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Original Research Paper

Implicit Caputo Tempered Fractional Differential Boundary Value Problems with Infinite Delay in Banach Spaces

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Abstract. The purpose of this article is to study the existence and Ulam-Hyers stability results for a class of boundary value problems with Caputo tempered fractional derivative and infinite delay. The results are based on Mönch's fixed point theorem. An illustrative example is given to demonstrate the applicability of our results.

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1 Introduction

Fractional calculus, an approach that involves extending differentiation and integration to non-integer orders, has gained significant interest in both theory and applications across various research fields. This has made it an important tool in tackling complex issues. To fully comprehend this approach, it is recommended to consult monographs such as [1–4, 20, 40, 43] and papers like [6, 10, 11, 14, 17, 18, 32, 34, 37]. In recent years, there has been a noticeable increase in research on fractional calculus, where authors have explored different outcomes for varying conditions and forms of fractional differential equations and inclusions. Additional information can be found in papers like [1, 25, 26, 35], as well as their respective references.

While solving differential equations precisely is difficult or impossible in several situations, along with nonlinear analysis and optimization, we investigate approximate solutions. It is important to stress that only stable estimates are acceptable. As a result, numerous methodologies for stability analysis are employed such as Lyapunov and exponential stability. Ulam, a mathematician, first raised the stability issue in functional equations in a 1940 lecture at Wisconsin University. S.M. Ulam posed the question, "Under what conditions does an additive mapping exist near an approximately additive mapping?" [42]. The succeeding year, Hyers addressed Ulam's issue for additive functions defined on Banach spaces in [23]. Rassias [36] showed the existence of unique linear mappings close to approximation additive mappings in 1978, generalizing Hyers' results. In comparison to Lyapunov and exponential stability analysis, Ulam-Hyers stability analysis focuses on the behavior of a function under perturbations, rather than the stability of a dynamical system or equilibrium point. The authors of [7, 8, 35] investigated the Ulam stabilities of fractional differential problems with different conditions. Considerable focus has been given to investigating the stability of various types of functional equations, specifically Ulam-Hyers and Ulam-Hyers-Rassias stability. This can be observed through the book by Abbas *et al.* [1], as well as the research conducted by Luo *et al.* [29] and Rus [38], which delved into the stability of operatorial equations using the Ulam-Hyers approach.

In [12], the authors considered the following fractional impulsive neutral integro-differential systems with infinite delay:

$$\begin{cases} D_\theta^q \left(\xi(\theta) - \chi\left(\theta, \xi_\theta\right)\right) = A(\theta, \xi)\left(\xi(\theta) - \chi\left(\theta, \xi_\theta\right)\right) \\ \qquad\qquad\qquad\qquad + \Psi\left(\theta, \xi_\theta, \displaystyle\int_0^\theta \widetilde{\Psi}\left(\theta, s, \xi_s\right) ds\right), \theta \in [0, b], \theta \neq \theta_\jmath, \\ \Delta\xi|_{\theta=\theta_\jmath} = I_\jmath\left(\xi\left(\theta_\jmath^-\right)\right), \quad \theta = \theta_\jmath; \jmath = 1, \ldots, m, \\ \xi(0) + g(\xi) = \Lambda, \quad \Lambda \in B_\vartheta, \end{cases}$$

where $0 < q < 1$, D_θ^q is the Caputo fractional derivative and $\xi_\theta(.)$ denote $\xi_\theta(s) = \xi(\theta + s), s \in (-\infty, 0]$. The results are obtained by a fixed point theorem.

In [27], the authors considered a class of problems for nonlinear Caputo tempered implicit fractional differential equations with boundary conditions and delay:

$${}_0^C D_\theta^{\beta,\gamma} \xi(\theta) = \Psi\left(\theta, \xi_\theta, {}_0^C D_\theta^{\beta,\gamma} \xi(\theta)\right), \quad \theta \in \Xi := [0, \varkappa],$$

$$\xi(\theta) = \Lambda(\theta), \quad \theta \in [-\kappa, 0],$$

$$\delta_1 \xi(0) + \delta_2 \xi(\varkappa) = \delta_3,$$

where $0 < \beta < 1$, $\gamma \geq 0$, ${}_0^C D_\theta^{\beta,\gamma}$ is the Caputo tempered fractional derivative, $\Psi : \Xi \times C([-\kappa, 0], \mathbb{R}) \times \mathbb{R}$ is a continuous function, $\xi \in C([-\kappa, 0], R)$, $0 < \varkappa < +\infty$, $\delta_1, \delta_2, \delta_3$ are real constants, and $\kappa > 0$ is the time delay. Their arguments are based on Banach, Schauder and Schaefer fixed point theorems.

Motivated by the above-mentioned works, we investigate the existence and Ulam stability of the solutions for the following boundary value problem with Caputo tempered fractional derivative and infinite delay:

$${}_0^C D_\theta^{\sigma,\varrho} \xi(\theta) = \Psi\left(\theta, \xi_\theta, {}_0^C D_\theta^{\sigma,\varrho} \xi(\theta)\right), \quad \theta \in \Xi := [0, \varkappa], \tag{1}$$

$$\xi(\theta) = \Lambda(\theta), \quad \theta \in (-\infty, 0], \tag{2}$$

$$\delta_1\xi(0)+\delta_2\xi(\varkappa)=\delta_3, \tag{3}$$

where ${}_0^C D_\theta^{\sigma,\varrho}$ represents the Caputo tempered fractional derivative of order $\sigma\in(0,1)$, $\varrho\geq 0$, $\Psi:\Xi\times\Bbbk\times E\to E$ is a given function, $\Lambda\in\Bbbk$, $\delta_1,\delta_2,\delta_3$ are real constants and $\Bbbk$ is called a phase space. We denote by ξ_θ the element of $\Bbbk$ defined by

$$\xi_\theta(s)=\xi(\theta+s):s\in(-\infty,0].$$

The following are the primary novelties of the current paper:

- Considering the diverse conditions imposed on the problem in [27], our research can be seen as a partial extension of the aforementioned studies. It expands the investigation to encompass a problem involving infinite delay, thus introducing the necessity for additional requirements in the concept of phase space.
- In our study, we address a problem within the abstract Banach space, which necessitates the utilization of the measure of noncompactness concept. This, in turn, introduces another tool, namely, Mönch's fixed point theorem.
- The study of Ulam-Hyers stability of an implicit problem that incorporates both infinite delay and boundary conditions.
- The potential of tempered fractional calculus, which includes the Caputo tempered fractional derivative, has gained recognition as a significant class of fractional calculus operators. It extends the scope of fractional calculus by encompassing various forms and exhibiting analytic kernels. Buschman's seminal work [16] established the definitions of fractional integration with weak singular and exponential kernels, and subsequent research in this area can be found in [9, 28, 30, 33, 39, 41]. Despite the limited attention received by the Caputo tempered fractional derivative in existing literature, it holds promise for making substantial contributions in this field. By exploring the properties and potential applications of the Caputo tempered fractional derivative, our study aims to enhance our understanding of this unique mathematical concept and contribute to the advancement of fractional calculus.

The following is how this paper is organized. Section 2 contains definitions and lemmas that will be utilized throughout the work. Section 3 derives the existence results for the problem (1)-(3). The fourth section discusses the Ulam-Hyers stability results for our problem. In the final part, we present an example to demonstrate our main results.

2 Preliminaries

This section aims to present the notations, definitions, and earlier findings that are essential for understanding the content of this paper. Specifically, we use the notation $C(\Xi, E)$ to denote the Banach space of continuous functions from $\Xi := [0, \varkappa]$ into E, with

$$\|\xi\|_\infty = \sup\{\|\xi(\theta)\| : \theta \in \Xi\}.$$

Let the space $(\Bbbk, \|\cdot\|_\Bbbk)$ a seminormed linear space of functions mapping $(-\infty, 0]$ into E and verifying the following axioms which were derived from Hale and Kato's originals [22]:

(Ax_1) If $\xi : (-\infty, 0] \to E$ and $\xi_0 \in \Bbbk$, then there exist constants $\wp_1, \wp_2, \wp_3 > 0$, such that for each $\theta \in \Xi$, we have:

- (i) ξ_θ is in $\Bbbk$;
- (ii) $\|\xi_\theta\|_\Bbbk \le \wp_1 \|\xi_0\|_\Bbbk + \wp_2 \sup\limits_{\theta \in [0,\varkappa]} \|\xi(\theta)\|$;
- (iii) $\|\xi(\theta)\| \le \wp_3 \|\xi_\theta\|_\Bbbk$.

(Ax_2) For the function $\xi(\cdot)$ *in* (Ax_1), ξ_θ is a $\Bbbk-$valued continuous function on Ξ.

(Ax_3) The space $\Bbbk$ is complete.

Consider the space

$$\Upsilon_E = \big\{\xi : (-\infty, \varkappa] \to E : \xi|_{(-\infty,0]} \in \Bbbk, \xi|_{[0,\varkappa]} \in C(\Xi, E)\big\}.$$

Υ_E is a Banach space with the norm

$$\|\xi\|_{\Upsilon_E} = \sup_{\theta \in (-\infty,\varkappa]} \|\xi(\theta)\|.$$

Definition 2.1 ([28,39,41]). Let the function $\Psi \in C(\Xi, E)$, $\varrho \geq 0$. The Riemann-Liouville tempered fractional integral of order σ is defined by

$$ {}_0I_\theta^{\sigma,\varrho}\Psi(\theta) = e^{-\varrho\theta}\ {}_0I_\theta^{\sigma}\left(e^{\varrho\theta}\Psi(\theta)\right) = \frac{1}{\Gamma(\sigma)}\int_0^\theta \frac{e^{-\varrho(\theta-s)}\Psi(s)}{(\theta-s)^{1-\sigma}}ds, \tag{4} $$

where ${}_0I_\theta^{\sigma}$ denotes the Riemann-Liouville fractional integral [24], defined by

$$ {}_0I_\theta^{\sigma}\Psi(\theta) = \frac{1}{\Gamma(\sigma)}\int_0^\theta \frac{\Psi(s)}{(\theta-s)^{1-\sigma}}ds. \tag{5} $$

Obviously, the tempered fractional integral (4) reduces to the Riemann-Liouville fractional integral (5) if $\varrho = 0$.

Definition 2.2 ([28,39]). For $\jmath-1 < \sigma < \jmath$; $\jmath \in \mathbb{N}$, $\varrho \geq 0$, the Riemann-Liouville tempered fractional derivative is defined by

$$ {}_0D_\theta^{\sigma,\varrho}\Psi(\theta) = e^{-\varrho\theta}{}_0D_\theta^{\sigma}\left(e^{\varrho\theta}\Psi(\theta)\right) = \frac{e^{-\varrho\theta}}{\Gamma(\jmath-\sigma)}\frac{d^\jmath}{d\theta^\jmath}\int_0^\theta \frac{e^{\varrho s}\Psi(s)}{(\theta-s)^{\sigma-\jmath+1}}ds, $$

where ${}_0D_\theta^{\sigma}\left(e^{\varrho\theta}\Psi(\theta)\right)$ denotes the Riemann-Liouville fractional derivative [24], given by

$$ \begin{aligned} {}_0D_\theta^{\sigma}\left(e^{\varrho\theta}\Psi(\theta)\right) &= \frac{d^\jmath}{d\theta^\jmath}\left({}_0I_\theta^{\jmath-\sigma}\left(e^{\varrho\theta}\Psi(\theta)\right)\right) \\ &= \frac{1}{\Gamma(\jmath-\sigma)}\frac{d^\jmath}{d\theta^\jmath}\int_0^\theta \frac{\left(e^{\varrho s}\Psi(s)\right)}{(\theta-s)^{\sigma-\jmath+1}}ds. \end{aligned} $$

Definition 2.3 ([28,41]). For $\jmath-1 < \sigma < \jmath$; $\jmath \in \mathbb{N}$, $\varrho \geq 0$, the Caputo tempered fractional derivative is defined as

$$ \begin{aligned} {}_0^CD_\theta^{\sigma,\varrho}\Psi(\theta) &= e^{-\varrho\theta}\ {}_0^CD_\theta^{\sigma}\left(e^{\varrho\theta}\Psi(\theta)\right) \\ &= \frac{e^{-\varrho\theta}}{\Gamma(\jmath-\sigma)}\int_0^\theta \frac{1}{(\theta-s)^{\sigma-\jmath+1}}\frac{d^\jmath\left(e^{\varrho s}\Psi(s)\right)}{ds^\jmath}ds, \end{aligned} $$

where ${}_0^CD_\theta^{\sigma,\varrho}\left(e^{\varrho\theta}\Psi(\theta)\right)$ denotes the Caputo fractional derivative [24], given by

$$ {}_0^CD_\theta^{\sigma}\left(e^{\varrho\theta}\Psi(\theta)\right) = \frac{1}{\Gamma(\jmath-\sigma)}\int_0^\theta \frac{1}{(\theta-s)^{\sigma-\jmath+1}}\frac{d^\jmath\left(e^{\varrho s}\Psi(s)\right)}{ds^\jmath}ds. $$

Lemma 2.4 ([28]). *For a constant C,*

$$ {}_0D_\theta^{\sigma,\varrho}C = Ce^{-\varrho\theta}\,{}_0D_\theta^{\sigma}e^{\varrho\theta}, \quad {}_0^CD_\theta^{\sigma,\varrho}C = Ce^{-\varrho\theta}\,{}_0^CD_\theta^{\sigma}e^{\varrho\theta}. $$

Obviously, ${}_0D_\theta^{\sigma,\varrho}(C) \neq {}_0^CD_\theta^{\sigma,\varrho}(C)$. And, ${}_0^CD_\theta^{\sigma,\varrho}(C)$ is no longer equal to zero, being different from ${}_0^CD_\theta^{\sigma}(C) \equiv 0$.

Lemma 2.5 ([28, 41]). *Let $\Psi \in C^{\jmath}(\Xi, E)$ and $\jmath - 1 < \sigma < \jmath$; $\jmath \in \mathbb{N}$. Then, the Caputo tempered fractional derivative and the Riemann-Liouville tempered fractional integral have the composite properties*

$$ {}_0I_\theta^{\sigma,\varrho}\left[{}_0^CD_\theta^{\sigma,\varrho}\Psi(\theta)\right] = \Psi(\theta) - \sum_{k=0}^{\jmath-1} e^{-\varrho\theta}\frac{\theta^k}{k!}\left[\left.\frac{d^k\left(e^{\varrho\theta}\Psi(\theta)\right)}{d\theta^k}\right|_{\theta=0}\right], $$

and

$$ {}_0^CD_\theta^{\sigma,\varrho}\left[{}_0I_\theta^{\sigma,\varrho}\Psi(\theta)\right] = \Psi(\theta), \ \textit{for } \sigma \in (0,1). $$

Definition 2.6 ([15]). Let $\mathfrak{W}$ be a Banach space and let $\Theta_{\mathfrak{W}}$ be the family of bounded subsets of $\mathfrak{W}$. The Kuratowski measure of noncompactness is the map $\zeta : \Theta_{\mathfrak{W}} \longrightarrow [0,\infty)$ defined by

$$ \zeta(\Omega) = inf\left\{\varepsilon > 0 : \Omega \subset \bigcup_{j=1}^{m}\Omega_j, diam(\Omega_j) \leq \varepsilon\right\}, $$

where $\Omega \in \Theta_{\mathfrak{W}}$.

The map ζ satisfies the following properties:

- $\zeta(\Omega) = 0 \Leftrightarrow \overline{\Omega}$ is compact (Ω is relatively compact);
- $\zeta(\Omega) = \zeta(\overline{\Omega})$;
- $\Omega_1 \subset \Omega_2 \Rightarrow \zeta(\Omega_1) \leq \zeta(\Omega_2)$;
- $\zeta(\Omega_1 + \Omega_2) \leq \zeta(\Omega_1) + \zeta(\Omega_2)$;
- $\zeta(c\Omega) = |c|\zeta(\Omega)$, $c \in \mathbb{R}$;
- $\zeta(conv\Omega) = \zeta(\Omega)$.

Lemma 2.7 ([21]). *Let $B \subset \Upsilon_E$ be a bounded and equicontinuous set. Then*

a) The function $\theta \to \zeta(B(\theta))$ is continuous on Ξ, and

$$\zeta_{\Upsilon_E}(B) = \sup_{\theta \in (-\infty, \varkappa]} \zeta(B(\theta)).$$

b) $\zeta\left(\int_0^{\varkappa} \xi(s)dy : \xi \in B\right) \leq \int_0^{\varkappa} \zeta(B(s))ds$, where

$$B(\theta) = \{\xi(\theta) : \xi \in B\}, \theta \in \Xi.$$

Theorem 2.8 (Mönch's fixed point Theorem [19]). *Let D be a nonempty, closed, bounded and convex subset of a Banach space $\mathfrak{W}$ such that $0 \in D$ and let $\mathcal{H} : D \longrightarrow D$ be a continuous mapping. If the implication*

$$B = \overline{conv}\mathcal{H}(B) \text{ or } B = \mathcal{H}(B) \cup \{0\} \Longrightarrow \zeta(B) = 0,$$

holds for every subset B of D, then $\mathcal{H}$ has at least one fixed point.

3 Existence Results

Consider the following fractional differential problem:

$$ {}_0^C D_\theta^{\sigma,\varrho}\xi(\theta) = \mu(\theta), \quad if \ \theta \in \Xi, \ 0 < \sigma < 1, \varrho \geq 0, \tag{6}$$

$$\xi(\theta) = \Lambda(\theta), \quad if \ \theta \in (-\infty, 0], \tag{7}$$

$$\delta_1\xi(0) + \delta_2\xi(\varkappa) = \delta_3, \tag{8}$$

where $\mu : \Xi \to E$ is a continuous function and $\Lambda \in \Bbbk$.

Lemma 3.1. *Let $\sigma \in (0, 1)$ and $\mu : \Xi \to E$ be continuous. Then, the problem (6)-(8) has a unique solution given by:*

$$\xi(\theta) = \begin{cases} \dfrac{\delta_3 e^{-\varrho\theta}}{\delta_1 + \delta_2 e^{-\varrho\varkappa}} - \dfrac{\delta_2 e^{-\varrho\theta} \int_0^{\varkappa} e^{-\varrho(\varkappa - s)}(\varkappa - s)^{\sigma-1}\mu(s)ds}{\Gamma(\sigma)(\delta_1 + \delta_2 e^{-\varrho\varkappa})} \\ +\dfrac{1}{\Gamma(\sigma)} \displaystyle\int_0^{\theta} e^{-\varrho(\theta - s)}(\theta - s)^{\sigma-1}\mu(s)ds, \ \theta \in \Xi, \\ \\ \Lambda(\theta), \qquad \theta \in (-\infty, 0]. \end{cases} \tag{9}$$

Proof. Suppose that ξ satisfies (6)-(8). Then, by applying the Riemann-Liouville tempered fractional integral of order σ and by Lemma 2.5, we get

$$ {}_0I_\theta^{\sigma,\varrho}\, {}_0^CD_\theta^{\sigma,\varrho}\xi(\theta) = {}_0I_\theta^{\sigma,\varrho}\mu(\theta). $$

This implies that

$$ \xi(\theta) - \xi(0)e^{-\varrho\theta} = \frac{1}{\Gamma(\sigma)}\int_0^\theta e^{-\varrho(\theta-s)}(\theta-s)^{\sigma-1}\mu(s)ds. $$

Then,

$$ \xi(\theta) = \xi(0)e^{-\varrho\theta} + \frac{1}{\Gamma(\sigma)}\int_0^\theta e^{-\varrho(\theta-s)}(\theta-s)^{\sigma-1}\mu(s)ds. $$

For $\theta = \varkappa$, we have

$$ \xi(\varkappa) = \xi(0)e^{-\varrho\varkappa} + \frac{1}{\Gamma(\sigma)}\int_0^\varkappa e^{-\varrho(\varkappa-s)}(\varkappa-s)^{\sigma-1}\mu(s)ds. $$

From condition (8), we get

$$ \delta_1\xi(0) + \delta_2\left(\xi(0)e^{-\varrho\varkappa} + \frac{1}{\Gamma(\sigma)}\int_0^\varkappa e^{-\varrho(\varkappa-s)}(\varkappa-s)^{\sigma-1}\mu(s)ds\right) = \delta_3. $$

Thus,

$$ \xi(0) = \frac{\delta_3 - \dfrac{\delta_2}{\Gamma(\sigma)}\displaystyle\int_0^\varkappa e^{-\varrho(\varkappa-s)}(\varkappa-s)^{\sigma-1}\mu(s)ds}{\delta_1 + \delta_2 e^{-\varrho\varkappa}}. $$

Finally, we obtain

$$ \begin{aligned} \xi(\theta) = {} & \frac{\delta_3 e^{-\varrho\theta}}{\delta_1 + \delta_2 e^{-\varrho\varkappa}} - \frac{\delta_2 e^{-\varrho\theta}}{\Gamma(\sigma)(\delta_1 + \delta_2 e^{-\varrho\varkappa})}\int_0^\varkappa e^{-\varrho(\varkappa-s)}(\varkappa-s)^{\sigma-1}\mu(s)ds \\ & + \frac{1}{\Gamma(\sigma)}\int_0^\theta e^{-\varrho(\theta-s)}(\theta-s)^{\sigma-1}\mu(s)ds. \end{aligned} $$

□

Definition 3.2. By a solution of the problem (1)-(3), we mean a function $\xi \in \Upsilon_E$ that satisfies the equation (1) and the conditions (2)-(3).

Lemma 3.3. *Let $\Psi : \Xi \times \Bbbk \times E \to E$ be a continuous function. Then, the problem (1)-(3) is equivalent to the following integral equation:*

$$\xi(\theta) = \begin{cases} \dfrac{\delta_3 e^{-\varrho\theta}}{\delta_1 + \delta_2 e^{-\varrho\varkappa}} - \dfrac{\delta_2 e^{-\varrho\theta} \displaystyle\int_0^{\varkappa} e^{-\varrho(\varkappa - s)}(\varkappa - s)^{\sigma-1}\Psi(s, \xi_s, \widehat{\Psi}(s))ds}{\Gamma(\sigma)(\delta_1 + \delta_2 e^{-\varrho\varkappa})} \\ + \dfrac{1}{\Gamma(\sigma)} \displaystyle\int_0^{\theta} e^{-\varrho(\theta - s)}(\theta - s)^{\sigma-1}\Psi(s, \xi_s, \widehat{\Psi}(s))ds, \ \theta \in \Xi, \\ \\ \Lambda(\theta), \qquad\qquad \theta \in (-\infty, 0]. \end{cases} \tag{10}$$

where $\widehat{\Psi} \in C(\Xi, E)$ satisfies the following functional equation

$$\widehat{\Psi}(\theta) = \Psi(\theta, \xi_\theta, \widehat{\Psi}(\theta)).$$

Let us set the following assumptions:

(*A1*) The function $\Psi : \Xi \times \Bbbk \times E \to E$ is continuous.

(*A2*) There exist constants $\lambda > 0$ and $0 < L < 1$ such that

$$\|\Psi(\theta, \alpha, \beta) - \Psi(\theta, \bar{\alpha}, \bar{\beta})\| \le \lambda\|\alpha - \bar{\alpha}\|_{\Bbbk} + L\|\beta - \bar{\beta}\|,$$

for any $\alpha, \bar{\alpha} \in \Bbbk$, $\beta, \bar{\beta} \in E$ and $\theta \in \Xi$.

(*A3*) For each $\theta \in \Xi$ and bounded sets $B_1 \subseteq \Bbbk$, $B_2 \subseteq E$, we have

$$\zeta(\Psi(\theta, B_1, B_2)) \le \lambda \sup_{s \in (-\infty, 0]} \zeta(B_1(s)) + L\zeta(B_2).$$

Remark 3.4 ([13])**.** It is worth noting that the hypotheses $(A2)$ and $(A3)$ are equivalent.

We are now in a position to prove the existence result of the problem (1)-(3) based on Mönch's fixed point theorem.

Theorem 3.5. *Assume that the hypotheses* $(A1)$-$(A2)$ *hold. If*

$$\frac{\lambda\varkappa^{\sigma}(\wp_2 + 1)(\delta_2 + \delta_1 + \delta_2 e^{-\varrho\varkappa})}{\Gamma(\sigma + 1)(1 - L)(\delta_1 + \delta_2 e^{-\varrho\varkappa})} < 1,$$

then the implicit fractional problem (1)-(3) has at least one solution.

Proof. Transform problem (1)-(3) into a fixed point problem by considering the operator $A : \Upsilon_E \longrightarrow \Upsilon_E$ by

$$A\xi(\theta) = \begin{cases} \dfrac{\delta_3 e^{-\varrho\theta}}{\delta_1 + \delta_2 e^{-\varrho\varkappa}} - \dfrac{\delta_2 e^{-\varrho\theta}\displaystyle\int_0^{\varkappa} e^{-\varrho(\varkappa-s)}(\varkappa-s)^{\sigma-1}\widehat{\Psi}(s)ds}{\Gamma(\sigma)(\delta_1+\delta_2 e^{-\varrho\varkappa})} \\ +\dfrac{1}{\Gamma(\sigma)}\displaystyle\int_0^{\theta} e^{-\varrho(\theta-s)}(\theta-s)^{\sigma-1}\widehat{\Psi}(s)ds, \ \theta \in \Xi, \\ \\ \Lambda(\theta), \qquad \theta \in (-\infty, 0]. \end{cases}$$

Let $w : (-\infty, \varkappa] \to E$ be a function given by

$$w(\theta) = \begin{cases} \dfrac{\delta_3 e^{-\varrho\theta}}{\delta_1 + \delta_2 e^{-\varrho\varkappa}}, \ \theta \in \Xi, \\ \\ \Lambda(\theta), \qquad \theta \in (-\infty, 0]. \end{cases}$$

For each $z \in C(\Xi, E)$, with $z(0) = 0$, we denote by $\bar{z}$ the function defined by

$$\bar{z}(\theta) = \begin{cases} z(\theta), \ \theta \in \Xi, \\ \\ 0, \qquad \theta \in (-\infty, 0], \end{cases}$$

where

$$\begin{aligned} z(\theta) = &-\frac{\delta_2 e^{-\varrho\theta}}{\Gamma(\sigma)(\delta_1+\delta_2 e^{-\varrho\varkappa})}\int_0^{\varkappa} e^{-\varrho(\varkappa-s)}(\varkappa-s)^{\sigma-1}\widehat{\Psi}(s)ds \\ &+\frac{1}{\Gamma(\sigma)}\int_0^{\theta} e^{-\varrho(\theta-s)}(\theta-s)^{\sigma-1}\widehat{\Psi}(s)ds. \end{aligned}$$

If $\xi(\cdot)$ satisfies the integral equation

$$\begin{aligned} \xi(\theta) = &\frac{\delta_3 e^{-\varrho\theta}}{\delta_1 + \delta_2 e^{-\varrho\varkappa}} - \frac{\delta_2 e^{-\varrho\theta}}{\Gamma(\sigma)(\delta_1+\delta_2 e^{-\varrho\varkappa})}\int_0^{\varkappa} e^{-\varrho(\varkappa-s)}(\varkappa-s)^{\sigma-1}\widehat{\Psi}(s)ds \\ &+\frac{1}{\Gamma(\sigma)}\int_0^{\theta} e^{-\varrho(\theta-s)}(\theta-s)^{\sigma-1}\widehat{\Psi}(s)ds, \end{aligned}$$

we can decompose $\xi(\cdot)$ as $\xi(\theta) = w(\theta) + \bar{z}(\theta)$, for $\theta \in \Xi$, which implies that $\xi_\theta = w_\theta + \bar{z}_\theta$ for every $\theta \in \Xi$.

Set

$$C_0 = \{z \in C(\Xi, E) : z_0 = 0\},$$

and let $\|\cdot\|_T$ be the norm in C_0 defined by

$$\|z\|_T = \|z_0\|_{\mathbb{k}} + \sup_{\theta\in\Xi} \|z(\theta)\| = \sup_{\theta\in\Xi} \|z(\theta)\|, \qquad z \in C_0,$$

where $(C_0, \|\cdot\|_T)$ is a Banach space. Let define the operator $S : C_0 \to C_0$ by

$$\begin{aligned} Sz(\theta) = &-\frac{\delta_2 e^{-\varrho\theta}}{\Gamma(\sigma)(\delta_1 + \delta_2 e^{-\varrho\varkappa})} \int_0^{\varkappa} e^{-\varrho(\varkappa-s)}(\varkappa - s)^{\sigma-1}\widehat{\Psi}(s)ds \\ &+ \frac{1}{\Gamma(\sigma)} \int_0^{\theta} e^{-\varrho(\theta-s)}(\theta - s)^{\sigma-1}\widehat{\Psi}(s)ds. \end{aligned}$$

To show that the operator A has a fixed point is equivalent to show that S has a fixed point. The proof will be given in several steps.

Step 1: The operator $S : C_0 \longrightarrow C_0$ is continuous.
Let $\{z_\jmath\}_{\jmath\in\mathbb{N}}$ be a sequence such that $z_\jmath \longrightarrow z$ in C_0. Then, for $\theta \in \Xi$, we have

$$\begin{aligned} &\|Sz_\jmath(\theta) - Sz(\theta)\| \\ &\quad\le \frac{\delta_2 e^{-\varrho\theta}}{\Gamma(\sigma)(\delta_1 + \delta_2 e^{-\varrho\varkappa})} \int_0^{\varkappa} e^{-\varrho(\varkappa-s)}(\varkappa - s)^{\sigma-1}\|\widehat{\Psi}_\jmath(s) - \widehat{\Psi}(s)\|ds \\ &\qquad+ \frac{1}{\Gamma(\sigma)} \int_0^{\theta} e^{-\varrho(\theta-s)}(\theta - s)^{\sigma-1}\|\widehat{\Psi}_\jmath(s) - \widehat{\Psi}(s)\|ds, \end{aligned}$$

where $\widehat{\Psi}_\jmath$ and $\widehat{\Psi}$ are two functions satisfying the following functional equations:

$$\widehat{\Psi}_\jmath(\theta) = \Psi(\theta, w_\theta + \bar{z}_{\jmath\theta}, \widehat{\Psi}_\jmath(\theta)),$$

and

$$\widehat{\Psi}(\theta) = \Psi(\theta, w_\theta + \bar{z}_\theta, \widehat{\Psi}(\theta)).$$

By $(A2)$, we have

$$\begin{aligned}\|\widehat{\Psi}_{\jmath}(\theta)-\widehat{\Psi}(\theta)\| &= \|\Psi(\theta,w_\theta+\bar{z}_{\jmath\theta},\widehat{\Psi}_{\jmath}(\theta))-\Psi(\theta,w_\theta+\bar{z}_\theta,\widehat{\Psi}(\theta))\| \\ &\leq \lambda\|\bar{z}_{\jmath\theta}-\bar{z}_\theta\|_{\mathbb{k}}+L\|\widehat{\Psi}_{\jmath}(\theta)-\widehat{\Psi}(\theta)\|.\end{aligned}$$

Thus,

$$\|\widehat{\Psi}_{\jmath}(\theta)-\widehat{\Psi}(\theta)\| \leq \frac{\lambda}{1-L}\|\bar{z}_{\jmath\theta}-\bar{z}_\theta\|_{\mathbb{k}}.$$

$$\begin{aligned}&\|Sz_{\jmath}(\theta)-Sz(\theta)\| \\ &\leq \frac{\lambda\wp_2\delta_2e^{-\varrho\theta}}{\Gamma(\sigma)(\delta_1+\delta_2e^{-\varrho\varkappa})(1-L)}\int_0^{\varkappa}e^{-\varrho(\varkappa-s)}(\varkappa-s)^{\sigma-1}\|\bar{z}_{\jmath}-\bar{z}\|_T\,ds \\ &\quad+\frac{\lambda\wp_2}{(1-L)\Gamma(\sigma)}\int_0^{\theta}e^{-\varrho(\theta-s)}(\theta-s)^{\sigma-1}\|\bar{z}_{\jmath}-\bar{z}\|_T\,ds.\end{aligned}$$

By applying the Lebesgue dominated convergence theorem, we get

$$\|Sz_{\jmath}(\theta)-Sz(\theta)\|\longrightarrow 0 \quad \text{as} \quad \jmath\longrightarrow\infty,$$

which implies that

$$\|Sz_{\jmath}-Sz\|_{\varkappa}\longrightarrow 0 \quad \text{as} \quad \jmath\longrightarrow\infty.$$

Hence, the operator S is continuous.

Let $\kappa>0$ such that

$$\kappa \geq \frac{\varkappa^{\sigma}\left(q_1^*+\lambda\wp_1\|\Lambda\|_{\mathbb{k}}+\dfrac{\lambda\wp_2\delta_3}{\delta_1+\delta_2e^{-\varrho\varkappa}}\right)(\delta_2+\delta_1+\delta_2e^{-\varrho\varkappa})}{\Gamma(\sigma+1)(\delta_1+\delta_2e^{-\varrho\varkappa})(1-L)-\lambda\varkappa^{\sigma}\wp_2\,(\delta_2+\delta_1+\delta_2e^{-\varrho\varkappa})}$$

where $q_1(\theta)=\|\Psi(\theta,0,0)\|$, with $q_1\in C(\Xi,E)$, such that

$$q_1^*=\sup_{\theta\in\Xi}q_1(\theta).$$

Define the ball

$$\digamma_{\kappa}=\{z\in C_0:\|z\|_{\varkappa}\leq\kappa\}.$$

It is clear that $\digamma_\kappa$ is a bounded, closed and convex subset of C_0.

Step 2: $S(\digamma_\kappa) \subset \digamma_\kappa$.
Let $z \in \digamma_\kappa$ and $\theta \in \Xi$. Then,

$$\begin{aligned}\|Sz(\theta)\| &\le \frac{\delta_2 e^{-\varrho\theta}}{\Gamma(\sigma)(\delta_1+\delta_2 e^{-\varrho\varkappa})}\int_0^{\varkappa} e^{-\varrho(\varkappa-s)}(\varkappa-s)^{\sigma-1}\|\widehat{\Psi}(s)\|ds \\ &\quad + \frac{1}{\Gamma(\sigma)}\int_0^{\theta} e^{-\varrho(\theta-s)}(\theta-s)^{\sigma-1}\|\widehat{\Psi}(s)\|ds.\end{aligned}$$

From hypothesis $(A2)$, we have

$$\begin{aligned}\|\widehat{\Psi}(\theta)\| &= \|\Psi(\theta, w_\theta+\bar{z}_\theta, \widehat{\Psi}(\theta))\| \\ &\le q_1(\theta) + \lambda\|w_\theta+\bar{z}_\theta\|_{\Bbbk} + L\|\widehat{\Psi}(\theta)\| \\ &\le q_1^* + \lambda[\|w_\theta\|_{\Bbbk} + \|\bar{z}_\theta\|_{\Bbbk}] + L\|\widehat{\Psi}(\theta)\| \\ &\le q_1^* + \lambda\wp_2\kappa + \lambda\wp_1\|\Lambda\|_{\Bbbk} + \frac{\wp_2\delta_3}{\delta_1+\delta_2 e^{-\varrho\varkappa}} + L\|\widehat{\Psi}(\theta)\|.\end{aligned}$$

Then,

$$\|\widehat{\Psi}(\theta)\| \le \frac{q_1^* + \lambda\wp_2\kappa + \lambda\wp_1\|\Lambda\|_{\Bbbk} + \dfrac{\lambda\wp_2\delta_3}{\delta_1+\delta_2 e^{-\varrho\varkappa}}}{1-L}.$$

Finally, we have

$$\begin{aligned}\|Sz(\theta)\| &\le \frac{\delta_2 e^{-\varrho\theta}\varkappa^\sigma\left(q_1^* + \lambda\wp_2\kappa + \lambda\wp_1\|\Lambda\|_{\Bbbk} + \dfrac{\lambda\wp_2\delta_3}{\delta_1+\delta_2 e^{-\varrho\varkappa}}\right)}{\Gamma(\sigma+1)(\delta_1+\delta_2 e^{-\varrho\varkappa})(1-L)} \\ &\quad + \frac{\varkappa^\sigma\left(q_1^* + \lambda\wp_2\kappa + \lambda\wp_1\|\Lambda\|_{\Bbbk} + \dfrac{\lambda\wp_2\delta_3}{\delta_1+\delta_2 e^{-\varrho\varkappa}}\right)}{\Gamma(\sigma+1)(1-L)} \\ &\le \frac{\varkappa^\sigma\left(q_1^* + \lambda\wp_2\kappa + \lambda\wp_1\|\Lambda\|_{\Bbbk} + \dfrac{\lambda\wp_2\delta_3}{\delta_1+\delta_2 e^{-\varrho\varkappa}}\right)(\delta_2+\delta_1+\delta_2 e^{-\varrho\varkappa})}{\Gamma(\sigma+1)(\delta_1+\delta_2 e^{-\varrho\varkappa})(1-L)} \\ &\le \kappa.\end{aligned}$$

Hence, $S(\digamma_\kappa) \subset \digamma_\kappa$.

Step 3: $S(\digamma_\kappa)$ is equicontinuous.
Let $\theta_1, \theta_2 \in \Xi$, where $\theta_1 < \theta_2$ and $z \in \digamma_\kappa$. Then,

$$
\begin{aligned}
&\|Sz(\theta_2) - Sz(\theta_1)\| \\
&\quad = \Bigg\| -\frac{\delta_2 e^{-\varrho t_2}}{\Gamma(\sigma)(\delta_1 + \delta_2 e^{-\varrho\varkappa})} \int_0^{\varkappa} e^{-\varrho(\varkappa - s)} (\varkappa - s)^{\sigma-1} \widehat{\Psi}(s) ds \\
&\qquad + \frac{1}{\Gamma(\sigma)} \int_0^{t_2} e^{-\varrho(t_2 - s)} (t_2 - s)^{\sigma-1} \widehat{\Psi}(s) ds \\
&\qquad + \frac{\delta_2 e^{-\varrho t_1}}{\Gamma(\sigma)(\delta_1 + \delta_2 e^{-\varrho\varkappa})} \int_0^{\varkappa} e^{-\varrho(\varkappa - s)} (\varkappa - s)^{\sigma-1} \widehat{\Psi}(s) ds \\
&\qquad + \frac{1}{\Gamma(\sigma)} \int_0^{t_1} e^{-\varrho(t_1 - s)} (t_1 - s)^{\sigma-1} \widehat{\Psi}(s) ds \Bigg\| \\
&\quad \leq \frac{\delta_2 \left\| e^{-\varrho\theta_2} - e^{-\varrho\theta_1} \right\|}{\Gamma(\sigma)(\delta_1 + \delta_2 e^{-\varrho\varkappa})} \int_0^{\varkappa} e^{-\varrho(\varkappa - s)} (\varkappa - s)^{\sigma-1} \|\widehat{\Psi}(s)\| ds \\
&\qquad + \frac{1}{\Gamma(\sigma)} \int_0^{\theta_1} \left[e^{-\varrho(\theta_2 - s)} (\theta_2 - s)^{\sigma-1} - e^{-\varrho(\theta_1 - s)} (\theta_1 - s)^{\sigma-1} \right] \|\widehat{\Psi}(s)\| ds \\
&\qquad + \frac{1}{\Gamma(\sigma)} \int_{\theta_1}^{\theta_2} e^{-\varrho(\theta_2 - s)} (\theta_2 - s)^{\sigma-1} \|\widehat{\Psi}(s)\| ds \\
&\quad \leq \frac{\delta_2 \varkappa^\sigma \left\| e^{-\varrho\theta_2} - e^{-\varrho\theta_1} \right\| \left(q_1^* + \lambda\wp_2\kappa + \lambda\wp_1 \|\Lambda\|_{\mathbb{k}} + \frac{\lambda\wp_2\delta_3}{\delta_1 + \delta_2 e^{-\varrho\varkappa}} \right)}{\Gamma(\sigma+1)(1-L)(\delta_1 + \delta_2 e^{-\varrho\varkappa})} \\
&\qquad + \frac{\left(q_1^* + \lambda\wp_2\kappa + \lambda\wp_1 \|\Lambda\|_{\mathbb{k}} + \frac{\lambda\wp_2\delta_3}{\delta_1 + \delta_2 e^{-\varrho\varkappa}} \right) (\theta_2^\sigma - \theta_1^\sigma)}{(1-L)\Gamma(\sigma+1)} \\
&\qquad + \frac{\left(q_1^* + \lambda\wp_2\kappa + \lambda\wp_1 \|\Lambda\|_{\mathbb{k}} + \frac{\lambda\wp_2\delta_3}{\delta_1 + \delta_2 e^{-\varrho\varkappa}} \right) (\theta_2 - \theta_1)^\sigma}{(1-L)\Gamma(\sigma+1)}.
\end{aligned}
$$

As $\theta_1 \longrightarrow \theta_2$, the right-hand side of the inequality above tend to zero. Therefore, the operator $S(\digamma_\kappa)$ is equicontinuous.

Step 4: The implication of Mönch's theorem is satisfied.
Let B be a subset of F_κ such that $B = S(B) \cup \{0\}$. Therefore, the function $\theta \longrightarrow b(\theta) = \zeta(B(\theta))$ is continuous on Ξ. Then, for $\theta \in \Xi$, we have

$$\begin{aligned} b(\theta) &= \zeta(B(\theta)) \\ &= \zeta\big\{Sz(\theta), \quad z \in B\big\} \\ &= \zeta\Bigg\{ - \frac{\delta_2 e^{-\varrho\theta}}{\Gamma(\sigma)(\delta_1 + \delta_2 e^{-\varrho\varkappa})} \int_0^{\varkappa} e^{-\varrho(\varkappa - s)}(\varkappa - s)^{\sigma-1}\widehat{\Psi}(s)ds \\ &+ \frac{1}{\Gamma(\sigma)} \int_0^{\theta} e^{-\varrho(\theta-s)}(\theta - s)^{\sigma-1}\widehat{\Psi}(s)ds, \quad z \in B\Bigg\} \\ &\leq \frac{\delta_2 e^{-\varrho\theta}}{\Gamma(\sigma)(\delta_1 + \delta_2 e^{-\varrho\varkappa})} \int_0^{\varkappa} e^{-\varrho(\varkappa-s)}(\varkappa - s)^{\sigma-1}\{\zeta(\widehat{\Psi}(s))ds, \quad z \in B\} \\ &+ \frac{1}{\Gamma(\sigma)} \int_0^{\theta} (\theta - s)^{\sigma-1}\{\zeta(\widehat{\Psi}(s))ds, \quad z \in B\}. \end{aligned}$$

By condition $(A3)$, we have

$$\begin{aligned} \zeta(\widehat{\Psi}(\theta)) &= \zeta(\Psi(\theta, w_\theta + \bar{z}_\theta, \widehat{\Psi}(\theta)) \\ &\leq \lambda \sup_{\theta \in (-\infty, 0]} \zeta(w_\theta + \bar{z}_\theta) + L\zeta(\widehat{\Psi}(\theta)) \\ &\leq \lambda \sup_{\theta \in (-\infty, \varkappa]} \zeta(w_\theta + \bar{z}_\theta) + L\zeta(\widehat{\Psi}(\theta)). \end{aligned}$$

Thus,

$$\zeta(\widehat{\Psi}(\theta)) \leq \frac{\lambda}{1 - L} \sup_{\theta \in (-\infty, \varkappa]} \zeta(w_\theta + \bar{z}_\theta).$$

Then,

$$\begin{aligned} \zeta(B(\theta)) \leq &\frac{\lambda\delta_2 \displaystyle\int_0^{\varkappa} (\varkappa - s)^{\sigma-1}\{\sup_{\theta \in (-\infty,\varkappa]} \zeta(w_s + \bar{z}_s)ds, \quad z \in B\}}{(1 - L)\Gamma(\sigma)(\delta_1 + \delta_2 e^{-\varrho\varkappa})} \\ &+ \frac{\lambda}{(1 - L)\Gamma(\sigma)} \int_0^{\theta} (\theta - s)^{\sigma-1}\{\sup_{\theta \in (-\infty,\varkappa]} \zeta(w_s + \bar{z}_s)ds, \quad z \in B\} \end{aligned}$$

$$
\begin{aligned}
&\leq \left[\frac{\lambda \varkappa^{\sigma} \delta_2}{(1-L)\Gamma(\sigma+1)(\delta_1+\delta_2 e^{-\varrho\varkappa})} + \frac{\lambda \varkappa^{\sigma}}{(1-L)\Gamma(\sigma+1)}\right] \zeta_{\Upsilon_E}(B) \\
&\leq \left[\frac{\lambda \varkappa^{\sigma}\left(\delta_2+\delta_1+\delta_2 e^{-\varrho\varkappa}\right)}{(1-L)\Gamma(\sigma+1)(\delta_1+\delta_2 e^{-\varrho\varkappa})}\right] \zeta_{\Upsilon_E}(B).
\end{aligned}
$$

Therefore,

$$
\zeta_{\Upsilon_E}(B) \leq \left[\frac{\lambda \varkappa^{\sigma}\left(\delta_2+\delta_1+\delta_2 e^{-\varrho\varkappa}\right)}{(1-L)\Gamma(\sigma+1)(\delta_1+\delta_2 e^{-\varrho\varkappa})}\right] \zeta_{\Upsilon_E}(B),
$$

which implies that $\zeta_{\Upsilon_E}(B) = 0$. We conclude by Mönch fixed point theorem that the operator S has at least one fixed point which is the fixed point of the operator A and the solution of the problem (1)-(3). □

4 Ulam-Hyers Stability

In this section, we will establish the Ulam stability for the problem (1)-(3).

Definition 4.1 ([1]). Problem (1)-(3) is Ulam-Hyers stable if there exists a real number $C_\Psi > 0$ such that for each $\varepsilon > 0$ and for each solution $\xi \in \Upsilon_E$ of the inequality

$$
\left\| {}_0^C D_\theta^{\sigma,\varrho} \xi(\theta) - \Psi\left(\theta, \xi_\theta,\ {}_0^C D_\theta^{\sigma,\varrho} \xi(\theta)\right)\right\| < \varepsilon, \quad \theta \in \Xi, \tag{11}
$$

there exists a solution $\bar{\xi} \in \Upsilon_E$ of the problem (1)-(3) with

$$
\|\xi(\theta) - \bar{\xi}(\theta)\| < C_\Psi \varepsilon, \quad \theta \in \Xi.
$$

Definition 4.2 ([1]). Problem (1)-(3) is generalized Ulam-Hyers stable if there exists $\Lambda_\Psi \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\Lambda_\Psi(0) = 0$ such that for each solution $\xi \in \Upsilon_E$ of the inequality (11) there exists a solution $\bar{\xi} \in \Upsilon_E$ of the problem (1)-(3) with

$$
\|\xi(\theta) - \bar{\xi}(\theta)\| < \Lambda_\Psi \varepsilon, \quad \theta \in \Xi.
$$

Remark 4.3. A function $\xi \in \Upsilon_E$ is a solution of the inequality (11) if and only if there exists a function $\ell \in C(\Xi, E)$ (which depend on ξ) such that

1. $\|\ell(\theta)\| \leq \varepsilon, \quad for\ each\ \theta \in \Xi.$
2. ${}^{C}_{0}D^{\sigma,\varrho}_{\theta}\xi(\theta) = \Psi\left(\theta, \xi_\theta, {}^{C}_{0}D^{\sigma,\varrho}_{\theta}\xi(\theta)\right) + \ell(\theta), \quad for\ each\ \theta \in \Xi.$

Lemma 4.4. *The solution of the following perturbed problem*

$$
\begin{gathered}
{}^{C}_{0}D^{\sigma,\varrho}_{\theta}\xi(\theta) = \Psi\left(\theta, \xi_\theta, {}^{C}_{0}D^{\sigma,\varrho}_{\theta}\xi(\theta)\right) + \ell(\theta), \qquad \theta \in \Xi := [0, \varkappa], \\
\xi(\theta) = \Lambda(\theta), \qquad \theta \in (-\infty, 0], \\
\delta_1\xi(0) + \delta_2\xi(\varkappa) = \delta_3,
\end{gathered}
$$

is given by

$$
\xi(\theta) = \begin{cases}
\dfrac{\delta_3 e^{-\varrho\theta}}{\delta_1 + \delta_2 e^{-\varrho\varkappa}} - \dfrac{\delta_2 e^{-\varrho\theta}\displaystyle\int_0^{\varkappa} e^{-\varrho(\varkappa-s)}(\varkappa - s)^{\sigma-1}\widehat{\Psi}(s)ds}{\Gamma(\sigma)(\delta_1 + \delta_2 e^{-\varrho\varkappa})} \\
+\dfrac{1}{\Gamma(\sigma)}\displaystyle\int_0^{\theta} e^{-\varrho(\theta-s)}(\theta - s)^{\sigma-1}\widehat{\Psi}(s)ds \\
-\dfrac{\delta_2 e^{-\varrho\theta}}{\Gamma(\sigma)(\delta_1 + \delta_2 e^{-\varrho\varkappa})}\displaystyle\int_0^{\varkappa} e^{-\varrho(\varkappa-s)}(\varkappa - s)^{\sigma-1}\ell(s)ds \\
+\dfrac{1}{\Gamma(\sigma)}\displaystyle\int_0^{\theta} e^{-\varrho(\theta-s)}(\theta - s)^{\sigma-1}\ell(s)ds \;\; \theta \in \Xi, \\
\\
\Lambda(\theta), \qquad\qquad \theta \in (-\infty, 0].
\end{cases}
$$

Moreover, the solution satisfies the following inequality

$$
\begin{aligned}
&\left\|\xi(\theta) - \left[\frac{\delta_3 e^{-\varrho\theta}}{\delta_1 + \delta_2 e^{-\varrho\varkappa}} - \frac{\delta_2 e^{-\varrho\theta}}{\Gamma(\sigma)(\delta_1 + \delta_2 e^{-\varrho\varkappa})}\int_0^{\varkappa} e^{-\varrho(\varkappa-s)}(\varkappa - s)^{\sigma-1}\widehat{\Psi}(s)ds\right.\right. \\
&\left.\left.+\frac{1}{\Gamma(\sigma)}\int_0^{\theta} e^{-\varrho(\theta-s)}(\theta - s)^{\sigma-1}\widehat{\Psi}(s)ds\right]\right\| \quad \leq \frac{\varkappa^\sigma\varepsilon}{\Gamma(\sigma+1)}\left[\frac{\delta_2}{\delta_1 + \delta_2 e^{-\varrho\varkappa}} + 1\right].
\end{aligned}
$$

Theorem 4.5. *Assume that the conditions* $(A1)$*-*$(A2)$ *hold. If*

$$
\frac{\lambda\varkappa^\sigma\left(\delta_1 + \delta_2 e^{-\varrho\varkappa} + \delta_2\right)}{\Gamma(\sigma+1)(1-L)(\delta_1 + \delta_2 e^{-\varrho\varkappa})} < 1,
$$

then the problem (1)*-*(3) *is Ulam-Hyers stable.*

Proof. Let $\xi \in \Upsilon_E$ be a solution of the inequality (11) and $\bar{\xi} \in \Upsilon_E$ the solution of the problem (1)-(3). Then,

$$\begin{aligned}\|\xi(\theta)-\bar{\xi}(\theta)\| \leq & \frac{\varkappa^{\sigma}\varepsilon}{\Gamma(\sigma+1)}\left[\frac{\delta_2}{\delta_1+\delta_2 e^{-\varrho\varkappa}}+1\right] \\ &+\frac{\delta_2 e^{-\varrho\theta}\displaystyle\int_0^{\varkappa} e^{-\varrho(\varkappa-s)}(\varkappa-s)^{\sigma-1}\|\widehat{\Psi}(s)-\widetilde{\Psi}(s)\|ds}{\Gamma(\sigma)(\delta_1+\delta_2 e^{-\varrho\varkappa})} \\ &+\frac{1}{\Gamma(\sigma)}\int_0^{\theta} e^{-\varrho(\theta-s)}(\theta-s)^{\sigma-1}\|\widehat{\Psi}(s)-\widetilde{\Psi}(s)\|ds,\end{aligned}$$

where $\widehat{\Psi}$ and $\widetilde{\Psi}$ are two functions satisfying the following functional equations:

$$\widehat{\Psi}(\theta)=\Psi(\theta,\xi_\theta,\widehat{\Psi}(\theta)),$$

and

$$\widetilde{\Psi}(\theta)=\Psi(\theta,\bar{\xi}_\theta,\widetilde{\Psi}(\theta)).$$

From hypothesis $(A2)$, we have

$$\begin{aligned}\|\widehat{\Psi}(\theta)-\widetilde{\Psi}(\theta)\| &= \|\Psi(\theta,\xi_\theta,\widehat{\Psi}(\theta))-\Psi(\theta,\bar{\xi}_\theta,\widetilde{\Psi}(\theta))\| \\ &\leq \lambda\|\xi_\theta-\bar{\xi}_\theta\|_{\mathbb{k}}+L\|\widehat{\Psi}(\theta)-\widetilde{\Psi}(\theta)\|,\end{aligned}$$

which implies that

$$\|\widehat{\Psi}(\theta)-\widetilde{\Psi}(\theta)\| \leq \frac{\lambda}{1-L}\|\xi_\theta-\bar{\xi}_\theta\|_{\mathbb{k}}.$$

Then,

$$\begin{aligned}\|\xi(\theta)-\bar{\xi}(\theta)\| \leq & \frac{\varkappa^{\sigma}\varepsilon}{\Gamma(\sigma+1)}\left[\frac{\delta_2}{\delta_1+\delta_2 e^{-\varrho\varkappa}}+1\right] \\ &+\frac{\lambda\varkappa^{\sigma}\delta_2}{\Gamma(\sigma+1)\,(\delta_1+\delta_2 e^{-\varrho\varkappa})\,(1-L)}\|\xi-\bar{\xi}\|_{\Upsilon_E} \\ &+\frac{\lambda\varkappa^{\sigma}}{\Gamma(\sigma+1)(1-L)}\|\xi-\bar{\xi}\|_{\Upsilon_E}.\end{aligned}$$

Thus,

$$\|\xi-\bar{\xi}\|_{\Upsilon_E} \leq \frac{\dfrac{\varkappa^{\sigma}\varepsilon}{\Gamma(\sigma+1)}\left[\dfrac{\delta_2}{\delta_1+\delta_2 e^{-\varrho\varkappa}}+1\right]}{1-\dfrac{\lambda\varkappa^{\sigma}(\delta_1+\delta_2 e^{-\varrho\varkappa}+\delta_2)}{\Gamma(\sigma+1)(1-L)(\delta_1+\delta_2 e^{-\varrho\varkappa})}} := C_f\varepsilon.$$

Consequently, the problem (1)-(3) is Ulam-Hyers stable. If we take $\Lambda_\Psi(\varepsilon)=C_\Psi\varepsilon$ and $\Lambda_\Psi(0)=0$, then we get the generalized Ulam-Hyers stability of the problem (1)-(3). $\square$

5 An Example

Set

$$E = l^1 = \left\{\xi=(\xi_1,\xi_2,\ldots,\xi_\jmath,\ldots), \sum_{\jmath=1}^{\infty}|\xi_\jmath|<\infty\right\},$$

where E is a Banach space with the norm $\|\xi\| = \sum_{\jmath=1}^{\infty}|\xi_\jmath|$.

Consider the following implicit problem:

$$ {}_0^C D_\theta^{\frac{1}{2},4}\xi(\theta) = \Psi\left(\theta,\xi_\theta,\ {}_0^C D_\theta^{\frac{1}{2},4}\xi(\theta)\right),\ \theta\in[0,1], \tag{12}$$

$$\xi(\theta)=\Lambda(\theta), \qquad \theta\in(-\infty,0], \tag{13}$$

$$\xi(0)+\xi(1)=0, \tag{14}$$

where $\Lambda\in\Bbbk_\gamma$, γ is a positive real constant and

$$\Bbbk_\gamma = \{\xi\in C((-\infty,0],E):\ \lim_{\theta\to-\infty} e^{\gamma\theta}\xi(\theta) \textit{ exists in } E\}. \tag{15}$$

The norm of $\Bbbk_\gamma$ is given by

$$\|\xi\|_\gamma = \sup_{\theta\in(-\infty,0]} e^{\gamma\theta}\|\xi(\theta)\|.$$

Let $\xi:(-\infty,1]\to E$ be such that $\xi_0\in\Bbbk_\gamma$. Then

$$\lim_{s\to-\infty} e^{\gamma s}\xi_\theta(s) = \lim_{s\to-\infty} e^{\gamma s}\xi(\theta+s)$$

$$
\begin{aligned}
&= \lim_{s\to-\infty} e^{\gamma(s-\theta)}\xi(s)\\
&= e^{-\gamma\theta}\lim_{s\to-\infty} e^{\gamma s}\xi_0(s)\\
&< \infty.
\end{aligned}
$$

Hence $\xi_\theta \in \Bbbk_\gamma$. Finally we prove that

$$\|\xi_\theta\|_\gamma \le \wp_1\|\xi_0\|_\gamma + \wp_2 \sup_{\theta\in[0,\varkappa]} \|\xi(\theta)\|,$$

where $\wp_1 = \wp_2 = 1$. We have

$$\|\xi_\theta(s)\| = \|\xi(\theta+s)\|.$$

If $\theta + s \le 0$, we get

$$\|\xi_\theta(s)\| \le \sup_{\theta\in(-\infty,0]} \|\xi(s)\|.$$

For $\theta + s \ge 0$, then we have

$$\|\xi_\theta(s)\| \le \sup_{s\in[0,\varkappa]} \|\xi(s)\|.$$

Thus, for all $\theta + s \in (-\infty, 1]$, we get

$$\|\xi_\theta(s)\| \le \sup_{s\in(-\infty,0]} \|\xi(s)\| + \sup_{s\in[0,\varkappa]} \|\xi(s)\|.$$

Then

$$\|\xi_\theta\|_\gamma \le \|\xi_0\|_\gamma + \sup_{s\in[0,\varkappa]} \|\xi(s)\|.$$

It is clear that $(\Bbbk_\gamma, \|\cdot\|)$ is a Banach space. We can conclude that $\Bbbk_\gamma$ a phase space.

Set

$$\Psi_\jmath\left(\theta, \xi_{\jmath\theta},\ {}_0^C D_\theta^{\frac{1}{2},4}\xi_\jmath(\theta)\right) = \frac{e^{-\pi\theta}\left(\|\xi_\theta\|_\Bbbk + \left|{}_0^C D_\theta^{\frac{1}{2},4}\xi_\jmath(\theta)\right|\right)}{110\left(1 + \|\xi_\theta\|_\Bbbk + \left\|{}_0^C D_\theta^{\frac{1}{2},4}\xi(\theta)\right\|\right)},$$

for $\theta \in [0,1]$, $\xi \in C((-\infty,1],E)$, where

$$\xi = (\xi_1, \xi_2, \ldots, \xi_\jmath, \ldots),$$

$$\Psi = (\Psi_1, \Psi_2, \ldots, \Psi_\jmath, \ldots),$$

and

$$ {}_0^C D_\theta^{\frac{1}{2},4} \xi = \left({}_0^C D_\theta^{\frac{1}{2},4} \xi_1 \ , \ {}_0^C D_\theta^{\frac{1}{2},4} \xi_2 \ , \ldots , \ {}_0^C D_\theta^{\frac{1}{2},4} \xi_\jmath, \ldots \right).$$

Clearly, Ψ is a continuous function, then the hypothesis $(A1)$ is satisfied. For any $\alpha, \bar{\alpha} \in \Bbbk$, $\beta, \bar{\beta} \in E$ and $\theta \in [0,1]$, we have

$$\|\Psi(\theta,\alpha,\beta) - \Psi(\theta,\bar{\alpha},\bar{\beta})\| \leq \frac{1}{110}\left[\|\alpha-\bar{\alpha}\|_{\Bbbk} + \|\beta-\bar{\beta}\|\right].$$

Then, the hypothesis $(A2)$ is satisfied with $\lambda = L = \frac{1}{110}$. Also we have

$$\frac{\lambda \varkappa^\sigma (\wp_2+1)\left(\delta_2+\delta_1+\delta_2 e^{-\varrho\varkappa}\right)}{\Gamma(\sigma+1)(1-L)(\delta_1+\delta_2 e^{-\varrho\varkappa})} < \frac{6}{109\sqrt{\pi}} < 1,$$

for $\varkappa = 1$, $\sigma = \frac{1}{2}$, $\wp_1 = \wp_2 = 1$ and $\delta_1 = \delta_2 = 1$. By Theorem 3.5, the problem (12)-(14) has at least one solution. Moreover, we have

$$\frac{\lambda \varkappa^\sigma \left(\delta_1+\delta_2 e^{-\varrho\varkappa}+\delta_2\right)}{\Gamma(\sigma+1)(1-L)(\delta_1+\delta_2 e^{-\varrho\varkappa})} < \frac{3}{109\sqrt{\pi}} < 1.$$

Thus, by Theorem 4.5, the problem (12)-(14) is Ulam-Hyers stable.

6 Conclusions

The current research focuses on exploring the existence criteria for solutions of a boundary value problem involving the Caputo tempered fractional derivative and infinite delay. To obtain the desired outcomes, we employed the fixed-point approach, specifically Mönch's fixed point theorem in conjunction with the concept of the measure of noncompactness. Furthermore, a section of the study is dedicated to investigating the Ulam-Hyers stability of our problem. To illustrate the practical application of the main findings, an example is provided. Our results, within this specific framework, are novel and make a significant contribution to the existing literature in this emerging field of study. Given the limited number of publications on implicit Caputo tempered fractional differential equations, we believe there are numerous potential research avenues to explore, such as coupled systems, problems with impulses, and more. Further, we have the opportunity to expand our work by incorporating various recently introduced fractional operators, including the Abu-Shady-Kaabar fractional derivative, which is a generalized fractional derivative. This particular fractional definition can achieve similar outcomes as the Caputo fractional operator in a straightforward manner, eliminating the necessity for modified numerical techniques. For more information, refer to [5,31]. We hope that this article will serve as a starting point for further exploration in these directions.

Declarations

Ethical approval: This article does not contain any studies with human participants or animals performed by any of the authors.

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