14] and [22]. In this paper, we study topological stability of an IFS by shadowing property. First, we give basic definitions.

Definition 2.1. Let (M,d) be a complete metric space and $\mathcal{F}$ be a family of continuous mappings $f_\lambda:M\to M$ for every $\lambda\in\Lambda$, where Λ is a finite nonempty set; that is, $\mathcal{F}=\Big\{f_\lambda,\ M\ :\lambda\in\Lambda=\{1,2,\dots,N\}\Big\}$.

Which called an "Iterated Functions System" or shortly, IFS.

In this paper, all of the functions of an IFS are homeomorphism or diffeomorphism. Let W be the set of non-negative integer numbers and Λ^W denote the set of all infinite sequences $\{\lambda_i\}_{i\in W}$ where $\lambda_i = \lambda(i) \in \Lambda$. Let us introduce the temporary notation F_{σ_n} for $f_{\lambda_n} o f_{\lambda_{n-1}} o \ldots o f_{\lambda_1} o f_{\lambda_0}$

Definition 2.2. Suppose that $\mathcal{F} = \Big\{ f_\lambda,\, M \,:\, \lambda \in \Lambda \Big\}$ is an IFS. The sequence $\{x_i\}_{i\in W} \subset M$ is said to be a "chain" for IFS $\mathcal{F}$ if, for every $i \in W$, there exists $\lambda_i = \lambda(i) \in \Lambda$ such that $x_{i+1} = f_{\lambda_i}(x_i)$.

Definition 2.3. Let $\mathcal{F} = \Big\{ f_\lambda,\, M \,:\, \lambda \in \Lambda \Big\}$ be an IFS. For the given $\delta > 0$, the sequence $\{x_i\}_{i\in W}$ is called a "$\delta-$chain" for IFS $\mathcal{F}$ if, for every $i \in W$, there exists $\lambda_i = \lambda(i) \in \Lambda$ such that $d\Big(x_{i+1}, f_{\lambda_i}(x_i)\Big) \leq \delta$.

Definition 2.4. Suppose that $\mathcal{F} = \Big\{ f_\lambda,\, M \,:\, \lambda \in \Lambda \Big\}$ is an IFS. We say that $\mathcal{F}$ has "shadowing property" if for every $\epsilon > 0$ there exists $\delta > 0$ such that for every δ-chain $\{x_i\}_{i\in W}$ with the sequence $\sigma = \{\lambda_i\}_{i\in W}$, there exists the chain $\{y_i\}_{i\in W}$ with the sequence $\bar{\sigma} = \{\bar{\lambda_i}\}_{i\in W}$ such that $d(x_i, y_i) \leq \epsilon$ for every $i \in W$. Sometimes is said that the chain $\{y_i\}_{i\in W}$ (ϵ)-shadows δ-chain $\{x_i\}_{i\in W}$.

Now suppose M is a C^∞ smooth m-dimensional closed(that is, compact and boundaryless) manifold, and r is a Riemannian metric on M. We consider the set of all homeomorphisms on M. The metric ρ_0 on this set defined as follows:
if f and g are the two homeomorphisms on M, we define

$$\rho_0(f,g) = Max\Big\{ r\Big(f(x), g(x)\Big),\; r\Big(f^{-1}(x), g^{-1}(x)\Big);\; for\ all\ x \in M \Big\}.$$

This set with the topology induced by the metric ρ_0 will be denoted by Homeo(M).

Definition 2.5. Suppose that $\mathcal{F} = \Big\{ f_\lambda,\, M \,:\, \lambda \in \Lambda \Big\}$ and $\mathcal{G} = \Big\{ g_{\overline{\lambda}},\, M \,:\, \overline{\lambda} \in \overline{\Lambda} \Big\}$ are two IFSs as the subsets of the space $Homeo(M)$. Let $\sigma \in \Lambda^W$ and $\overline{\sigma} \in \overline{\Lambda}^W$. We consider the sequences $\mathcal{F}_\sigma = \{f_{\lambda_i}\}_{i\in W}$ and $\mathcal{G}_{\overline{\sigma}} =$

$\{g_{\overline{\lambda}_i}\}_{i\in W}$ where $\lambda_i \in \sigma$ and $\overline{\lambda}_i \in \overline{\sigma}$ for every $i \in W$. We will denote by $\mathcal{D}_0$ the measure distance between the two IFSs relative to the sequences σ and $\overline{\sigma}$ and define as follows:

$$\mathcal{D}_0\Big(\mathcal{F}_\sigma, \mathcal{G}_{\overline{\sigma}}\Big) = sup\Big\{\rho_0(f_{\lambda_i}, g_{\overline{\lambda}_i}) : \quad \lambda_i \in \sigma,\ \overline{\lambda}_i \in \overline{\sigma}\ \ for\ all\ i \in W\Big\}.$$

Definition 2.6. Let $\mathcal{F} = \Big\{f_\lambda,\ M\ : \lambda \in \Lambda\Big\}$ be an IFS. We say that an IFS $\mathcal{F}$ is "topologically stable" if for a given $\epsilon > 0$, there exists $\delta > 0$ such that if $\mathcal{G} = \Big\{g_{\overline{\lambda}},\ M\ : \overline{\lambda} \in \overline{\Lambda}\Big\}$ be an IFS that for every σ and $\overline{\sigma}$ with $\mathcal{D}_0\Big(\mathcal{F}_\sigma, \mathcal{G}_{\overline{\sigma}}\Big) < \delta$ then there exists a continuous mapping h of M onto M with the following properties:

$$\begin{cases} i) \quad F_{\sigma_n} oh = hoG_{\overline{\sigma}_n}, \quad \forall n \in W, \\ ii) \quad r\Big(x, h(x)\Big) < \epsilon, \quad \forall x \in M. \end{cases}$$

Definition 2.7. Let $\mathcal{F} = \Big\{f_\lambda,\ M\ : \lambda \in \Lambda\Big\}$ be an IFS and $\sigma \in \Lambda^W$ be given. The IFS $\mathcal{F}$ is called "weakly topologically stable" if for a given $\epsilon > 0$, there exists $\delta > 0$ such that if $\mathcal{G} = \Big\{g_{\overline{\lambda}},\ M\ : \overline{\lambda} \in \overline{\Lambda}\Big\}$ be an IFS that for every $\overline{\sigma}$ with $\mathcal{D}_0\Big(\mathcal{F}_\sigma, \mathcal{G}_{\overline{\sigma}}\Big) < \delta$ then there exists a continuous mapping h of M onto M with the following properties:

$$\begin{cases} i) \quad F_{\sigma_n} oh = hoG_{\overline{\sigma}_n}, \quad \forall n \in W, \\ ii) \quad r\Big(x, h(x)\Big) < \epsilon, \quad \forall x \in M. \end{cases}$$

Now, we are going to find the relation between topological stability and shadowing property in IFSs. In [16], Pilyugin proved that a topologically stable homeomorphism has SP. In this paper, we also show this demonstration for IFSs using his methods. But since we deal with the set of functions, proving is harder and more complex. In the proof of the following lemma, the method of proof of shadowing property on N subset of M is a bit different from Pilyugin's method because of presenting of a chain.

Lemma 2.8. *Consider IFS* $\mathcal{F} = \Big\{f_\lambda,\ M\ : \lambda \in \Lambda\Big\}$. *Suppose that* $\mathcal{F}$ *has finite shadowing property on* $N (N \subset M)$*; that is, for a given* $\epsilon > 0$ *there exists* $\delta > 0$ *such that for every set* $\Big\{x_0, \ldots, x_m\Big\} \subset N$ *that satisfies* *in the inequality* $r\Big(x_{i+1}, f_{\lambda_i}(x_i)\Big) \leq \delta$ *for every* $i = 0, \dots, m-1$, *then there exists a chain* $\{y_i\}_{i\in W} \subset M$ *such that for every* $i = 0, \dots, m-1$, $r(x_i, y_i) < \epsilon$. *Thus,* $\mathcal{F}$ *has shadowing property on* N.

Proof. For a given $\epsilon > 0$, by using the uniformly continuous f_λ, $\lambda \in \Lambda$, there exists $\delta_\lambda > 0$ such that for every $x, y \in M$ if $r(x, y) < \delta_\lambda$, then $r\Big(f_\lambda(x), f_\lambda(y)\Big) < \frac{\epsilon}{4}$. Put $\delta_0 = min\{\delta_\lambda : \lambda \in \Lambda\}$. Thus, for every $x, y \in M$ and for each $\lambda \in \Lambda$, $r\Big(f_\lambda(x), f_\lambda(y)\Big) < \frac{\epsilon}{4}$ if $r(x, y) < \delta_0$. For the obtained δ_0, duo to finite shadowing property of $\mathcal{F}$, there exists $\delta > 0$ such that for every set $\Big\{x_0, \dots, x_m\Big\} \subset N$ that satisfies in the inequality $r\Big(x_{i+1}, f_{\lambda_i}(x_i)\Big) \leq \delta$ for every $i = 0, \dots, m-1$, then there exists a chain $\{y_i\}_{i\in W} \subset M$ such that for every $i = 0, \dots, m-1$, $r(x_i, y_i) < \delta_0$. We can assume that $\delta < \frac{\epsilon}{4}$. Now, suppose that $\xi = \{x_i\}_{i\in W} \subset N$ is a δ-chain for IFS $\mathcal{F}$. Let $m > 0$ be a constant number. Consider the set $\{x_i : 0 \leq i \leq m\}$ with $r\Big(x_{i+1}, f_{\lambda_i}(x_i)\Big) \leq \delta$ for every i, $0 \leq i \leq m-1$, then there exists a chain $\{{y'}_{m,i}\}_{i\in W} \subset M$ that

$$r\Big(x_i, {y'}_{m,i}\Big) < \frac{\epsilon}{4} \qquad \forall i, \quad 0 \leq i \leq m-1. \tag{1}$$

Now, we choose the value k in W. According to what was said, we see that for the arbitrary positive integer number m there exists ${y'}_{m,k} \in M$. Consider the sequence $\{{y'}_{m,k}\}_{m=1}^{\infty} \subset M$. This space is a compact metric space then this sequence has limit points in this space and since the manifold M is Hausdorff, limit point is unique alike y_k; $y_k \in M$. With repeating of the previous process for every value k in W, we obtain the sequence $\{y_k\}_{k\in W} \subset M$. We claim that this sequence is a chain for IFS $\mathcal{F}$ and for each $k \in W$, $r(x_k, y_k) < \frac{\epsilon}{4}$. Since the metric r and f_λ, $\lambda \in \Lambda$, are continuous, for every $k \in W$ we have

$$\begin{aligned} r\Big(y_{k+1}, f_{\lambda_k}(y_k)\Big) &= \lim_{m\to+\infty} r\Big({y'}_{m,k+1}, f_{\lambda_k}({y'}_{m,k})\Big) \\ &\leq \lim_{m\to+\infty} r\Big({y'}_{m,k+1}, x_{k+1}\Big) \\ &+ \lim_{m\to+\infty} r\Big(x_{k+1}, f_{\lambda_k}({y'}_{m,k})\Big). \end{aligned}$$

Passing to the limit as $m \to +\infty$ in the inequality 1, we get

$$r(x_k, y_k) < \frac{\epsilon}{4}, \qquad \forall k \in W. \tag{2}$$

Since r is metric so we can write

$$\begin{aligned}\lim_{m\to+\infty} r\Big(x_{k+1}, f_{\lambda_k}(y'_{m,k})\Big) \leq\ & \lim_{m\to+\infty} r\Big(x_{k+1}, f_{\lambda_k}(x_k)\Big)\\ &+\lim_{m\to+\infty} r\Big(f_{\lambda_k}(x_k), f_{\lambda_k}(y_k)\Big)\\ &+\lim_{m\to+\infty} r\Big(f_{\lambda_k}(y_k), f_{\lambda_k}(y'_{m,k})\Big).\end{aligned}$$

We know that ξ is a δ-chain, the function $f_{\lambda_k}(\lambda_k \in \Lambda)$ is uniformly continuous, $\delta < \frac{\epsilon}{4}$, and the sequence $\{y'_{m,k}\}_{m=1}^{\infty}$ is convergent to y_k. Regarding these facts, we will get the following relation from the latter inequality:

$$\lim_{m\to+\infty} r\Big(x_{k+1}, f_{\lambda_k}(y'_{m,k})\Big) \leq \epsilon + \epsilon + \epsilon = 3\frac{\epsilon}{4}. \tag{3}$$

Also from the relation 2, we have

$$\lim_{m\to+\infty} r\Big(y'_{m,k+1}, x_{k+1}\Big) = r\Big(y_{k+1}, x_{k+1}\Big) < \frac{\epsilon}{4}. \tag{4}$$

Thus, for each $k \in W$, we obtain the following relation from the relations 3 and 4:

$$r\Big(y_{k+1}, f_{\lambda_k}(y_k)\Big) \leq \epsilon.$$

According to the making of the sequence $\{y_k\}_{k\in W}$, we see that if $\epsilon > 0$ be a very small value, then the obtained sequence $\{y_k\}_{k\in W}$ is valid for every arbitrary value $\epsilon > 0$. Therefore for this sequence $\{y_k\}_{k\in W}$, the previous relation is true for every arbitrary value $\epsilon > 0$. So we will get $r\Big(y_{k+1}, f_{\lambda_k}(y_k)\Big) = 0$ and hence $y_{k+1} = f_{\lambda_k}(y_k)$ and this means that, $\{y_k\}_{k\in W}$ is a chain for IFS $\mathcal{F}$ and by considering the relation 2, $r(x_k, y_k) < \epsilon$ for every $k \in W$, so our claim is proved. Therefore, for a given $\epsilon > 0$ we found $\delta > 0$ such that for every δ-chain $\{x_k\}_{k\in W}$, there exists a chain $\{y_k\}_{k\in W}$ with $r(x_k, y_k) < \epsilon$, that is, $\mathcal{F}$ has shadowing property on N. □

Lemma 2.9. *Assume dim*$(M) \geq 2$. *Consider a finite collection* $\Big\{(p_i, q_i) \in M \times M : \quad i = 1, \ldots, k\Big\}$ *such that*

$$\begin{cases} i) & p_i \neq p_j,\ q_i \neq q_j \quad \text{for } 1 \leq i < j \leq k,\\ ii) & r\Big(p_i, q_i\Big) < \delta, \quad \text{for } i = 1, \ldots, k, \text{ with small positive } \delta.\end{cases}$$

Then, there exists a diffeomorphism f of M with the following properties:

$$\begin{cases} i) & \rho_0(f, id) < 2\delta, \quad (here\ id\ is\ the\ identity\ mapping\ of\ M), \\ ii) & f(p_i) = q_i, \quad for\ i = 1, \ldots, k. \end{cases}$$

Proof. It has proved at Lemma 2.1.1. in [16]. $\square$

Lemma 2.10. *Suppose $\xi = \{x_i\}_{i \in W}$ is a δ-chain for the IFS $\mathcal{F} = \Big\{f_\lambda,\ M\ : \lambda \in \Lambda\Big\} \subset Homeo(M)$ with the sequence σ. Consider the integer number $m \geq 0$ and also the number $\eta > 0$. Then, there exists a set of the points $\Big\{y_0, \ldots, y_m\Big\}$ such that it satisfies in the following conditions:*

1. *for every i, $0 \leq i \leq m$, $r(x_i, y_i) < \eta$,*
2. *for every i, $0 \leq i \leq m-1$, and $\lambda_i \in \sigma$, $r\Big(y_{i+1}, f_{\lambda_i}(y_i)\Big) < 3\delta$,*
3. *for every i, j, $0 \leq i < j \leq m$, $y_i \neq y_j$.*

Proof. We prove the statement by using induction on m. If $m = 0$, then it is sufficient to consider the singleton set $\Big\{y_0 = x_0\Big\}$ then, $r(x_0, y_0) = r(x_0, x_0) = 0 < \eta$, so for $m = 0$, the lemma is true.
Suppose that the statement is true for $m - 1$. Now we prove the lemma for m.
For a given $\eta > 0$, assume that $\eta < \delta$ since if $\eta \geq \delta$, then there exist $q, p \in \mathbb{N}$ such that $\eta = q\delta + p$ where $0 < p < \delta$ or $p = 0$.
If $0 < p < \delta$ then, assume $\eta = p$ and if $p = 0$ then, take the new η to be less than $\frac{\eta}{q}$. Since $\mathcal{F} \subset Homeo(M)$ and M is compact, each function f_λ, $\lambda \in \Lambda$, is uniformly continuous. Consequently, for δ and f_λ there exists $\delta_\lambda(\delta) > 0$ such that for every $x, y \in M$ that $r(x, y) < \delta_\lambda$ then $r\Big(f_\lambda(x), f_\lambda(y)\Big) < \delta$. Put $\delta_0 = min\{\delta_\lambda\ :\ \lambda \in \Lambda\}$. We can consider $\delta_0 < \eta$, that is, $\delta_0 \in (0, \eta)$ because if $\delta_0 \geq \eta$, then it is sufficient to take the new δ_0, δ_0', to be less than η. Thus, for every $x, y \in M$ that $r(x, y) < \delta_0'$ we have $r(x, y) < \delta_0' < \eta \leq \delta_0$, according to the assumption of uniformly continuous, $r\Big(f_\lambda(x), f_\lambda(y)\Big) < \delta$ for every $\lambda \in \Lambda$. By using the assumption of induction, we can find a set of the points $\Big\{y_0, \ldots, y_{m-1}\Big\}$ such that

1. for every i, $0 \leq i \leq m-1$, $r(x_i, y_i) < \delta_0$,

2. for every i, $0 \le i \le m-2$, and $\lambda_i \in \sigma$, $r\Big(y_{i+1}, f_{\lambda_i}(y_i)\Big) < 3\delta$,

3. for every i, j, $0 \le i < j \le m-1$, $y_i \ne y_j$.

Since the functions of IFS$\mathcal{F}$ are uniformly continuous, we can choose a point y_m such that $r(x_m, y_m) < \delta_0$ and also $y_m \ne y_i$ for every $i = 0, \dots, m-1$. Now, for $\lambda_{m-1} \in \sigma$, we have

$$\begin{aligned} r\Big(f_{\lambda_{m-1}}(y_{m-1}), y_m\Big) &< r\Big(f_{\lambda_{m-1}}(y_{m-1}), x_m\Big) + r(x_m, y_m) \\ &< r\Big(f_{\lambda_{m-1}}(y_{m-1}), f_{\lambda_{m-1}}(x_{m-1})\Big) \\ &+ r\Big(f_{\lambda_{m-1}}(x_{m-1}), x_m\Big) + r(x_m, y_m). \end{aligned}$$

We know that the function f_λ, $\lambda \in \Lambda$, is uniformly continuous and $r(y_{m-1}, x_{m-1}) < \delta_0$ then $r\Big(f_{\lambda_{m-1}}(y_{m-1}), f_{\lambda_{m-1}}(x_{m-1})\Big) < \delta$.
Also $\lambda_{m-1} \in \sigma$ and $\xi = \{x_i\}_{i \in W}$ is a δ-chain and $r(x_m, y_m) < \delta_0$ and $\delta_0 < \eta < \delta$ thus, the previous relation is $r\Big(f_{\lambda_{m-1}}(y_{m-1}), y_m\Big) < \delta + \delta + \delta = 3\delta$. Thus, the statement of induction for m was proved. □ The method of finding the required IFS in the demonstration of the following lemma is complicated.

Lemma 2.11. *Suppose $dimM \ge 2$ and let $\mathcal{F} = \Big\{f_\lambda, M \ : \lambda \in \Lambda\Big\} \subset Homeo(M)$ be an IFS. Let $m \in \mathbb{N}$ and $\Delta > 0$ be given. Then there exists $\delta > 0$ with the following property:*
If $\xi = \{x_i\}_{i \in W}$ is a δ-chain for IFS $\mathcal{F}$ with the sequence σ, then there exist an IFS $\mathcal{G} \subset Homeo(M)$ and a sequence $\overline{\sigma}$ such that $\mathcal{D}_0\Big(\mathcal{F}_\sigma, \mathcal{G}_{\overline{\sigma}}\Big) < \Delta$. Also, there exists a chain $\{y_i\}_{i \in W}$ for IFS $\mathcal{G}$ with $r(x_i, y_i) < \Delta$ for $i = 0, \dots, m$.

Proof. Since $\mathcal{F} \subset Homeo(M)$, the function f_λ^{-1} for every $\lambda \in \Lambda$ is continuous and consequently, it is uniformly continuous on the compact space M. Thus for a given $\Delta > 0$ and for each $\lambda \in \Lambda$ there exists $\delta_\lambda(\Delta) > 0$ such that for every $x, y \in M$ with $r(x, y) < \delta_\lambda$ then $r\Big(f_\lambda^{-1}(x), f_\lambda^{-1}(y)\Big) < \Delta$. Put $\delta_0 = min\{\frac{\Delta}{2}, \delta_\lambda;\ \lambda \in \Lambda\}$. Then, put $\delta = \frac{\delta_0}{6}$ and suppose that $\xi = \{x_i\}_{i \in W}$ is a δ-chain for IFS $\mathcal{F}$ with the sequence σ. By Lemma 2.10, there exists a finite sequence $\Big\{y_0,\dots,y_m\Big\}$ with the following conditions

1. for every i, $0\leq i\leq m$, $r(x_i,y_i)<\Delta$,

2. for every i, $0\leq i\leq m-1$, and $\lambda_i\in\sigma$, $r\Big(y_{i+1},f_{\lambda_i}(y_i)\Big)<3\delta=\frac{\delta_0}{2}$,

3. for every i,j, $0\leq i<j\leq m$, $y_i\neq y_j$.

Now, for every i, $0\leq i\leq m-1$, and $\lambda_i\in\sigma$, consider the singleton set $\Big\{\Big(f_{\lambda_i}(y_i),y_{i+1}\Big)\Big\}$. Considering condition(2) and using Lemma 2.9, there exists a diffeomorphism h_{λ_i} of M such that $\rho_0\Big(h_{\lambda_i},id\Big)<\delta_0$ and also $h_{\lambda_i}\Big(f_{\lambda_i}(y_i)\Big)=y_{i+1}$. Now, for every i, $i=0,\dots,m-1$, $\lambda_i\in\sigma$, and $\lambda\in\Lambda$, put $g_{\lambda_i,\lambda}=h_{\lambda_i}of_\lambda$. We prove that $\rho_0\Big(g_{\lambda_i,\lambda},f_\lambda\Big)<\Delta$.
Suppose $x\in M$. For each i, $i=0,\dots,m-1$, $\lambda_i\in\sigma$, and $\lambda\in\Lambda$, we have

$$\begin{aligned}
r\Big(g_{\lambda_i,\lambda}(x),f_\lambda(x)\Big) &= r\Big((h_{\lambda_i}of_\lambda)(x),f_\lambda(x)\Big)\\
&= r\Big(h_{\lambda_i}\Big(f_\lambda(x)\Big),f_\lambda(x)\Big)\\
&= r\Big(h_{\lambda_i}\Big(f_\lambda(x)\Big),id\Big(f_\lambda(x)\Big)\Big)\\
&< \rho_0\Big(h_{\lambda_i},id\Big)<\delta_0<\Delta,
\end{aligned}$$

and

$$\begin{aligned}
r\Big(g^{-1}_{\lambda_i,\lambda}(x),f^{-1}_\lambda(x)\Big) &= r\Big((f^{-1}_\lambda oh^{-1}_{\lambda_i})(x),f^{-1}_\lambda(x)\Big)\\
&= r\Big(f^{-1}_\lambda\Big(h^{-1}_{\lambda_i}(x)\Big),f^{-1}_\lambda(x)\Big)<\Delta,
\end{aligned}$$

because $r\Big(h^{-1}_{\lambda_i}(x),x\Big)=r\Big(h^{-1}_{\lambda_i}(x),I^{-1}(x)\Big)<\rho_0\Big(h_{\lambda_i},id\Big)<\delta_0$ and the function f^{-1}_λ is uniformly continuous.
Therefore,

$$\rho_0\Big(g_{\lambda_i,\lambda},f_\lambda\Big)=Max\Big\{r\Big(g_{\lambda_i,\lambda}(x),f_\lambda(x)\Big),\,r\Big(g^{-1}_{\lambda_i,\lambda}(x),f^{-1}_\lambda(x)\Big)\Big\}<\Delta$$

We consider the first m elements of the sequence σ, that is $\lambda_0, \lambda_1, \ldots, \lambda_{m-1}$, and we put them in a set which is called $\overline{\Lambda}$. Then consider IFS $\mathcal{G}$ as $\mathcal{G} = \Big\{g_{\lambda_i,\lambda},\ M\ :\ \lambda_i \in \overline{\Lambda},\ \lambda \in \Lambda\Big\}$. Every element of $\mathcal{G}$ is a homeomorphism on M since it is the composition of the homeomorphisms so we can consider IFS $\mathcal{G}$ as a subset of $Homeo(M)$. Fix the value k, $0 \leq k \leq m-1$, we make the sequence $\overline{\sigma} = \Big\{(\lambda_k, \lambda_j)\Big\}_{j\in W}$ where $\lambda_j \in \sigma$. Now, set $\mathcal{G}_{\overline{\sigma}} = \{g_{\lambda_i,\lambda}\}_{(\lambda_i,\lambda)\in\overline{\sigma}} = \{g_{\lambda_k,\lambda_j}\}_{j\in W}$. According to the above, we have $\mathcal{D}_0\Big(\mathcal{F}_\sigma, \mathcal{G}_{\overline{\sigma}}\Big) < \Delta$. Now, we are going to extend the set $\Big\{y_0, \ldots, y_m\Big\}$ such that $\{y_j\}_{j\in W}$ be a chain for IFS $\mathcal{G}$. It is sufficient to put $y_{j+1} = g_{\lambda_k,\lambda_j}(y_j)$ for every positive integer j with condition $j \geq m$. Thus, $\{y_j\}_{j\in W}$ is a chain for IFS $\mathcal{G}$ such that for every j, $j = 0, \ldots, m$, $r(x_j, y_j) < \Delta$ by condition(1). So, the proof is completed. □

Theorem 2.12. *Suppose that $dimM \geq 2$ and $\mathcal{F} = \Big\{f_\lambda,\ M\ : \lambda \in \Lambda\Big\}$ be an IFS. If $\mathcal{F}$ is topologically stable, then $\mathcal{F}$ has shadowing property.*

Proof. Let $\epsilon > 0$ be given. For $\frac{\epsilon}{2}$, since $\mathcal{F}$ is the topologically stable, there exists a $\Delta > 0$ with the mentioned properties in the definition of the topological stability.
We assume that $\Delta < \frac{\epsilon}{2}$ because if $\Delta \geq \frac{\epsilon}{2}$, then there exist the natural numbers q and p such that $\Delta = q.\frac{\epsilon}{2} + p$, consequently, we can consider Δ being equal to p or less than $\frac{\Delta}{q}$. For Δ and an arbitrary natural number m, using Lemma 2.11 and the introduced notations in this lemma, there exists $\delta > 0$ such that for every δ-chain $\xi = \{x_i\}_{i\in W}$ of IFS $\mathcal{F}$ with the sequence σ exist an IFS $\mathcal{G} = \Big\{g_{\lambda_i,\lambda},\ M\ :\ \lambda_i \in \overline{\Lambda},\ \lambda \in \Lambda\Big\} \subset Homeo(M)$ and a sequence $\overline{\sigma}$ such that $\mathcal{D}_0\Big(\mathcal{F}_\sigma, \mathcal{G}_{\overline{\sigma}}\Big) < \Delta$ and also there exists a chain $\{y_i\}_{i\in W}$ for IFS $\mathcal{G}$ with $r(x_i, y_i) < \Delta$ for all i, $0 \leq i \leq m$. Since $\mathcal{D}_0\Big(\mathcal{F}_\sigma, \mathcal{G}_{\overline{\sigma}}\Big) < \Delta$, then for this σ on the basis of topological stability of IFS $\mathcal{F}$, there exists a continuous mapping h of M onto M such that

$$\begin{cases} i) & F_{\sigma_n} oh = hoG_{\overline{\sigma}_n}, \quad \forall n \in W, \\ ii) & r\Big(x, h(x)\Big) < \frac{\epsilon}{2}, \quad \forall x \in M. \end{cases}$$

Now, for every $i \in W$ put $z_i = h(y_i)$. We prove that the sequence $\{z_i\}_{i\in W}$ is a chain for IFS $\mathcal{F}$.

On the basis of the proof of Lemma 2.11, $\{y_i\}_{i\in W}$ is a chain for IFS $\mathcal{G}$ with the sequence $\Big\{(\lambda_0,\lambda_0),(\lambda_1,\lambda_1),\ldots,(\lambda_{m-1},\lambda_{m-1}),(\lambda_k,\lambda_m),(\lambda_k,\lambda_{m+1}),\ldots\Big\}$ where $\lambda_k \in \overline{\Lambda}$ is fixed. Considering this matter and the part(i) of the above relations, we see that

$$
\begin{aligned}
z_0 &= h(y_0)\\
z_1 &= h(y_1) = h\Big(g_{\lambda_0,\lambda_0}(y_0)\Big) = f_{\lambda_0}\Big(h(y_0)\Big) = f_{\lambda_0}(z_0)\\
z_2 &= h(y_2) = h\Big(g_{\lambda_1,\lambda_1}(y_1)\Big) = h\Big((g_{\lambda_1,\lambda_1} o g_{\lambda_0,\lambda_0})(y_0)\Big)\\
&= (f_{\lambda_1} o f_{\lambda_0})\Big(h(y_0)\Big) = f_{\lambda_1}\Big(f_{\lambda_0}(h(y_0))\Big) = f_{\lambda_1}\Big(f_{\lambda_0}(z_0)\Big) = f_{\lambda_1}(z_1)\\
&\vdots\\
z_{i+1} &= h(y_{i+1}) = h\Big(G_{\overline{\sigma}_i}(y_0)\Big) = F_{\sigma_i}\Big(h(y_0)\Big) = F_{\sigma_i}(z_0)\\
&= (f_{\lambda_i} o f_{\lambda_{k-1}} o \ldots o f_{\lambda_1} o f_{\lambda_0})(z_0) = (f_{\lambda_i} o f_{\lambda_{i-1}} o \ldots o f_{\lambda_1})(f_{\lambda_0}(z_0))\\
&= (f_{\lambda_i} o f_{\lambda_{i-1}} o \ldots o f_{\lambda_1})(z_1) = \ldots = f_{\lambda_i}(z_i)
\end{aligned}
$$

The relation $z_{i+1} = f_{\lambda_i}(z_i)$ shows that $\{z_i\}_{i\in W}$ is a chain for IFS $\mathcal{F}$. Also, for every i, $i = 0,\ldots,m$, we obtain that

$$
\begin{aligned}
r(x_i,z_i) &= r\Big(x_i,h(y_i)\Big)\\
&\leq r(x_i,y_i) + r\Big(y_i,h(y_i)\Big)\\
&\leq \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon
\end{aligned}
$$

Thus, for given $\epsilon > 0$ we found $\delta > 0$ such that if $\xi = \{x_i\}_{i\in W}$ is a δ-chain for IFS $\mathcal{F}$, then there exists a chain $\{z_i\}_{i\in W}$ for $\mathcal{F}$ with $r(x_i,z_i) \leq \epsilon$, for every i, $i = 0,\ldots,m-1$. Therefore, according to Lemma 2.8, $\mathcal{F}$ has shadowing property. $\square$

3 The Weak Converse Demonstration of the Previous Section

In this section, we define expansiveness property for an IFS and then prove the converse demonstration of the previous section by Lemmas 3.2 and 3.4 and Theorem 3.5.

Definition 3.1. Consider IFS $\mathcal{F} = \Big\{f_\lambda,\, M\ :\ \lambda \in \Lambda\Big\}$. Assume that the sequence $\sigma = \Big\{\lambda_0, \lambda_1, \dots\Big\}$ be given. We say that $\mathcal{F}$ is "expansive relative to σ" if there exists $\Delta > 0$ such that for two arbitrary points x and y in M with $r\Big(F_{\sigma_n}(x), F_{\sigma_n}(y)\Big) \leq \Delta$, for each $n \in W$, then $x = y$. The number Δ is called "expansive constant relative to σ".

Lemma 3.2. *Suppose that IFS* $\mathcal{F} = \Big\{f_\lambda,\, M\ :\ \lambda \in \Lambda\Big\} \subset Homeo(M)$ *is expansive relative to* $\sigma = \Big\{\lambda_0, \lambda_1, \dots\Big\}$ *with constant expansive* η. *Let* $\mu > 0$ *be given. Thus, there exists* $a \geq 1$ *such that if for every* $x, y \in M$ *that verify to the relation* $r\Big(F_{\sigma_n}(x), F_{\sigma_n}(y)\Big) \leq \eta$, *for each integer number* n *with* $0 \leq n < a$, *then* $r(x, y) < \mu$.

Proof. Let $\mu > 0$ be given. By demonstration of contradiction, we assume that there exists no $a \geq 1$ that satisfies in the properties of the lemma, thus for each $a \geq 1$ there exist the points x_a and y_a such that $r\Big(F_{\sigma_n}(x_a), F_{\sigma_n}(y_a)\Big) \leq \eta$ for all integer number n with $0 \leq n < a$ and also $r\Big(x_a, y_a\Big) \geq \mu$. Choose two arbitrary subsequences $\Big\{x_{a_i}\Big\}_{i=1}^{+\infty}$ and $\Big\{y_{a_i}\Big\}_{i=1}^{+\infty}$ with condition $a_i \to +\infty$ as $i \to +\infty$. Since M is a compact metric space so it has Bolzano-Weierstrass property and since it's Hausdorff space so these subsequences have unique limit points in this space. Therefore, the points x and y there exist such that $x_{a_i} \to x$ and $y_{a_i} \to y$ as $i \to +\infty$.

Since $\mathcal{F} \subset Homeo(M)$, the function F_{σ_n}, for every n, is the composition of continuous functions and consequently, itself is a continuous function. Therefore, we have $F_{\sigma_n}(x_{a_i}) \to F_{\sigma_n}(x)$ and $F_{\sigma_n}(y_{a_i}) \to F_{\sigma_n}(y)$ as $i \to +\infty$. Thus, for a given $\epsilon > 0$, there exist $k_1, k_2 \in \mathbb{N}$ such that for every $i \geq k_1$, $r\Big(F_{\sigma_n}(x_{a_i}), F_{\sigma_n}(x)\Big) < \frac{\epsilon}{2}$ and for every $i \geq k_2$, $r\Big(F_{\sigma_n}(y_{a_i}), F_{\sigma_n}(y)\Big) < \frac{\epsilon}{2}$. Consequently, for every $i \geq k = max\{k_1, k_2\}$ also the above relations are true. We have

$$\begin{aligned} r\Big(F_{\sigma_n}(x), F_{\sigma_n}(y)\Big) &\leq r\Big(F_{\sigma_n}(x), F_{\sigma_n}(x_{a_i})\Big) + r\Big(F_{\sigma_n}(x_{a_i}), F_{\sigma_n}(y_{a_i})\Big) \\ &+ r\Big(F_{\sigma_n}(y_{a_i}), F_{\sigma_n}(y)\Big). \end{aligned}$$

Now, for sufficiently large a_i when $i \to +\infty$, we see that

$$r\Big(F_{\sigma_n}(x), F_{\sigma_n}(y)\Big) < \frac{\epsilon}{2} + \eta + \frac{\epsilon}{2} = \epsilon + \eta.$$

Whereas $a_i \to +\infty$ as $i \to +\infty$ so the previous relation for every $n \in W$ is true. Since $\epsilon > 0$ is a small arbitrary number then for each $n \in W$, we have $r\Big(F_{\sigma_n}(x), F_{\sigma_n}(y)\Big) \leq \eta$. Whereof $\mathcal{F}$ is expansive relative to σ with constant expansive η hence we obtain $x = y$ and so $r(x, y) = 0$. So

$$\begin{aligned} r\Big(x_{a_i}, y_{a_i}\Big) &< r\Big(x_{a_i}, x\Big) + r(x, y) + r\Big(y, y_{a_i}\Big) \\ &= r\Big(x_{a_i}, x\Big) + r\Big(y, y_{a_i}\Big). \end{aligned}$$

As $i \to +\infty$, on the basis of the convergence of the subsequences $\Big\{x_{a_i}\Big\}_{i=1}^{+\infty}$ and $\Big\{y_{a_i}\Big\}_{i=1}^{+\infty}$ to x and y, respectively, for given $\epsilon > 0(\epsilon < \mu)$ there exists $k_3 \in \mathbb{N}$ such that for every $i \geq k_3$, $r\Big(x_{a_i}, x\Big) < \frac{\epsilon}{2}$ and $r\Big(y_{a_i}, y\Big) < \frac{\epsilon}{2}$. Thus, for sufficiently large a_i when $i \to +\infty$, the previous relation will be as follows

$$r\Big(x_{a_i}, y_{a_i}\Big) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon < \mu$$

This contradicts absurd hypothesis and the assertion is proved. □

Definition 3.3. We say that IFS $\mathcal{F}$ has "shadowing uniqueness property relative to σ" if there exists a constant number $\epsilon > 0$ such that for every δ-chain $\xi = \{x_i\}_{i \in W}$ with the sequence σ there exists only one chain $\{y_i\}_{i \in W}$ with the same sequence σ that $r\Big(x_i, y_i\Big) < \epsilon$ for every $i \in W$.

In the following lemma, we show that if an IFS has shadowing property and is the expansive relative to given σ, IFS has shadowing uniqueness property relative to σ.

Lemma 3.4. *Suppose that $\mathcal{F}$ is an expansive IFS relative to $\sigma = \Big\{\lambda_0, \lambda_1, \dots\Big\}$ with constant expansive η. Also, $\mathcal{F}$ has the shadowing property. Then $\mathcal{F}$ has the shadowing uniqueness property relative to σ.*

Proof. Put $\epsilon = \frac{\eta}{2}$. Assume that $\xi = \{x_i\}_{i\in W}$ is a δ-chain with the given sequence σ. By considering the process of the proof of Theorem(3.4) in [8], we see that there exists a chain $\{y_i\}_{i\in W}$ with the same sequence σ such that $r\Big(x_i, y_i\Big) < \epsilon$ for all $i \in W$.
Now, we prove the uniqueness. Let $\{y_i\}_{i\in W}$ and $\{z_i\}_{i\in W}$ be two chains with the given sequence σ that for every $i \in W$, $r\Big(x_i, y_i\Big) < \epsilon$ and $r\Big(x_i, z_i\Big) < \epsilon$. Thus, for each $i \in W$, we obtain the following relation

$$\begin{aligned} r\Big(F_{\sigma_i}(y_0), F_{\sigma_i}(z_0)\Big) &= r\Big(y_i, z_i\Big) \\ &\leq r\Big(y_i, x_i\Big) + r\Big(x_i, z_i\Big) \\ &< \epsilon + \epsilon = 2\epsilon = \eta. \end{aligned}$$

Since $\mathcal{F}$ is expansive relative to σ with constant expansive η thus $y_0 = z_0$. We know that $\mathcal{F}$ is an IFS, then for each $i \in W$, F_{σ_i} is a function and so $F_{\sigma_i}(y_0) = F_{\sigma_i}(z_0)$. That is, $y_i = z_i$. Therefore, the chain $\{y_i\}_{i\in W}$ is unique and the statement is proved. □ Now, we prove the following technical theorem whose process of proof is especially complicated and delicate. This theorem has a critical role in the proof of the converse demonstration of Theorem 2.12(weakly) with further conditions.

Theorem 3.5. *Suppose that $\mathcal{F} = \Big\{f_\lambda, M : \lambda \in \Lambda\Big\} \subset Homeo(M)$ is an expansive IFS relative to $\sigma = \Big\{\lambda_0, \lambda_1, \dots\Big\}$ with constant expansive η. Also, $\mathcal{F}$ has the shadowing property. Then there exist $\epsilon > 0$, $3\epsilon < \eta$, and $\delta > 0$ with the following property:*
If $\mathcal{G} = \Big\{g_{\overline{\lambda}}, M : \overline{\lambda} \in \overline{\Lambda}\Big\} \subset Homeo(M)$ be an IFS and $\overline{\sigma}$ be a sequence in $\overline{\Lambda}^W$ such that $\mathcal{D}_0\Big(\mathcal{F}_\sigma, \mathcal{G}_{\overline{\sigma}}\Big) < \delta$ then there exists a continuous function $h : M \to M$ such that:

$$\begin{cases} i) \quad r\Big(G_{\overline{\sigma}_n}(x), F_{\sigma_n}(h(x))\Big) < \epsilon, & \forall x \in M \text{ and } \forall n \in W, \\ ii) \quad r\Big(x, h(x)\Big) < \epsilon, & \forall x \in M. \end{cases}$$

Moreover, if $\epsilon > 0$ is sufficiently small, then the function h is surjective and also $F_{\sigma_n} o h = h o G_{\overline{\sigma}_n}$ for every $n \in W$.

Proof. We consider $\epsilon > 0$ with condition $3\epsilon < \eta$. Since $\mathcal{F}$ has the shadowing property, there exists $\delta > 0$ such that every δ-chain ϵ-is shadowed

by a chain. Let us now assume that
$\mathcal{G} = \Big\{g_{\overline{\lambda}},\ M\ : \overline{\lambda} \in \overline{\Lambda}\Big\} \subset Homeo(M)$ be an IFS and let $\overline{\sigma}$ be a sequence in $\overline{\Lambda}^W$ such that $\mathcal{D}_0\Big(\mathcal{F}_\sigma, \mathcal{G}_{\overline{\sigma}}\Big) < \delta$. Fix $x \in M$. We make the sequence $\xi = \Big\{x, G_{\overline{\sigma}_0}(x), G_{\overline{\sigma}_1}(x), \ldots, G_{\overline{\sigma}_n}(x), \ldots\Big\}$. We claim that ξ is a δ-chain for IFS $\mathcal{F}$. The proof of the claim is as follows:

$$r\Big(G_{\overline{\sigma}_0}(x), f_{\lambda_0}(x)\Big) = r\Big(g_{\overline{\lambda}_0}(x), f_{\lambda_0}(x)\Big) < \rho_0\Big(g_{\overline{\lambda}_0}, f_{\lambda_0}\Big) < \mathcal{D}_0\Big(\mathcal{F}_\sigma, \mathcal{G}_{\overline{\sigma}}\Big) < \delta,$$

and for every $n \geq 1$, we also have

$$\begin{aligned} r\bigg(G_{\overline{\sigma}_n}(x), f_{\lambda_n}\Big(G_{\overline{\sigma}_{n-1}}(x)\Big)\bigg) &= r\bigg(g_{\overline{\lambda}_n}\Big(G_{\overline{\sigma}_{n-1}}(x)\Big), f_{\lambda_n}\Big(G_{\overline{\sigma}_{n-1}}(x)\Big)\bigg) \\ &< \rho_0\Big(g_{\overline{\lambda}_0}, f_{\lambda_0}\Big) < \mathcal{D}_0\Big(\mathcal{F}_\sigma, \mathcal{G}_{\overline{\sigma}}\Big) < \delta. \end{aligned}$$

Thus the claim is proved. By using Lemma 3.4 and according to the proof of this lemma, IFS $\mathcal{F}$ has the shadowing uniqueness property relative to σ with the constant $\epsilon = \frac{\eta}{2}$. Thus there is a unique chain $\{y_n\}_{n\in W}$ such that $r(x, y_0) < \epsilon$ and $r\Big(G_{\overline{\sigma}_n}(x), y_{n+1}\Big) < \epsilon$ for every $n \in W$.
Therefore, for every $x \in M$, we obtain the unique chain $\{y_n\}_{n\in W}$, it follows that the function defined $h : M \to M$ with the criterion $h(x) = y_0$ is well-defined. By substitution $h(x) = y_0$ we will get $r\Big(x, h(x)\Big) < \epsilon$. Since $\{y_n\}_{n\in W}$ is a chain with the sequence σ, for every $n \in W$, we obtain $y_{n+1} = F_{\sigma_n}(y_0)$.
Then, for every $n \in W$, we rewrite the relation $r\Big(G_{\overline{\sigma}_n}(x), y_{n+1}\Big) < \epsilon$ as $r\Big(G_{\overline{\sigma}_n}(x), F_{\sigma_n}(y_0)\Big) = r\Big(G_{\overline{\sigma}_n}(x), F_{\sigma_n}(h(x))\Big) < \epsilon$.
Now, we show that the function h is continuous. Whereas the space M is compact, continuity is equivalent to uniform continuity thus, we prove that h is uniformly continuous on M. Suppose that $\mu > 0$ be given. By using Lemma 3.2 for this σ there is $a \geq 1$ such that if $u, v \in M$ that we have $r\Big(F_{\sigma_n}(u), F_{\sigma_n}(v)\Big) \leq \eta$ for every n, $0 \leq n < a$, then $r(u, v) < \mu$. We know that for every $n \in W$, the functions F_{σ_n} and $G_{\overline{\sigma}_n}$ are uniformly continuous on M so for every integer number n, $0 \leq n < a$, there exist the positive numbers β_n and α_n, respectively, dependent on the given values $\frac{\eta}{3}$ and ϵ, respectively, such that if $r(x, y) < \beta_n$, then

$r\Big(F_{\sigma_n}(x), F_{\sigma_n}(y)\Big) < \frac{\eta}{3}$ and if $r(x,y) < \alpha_n$, then $r\Big(G_{\overline{\sigma}_n}(x), G_{\overline{\sigma}_n}(y)\Big) < \epsilon$. Put $\beta = min\Big\{\beta_n \mid 0 \leq n < a\Big\}$ and $\alpha = min\Big\{\alpha_n \mid 0 \leq n < a\Big\}$. Now, we choose the positive number $\gamma < min\{\beta, \alpha\}$. Subsequently, for every $x, y \in M$ with $r(x,y) < \gamma$, the relations $r\Big(F_{\sigma_n}(x), F_{\sigma_n}(y)\Big) < \frac{\eta}{3}$ and $r\Big(G_{\overline{\sigma}_n}(x), G_{\overline{\sigma}_n}(y)\Big) < \epsilon$ are true for every n, $0 \leq n < a$, and thereby, we see that

$$\begin{aligned} r\Bigg(F_{\sigma_n}\Big(h(x)\Big), F_{\sigma_n}\Big(h(y)\Big)\Bigg) &\leq r\Bigg(F_{\sigma_n}\Big(h(x)\Big), G_{\overline{\sigma}_n}(x)\Bigg) \\ &+ r\Big(G_{\overline{\sigma}_n}(x), G_{\overline{\sigma}_n}(y)\Big) \\ &+ r\Bigg(G_{\overline{\sigma}_n}(y), F_{\sigma_n}\Big(h(y)\Big)\Bigg) \\ &< \epsilon + \epsilon + \epsilon = 3\epsilon < \eta. \end{aligned}$$

The previous relation is true for very n, $0 \leq n < a$, so $r\Big(h(x), h(y)\Big) < \mu$. Thus, for every $\mu > 0$ there exists $\gamma > 0$ such that every $x, y \in M$ with $r(x,y) < \gamma$ implies $r\Big(h(x), h(y)\Big) < \mu$ and this means that the function h is uniformly continuous on M and consequently, it is continuous.
Now, assume that $\epsilon > 0$ is sufficiently small. Since M is a compact metric space, thus for the given $\epsilon > 0$ there exist $x_1, x_2, \ldots, x_n$ in M that $M = \bigcup_{i=1}^{n} B_\epsilon(x_i)$. Choose $y \in M$. Therefore, there is $x_j \in M$ such that $y \in B_\epsilon(x_j)$. Hence,

$$r\Big(y, h(x_j)\Big) \leq r(y, x_j) + r\Big(x_j, h(x_j)\Big) < \epsilon + \epsilon = 2\epsilon.$$

Also, for every $x \in M$, we have

$$\begin{aligned} r\Bigg(F_{\sigma_n}\Big(h(x)\Big), h\Big(G_{\overline{\sigma}_n}(x)\Big)\Bigg) &\leq r\Bigg(F_{\sigma_n}\Big(h(x)\Big), G_{\overline{\sigma}_n}(x)\Bigg) \\ &+ r\Bigg(G_{\overline{\sigma}_n}(x), h\Big(G_{\overline{\sigma}_n}(x)\Big)\Bigg) \\ &< \epsilon + \epsilon = 2\epsilon \end{aligned}$$

Since $\epsilon > 0$ is sufficiently small, we can calculate the limitation as $\epsilon \to 0$ in the two previous relations. From the former relation, we will obtain the relation $r\Big(y, h(x_j)\Big) = 0$, that is, $y = h(x_j)$, and from the latter relation, for every $x \in M$, we get the relation $r\Bigg(F_{\sigma_n}\Big(h(x)\Big), h\Big(G_{\overline{\sigma}_n}(x)\Big)\Bigg) = 0$. Consequently, these relations show that the function h is surjective and $F_{\sigma_n} oh = hoG_{\overline{\sigma}_n}$ for every $n \in W$. $\square$ Considering the previous theorem and the definition of weakly topologically stable, we have the following corollary;

Corollary 3.6. *If IFS $\mathcal{F}$ has the shadowing property and moreover $\mathcal{F}$ is the expansive relative to the sequence σ with sufficiently small expansive constant, then $\mathcal{F}$ is weakly topologically stable(relative to this σ).*

4 Shadowing Property is not Enough for Structural Stability

Now, we are going to study the relation between shadowing property and structural stability in an IFS. First, we define the space of diffeomorphisms on M. Let the functions f and g be C^1-diffeomorphisms on M. We define the metric ρ_1 as follows;

$$\rho_1(f, g) = \rho_0(f, g) + Max\Bigg\{ \parallel Df(x) - Dg(x) \parallel;\ \forall x \in M\Bigg\};$$

that here

$$\parallel Df(x) - Dg(x) \parallel = Max\Bigg\{ \mid Df(x)u - Dg(x)u \mid;\ \forall u \in T_xM : \mid u \mid = 1\Bigg\}$$

The set of C^1-diffeomorphisms on M with the induced topology of the metric ρ_1 is denoted by $Diff^1(M)$.

We know that there is no relation of equivalence between the set of all structurally stable diffeomorphisms and the set of all diffeomorphisms with the SP. In fact, the SS is stronger than the SP. Robinson proves that a structurally stable diffeomorphism on a closed manifold has the SP. The previous converse demonstration has been rejected by giving a counterexample; for example, in the article [19]. In the following, we show that this equivalence is not also true for IFSs by giving an example.

Definition 4.1. Let IFSs $\mathcal{F} = \Big\{f_\lambda,\ M\ : \lambda \in \Lambda\Big\}$ and $\mathcal{G} = \Big\{g_{\overline{\lambda}},\ M\ : \overline{\lambda} \in \overline{\Lambda}\Big\}$ be subsets of $Diff^1(M)$. Let $\sigma \in \Lambda^W$ and $\overline{\sigma} \in \overline{\Lambda}^W$. We consider the sequences $\mathcal{F}_\sigma = \{f_{\lambda_i}\}_{i\in W}$ and $\mathcal{G}_{\overline{\sigma}} = \{g_{\overline{\lambda}_i}\}_{i\in W}$ where $\lambda_i \in \sigma$ and $\overline{\lambda}_i \in \overline{\sigma}$ for every $i \in W$. The measure distance between the two IFSs relative to the sequences σ and $\overline{\sigma}$ will be denoted by $\mathcal{D}_1$ and is defined as follows:

$$\mathcal{D}_1\Big(\mathcal{F}_\sigma, \mathcal{G}_{\overline{\sigma}}\Big) = sup\Big\{\rho_1(f_{\lambda_i}, g_{\overline{\lambda}_i}) : \qquad \lambda_i \in \sigma,\ \overline{\lambda}_i \in \overline{\sigma}\ \ for\ all\ i \in W\Big\}$$

Notice that $\mathcal{D}_1$ is well-defined and metric.

Definition 4.2. Assume that $\mathcal{F} = \{f_\lambda, M : \lambda \in \Lambda\} \subset Diff^1(M)$ be an IFS. We say that IFS $\mathcal{F}$ is "structurally stable" if for given $\epsilon > 0$ there is $\delta > 0$ such that for any IFS $\mathcal{G} = \Big\{g_{\overline{\lambda}},\ M\ : \overline{\lambda} \in \overline{\Lambda}\Big\} \subset Diff^1(M)$ and for every sequence σ and $\overline{\sigma}$ with the condition $\mathcal{D}_1\Big(\mathcal{F}_\sigma, \mathcal{G}_{\overline{\sigma}}\Big) < \delta$, there exists a homeomorphism $h : M \to M$ with the following properties:

$$\begin{cases} i) \quad F_{\sigma_n} oh = hoG_{\overline{\sigma}_n}, \quad \forall n \in W, \\ ii) \quad r\Big(x, h(x)\Big) < \epsilon, \quad \forall x \in M. \end{cases}$$

We show that in an IFS, shadowing property doesn't imply structural stability by giving an example.

Example 4.3. We define the function $F : T^2 \times T^2 \to T^2 \times T^2$(T^2 is a two-dimensional torus) with the following criterion:

$$F(x, y, u, v) = \Big(2x - c(u, v)f(x) + y, x - c(u, v)f(x) + y, 2u + v, u + v\Big),$$

where $f(x) = \frac{1}{2\pi}\sin 2\pi x$ and c is a C^∞ function from T^2 to $\mathbb{R}$ such that the first order derivatives are small and also $0 < c(u, v) \leq 1$ for every $(u, v) \in T^2$. Moreover, $c(u, v) = 1$ if and only if (u, v) belong to nontrivial and the minimal set of the function G, $G : T^2 \to T^2$ with the criterion $G(u, v) = (2u + v, u + v)$.
In the criterion of the function F, we once put $c(u, v) = \cos^2 \pi(u + v)$ and call the obtained function F_1, again we set $c(u, v) = \cos^2 \pi(u - v)$ and call this obtained function F_2. Now, consider $\mathcal{F} = \Big\{F_1, F_2\ ;\ T^2 \times T^2\Big\}$. Clearly, $\mathcal{F}$ is an IFS and since T^2 is a compact metric space then $T^2 \times T^2$

is compact metric space.
First, we show that IFS $\mathcal{F}$ has shadowing property. On the basis of Corollary(3.4) in [9], it is sufficient we prove that for a given $\epsilon > 0$ there is $\delta > 0$ such that

$$B\Big(F_i(X), \epsilon + \delta\Big) \subseteq F_i\Big(B(X, \epsilon)\Big); \quad \forall X \in T^2 \times T^2,\ i = 1, 2. \tag{5}$$

The functions F_1 and F_2 are diffeomorphisms by the criteria of the functions, so these functions are uniformly continuous on the compact space $T^2 \times T^2$. Assume that $\epsilon > 0$ be given. Put $\delta = \epsilon$. For an arbitrary and assumed value $X \in T^2 \times T^2$ and i, $i = 1, 2$, consider $Z \in B\Big(F_i(X), 2\epsilon\Big)$. According to the uniform continuity of the function F_i, for this value ϵ, there exists $\delta_1 > 0$ such that for every $Y \in T^2 \times T^2$ with $r(Y, X) < \delta_1$ then $r\Big(F_i(Y), F_i(X)\Big) < \epsilon$. Since F_i is one to one function, we put $Z^* = F_i^{-1}(Z)$. F_i is uniformly continuous, there is δ_2 such that for every $Y \in T^2 \times T^2$ with $r(Y, Z^*) < \delta_2$ then $r\Big(F_i(Y), F_i(Z^*)\Big) < \epsilon$. We can choose the values δ_1 and δ_2 such that $\delta_1, \delta_2 < \frac{\epsilon}{2}$. Put $\delta^* = min\{\delta_1, \delta_2\}$. Assume that $Y \in T^2 \times T^2$ with $r(Y, X) < \delta^*$ and $r(Y, Z^*) < \delta^*$, then we see that

$$r(Z^*, X) < r(Z^*, Y) + r(Y, X) < 2\delta^* < \epsilon.$$

Hence, $Z^* \in B(X, \epsilon)$ and whereas $F_i(Z^*) = Z$, we obtain that $Z \in F_i\Big(B(X, \epsilon)\Big)$ and consequently, the relation 5 is proved.
Second, we claim that IFS $\mathcal{F}$ isn't structurally stable. By reduction ad absurdum, we assume that IFS $\mathcal{F}$ is structurally stable. Thus, for given $\epsilon > 0$ there exists $\delta > 0$ such that if $\mathcal{G} = \Big\{G_1, G_2\ ;\ T^2 \times T^2\Big\}$ be an IFS including C^1-diffeomorphisms functions with $\mathcal{D}_1\Big(\mathcal{F}, \mathcal{G}\Big) < \delta$ then for the sequence $\sigma = \{1, 1, \ldots\}$ and $n = 1$ there is a homeomorphism $h : T^2 \times T^2 \to T^2 \times T^2$ such that $F_1 oh = hoG_1$. Since G_1 is an arbitrary function with $\rho_1\Big(F_1, G_1\Big) < \delta$, according to the previous relation, we conclude that the function F_1 is structurally stable. But we conclude from the article [21] that the function F_1 isn't structurally stable and this is a contradiction. Thus, our claim is proved.

5 Discussion

Certainly, structural stability doesn't imply shadowing property in an IFS, because if it be true then it for an IFS including a diffeomorphism must be true. But on the basis of scientific texts' theorems, it's false. Now, a question arises that under what conditions a structurally stable IFS has shadowing property.

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