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The Weak Integral Closure of a Filtration Relative to a Module

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Abstract. In this paper, we introduce the weak integral closure of a filtration relative to a module and the asymptotic prime divisors of a filtration relative to a module. Based on these concepts, we prove some new results.

One of the most important points of this paper is to find the effect of contraction, and extension on the integral closure of a filtration relative to a module. Finally, the theorems we prove enable us to characterize the asymptotic prime divisors of contraction of a filtration f relative to a module M according to the asymptotic prime divisors of the filtration f relative to a module M.

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1 Introduction

Throughout this paper, R is a commutative ring with a non-zero identity and M is an $R-$module.

The concepts of reduction and integral closure of an ideal were introduced by D. G. Northcott and D. Rees in [5]. Let R be a commutative Noetherian ring and I and J be ideals of R. Then I is a reduction of J if $I \subseteq J$ and there exists a positive integer s such that $IJ^s = J^{s+1}$. An element x of R is said to be integrally dependent on I if there exist a positive integer m and elements $a_1, \ldots, a_m \in R$ with $a_i \in I^i$ for $i = 1, 2, \ldots, m$ such that

$$x^m + a_1 x^{m-1} + \cdots + a_m = 0.$$

We know, the set of all elements of R which are integrally dependent on I, is an ideal of R. This ideal is called the integral closure of I and is denoted by $\overline{I}$. In fact, $\overline{I}$ is the largest ideal of R which has I as a reduction.

In [8], R. Y. Sharp, Y. Tiraş, and M. Yassi introduced concepts of reduction and integral closure of an ideal I of a commutative ring R relative to a Noetherian module M. Here, we recall some of these definitions. Let I and J be ideals of R and M be a Noetherian $R-$module. Then I is said to be a reduction of J relative to M if $I \subseteq J$ and there exists a positive integer s such that $IJ^sM = J^{s+1}M$. Also, an element x of R is said to be integrally dependent on I relative to M if there exists a positive integer m such that

$$x^m M \subseteq \sum_{i=1}^{m} x^{m-i} I^i M.$$

The set of all elements of R which are integrally dependent on I relative to M, is an ideal of R. This ideal is called the integral closure of I relative to M and is denoted by $I^{-(M)}$. It is the largest ideal of R which has I as a reduction relative to M.

A filtration $f = \{I_n\}_{n \geq 0}$ on R is a sequence of ideals of R such that $I_0 = R$, $I_{n+1} \subseteq I_n$, and $I_n I_m \subseteq I_{n+m}$, for all non-negative integers m and n. For an ideal I of R, the filtration $f = \{I^n\}_{n \geq 0}$ is called the I-adic filtration on R.

Let $f = \{I_n\}_{n\geq 0}$ be a filtration on R and k be a positive integer. We know $\{I_{nk}\}_{n\geq 0}$ is a filtration on R and it is denoted by $f^{(k)}$. Further for every $n \geq 0$, $I_{n0} = R$ and this shows $f^{(0)}$ is also a filtration on R.

Let $f = \{I_n\}_{n\geq 0}$ and $g = \{J_n\}_{n\geq 0}$ be two filtrations on R. We know $f \leq g$, if $I_n \subseteq J_n$ for all n. Also two filtrations $\{\sum_{i=0}^{n} I_{n-i}J_i\}_{n\geq 0}$ and $\{I_nJ_n\}_{n\geq 0}$ are denoted by $f+g$ and fg respectively. Further, it is easy to see that $\{I_n \cap J_n\}_{n\geq 0}$ is a filtration. This filtration is denoted by $f \cap g$.

Now, let R be a Noetherian ring. A filtration $f = \{I_n\}_{n\geq 0}$ on R is called a Noetherian filtration if there exists a positive integer k such that

$$I_n = \sum_{i=1}^{k} I_{n-i}I_i,$$

for all $n \geq 1$. Clearly, if R is Noetherian, then the I-adic filtration is Noetherian (see [7]).

The weak integral closure of a filtration $f = \{I_n\}_{n\geq 0}$ is defined in [6]. For a non-negative integer n, let $(I_n)_w$ be the set of all $x \in R$ such that for each, there exists a positive integer m such that x satisfies an equation of the form $x^m + a_1x^{m-1} + ... + a_m = 0$, where $a_i \in I_{ni}$ for every $1 \leq i \leq m$. We know from [6, 2.2] that the sequence $\{(I_n)_w\}_{n\geq 0}$ of ideals of R is a filtration on R. This filtration is called the weak integral closure of the filtration $f = \{I_n\}_{n\geq 0}$ and is denoted by f_w. According to our notations in this paper, we prefer to denote the weak integral closure of the filtration f by f^-.

In [2], H. Dichi defined the integral closure of a filtration. An element $x \in R$ is said to be integral over a filtration $f = \{I_n\}_{n\geq 0}$ on R if there exists a positive integer m such that

$$x^m + a_1x^{m-1} + \cdots + a_m = 0,$$

where $a_i \in I_i$ for every $1 \leq i \leq m$. The set of all elements $x \in R$, which are integral over $f = \{I_n\}_{n\geq 0}$, is an ideal. This ideal is called the integral closure of a filtration $f = \{I_n\}_{n\geq 0}$ and is denoted by $Clos_R(f)$.

Even in [1], for a filtration $f = \{I_n\}_{n\geq 0}$, the set of all elements $x \in R$, which are integral over the filtration $f^{(k)} = \{I_{nk}\}_{n\geq 0}$, is denoted by $P_k(f)$. Further in [1, 2.1], It has been shown that the sequence

$\{P_k(f)\}_{k\geq 0}$ of ideals of R is a filtration on R. It might be beneficial to consider the following equation

$$P_k(f) = Clos_R(f^{(k)}) = (I_k)_w$$

for all $k \geq 0$.

Also, the integral closure of a filtration relative to a module is introduced in [2]. An element $x \in R$ is integral over a filtration $f = \{I_n\}_{n\geq 0}$ relative to an $R-$module M, if there exists a positive integer m such that

$$x^m + a_1x^{m-1} + \cdots + a_m \in (0 :_R M),$$

where $a_i \in I_i$ for every $1 \leq i \leq m$. The set of all elements of R which are integral over a filtration $f = \{I_n\}_{n\geq 0}$ relative to a module M is an ideal. This ideal is called the integral closure of a filtration $f = \{I_n\}_{n\geq 0}$ relative to M and is denoted by $Clos_R(f, M)$.

In [3] and [4], the integral closure of a filtration relative to an injective module and a Noetherian module have been defined. As we saw above, if f^- is the weak integral closure of a filtration f then $f^- = \{Clos_R(f^{(k)})\}_{k\geq 0}$. Drawing inspiration from this point, we were able to define the weak integral closure of a filtration relative to an arbitrary module. The definition of asymptotic prime divisors of a filtration f on a Noetherian ring R was first introduced in [6].

In this paper, after verifying some classical properties for the weak integral closure of a filtration relative to a module, we defined the asymptotic prime divisors of a filtration f on a Noetherian ring R relative to an $R-$module M. Then, we proved some theorems about the asymptotic prime divisors of a filtration relative to a module.

2 Auxiliary Results

We have seen the definition of the integral closure of a filtration on R relative to an $R-$module M. When M is a finitely generated R-module, we have the following proposition.

Proposition 2.1. (See [2, 2.6].) *Let $f = \{I_n\}_{n\geq 0}$ be a filtration on R and let M be a finitely generated R-module. Then x is integral over f relative to M if and only if there exists a positive integer m such that*

$$x^m M \subseteq \sum_{i=1}^{m} x^{m-i} I_i M.$$

Remark 2.2. Let M be an $R-$module. In the remainder of this paper, as shown in [8], the commutative ring $R/Ann_R(M)$ is denoted by $\widetilde{R}$. For an ideal I of R, the ideal $I + Ann_R(M)/Ann_R(M)$ of $\widetilde{R}$ is denoted by $\widetilde{I}$. Also, an element $x + Ann_R(M) \in R/Ann_R(M)$ is denoted by $\widetilde{x}$. If $f = \{I_n\}_{n\geq 0}$ is a filtration of ideals of R then $\{\widetilde{I_n}\}_{n\geq 0}$ is a filtration of ideals of $\widetilde{R}$ and this is denoted by $\widetilde{f}$.

Proposition 2.3. (See [2, 2.2, 2.3 (iii)].) *Let $f = \{I_n\}_{n\geq 0}$ be a filtration on R. Let M be an $R-$module. Then we have the following statements.*

(a) *$x \in Clos_R(f, M)$ if and only if $\widetilde{x} \in Clos_{\widetilde{R}}(\widetilde{f})$.*

(b) *$Clos_{\widetilde{R}}(\widetilde{f}) = Clos_R(f, M)/(0 :_R M)$.*

Theorem 2.4. *Let $f = \{I_n\}_{n\geq 0}$ be a filtration on R and M be an $R-$module. Then $\{Clos_R(f^{(k)}, M)\}_{k\geq 0}$ is a filtration on R.*

Proof. Let ${U_k}^f = Clos_R(f^{(k)}, M)$ for every $k \geq 0$. It is easy to see that ${U_0}^f = R$ and ${U_{k+1}}^f \subseteq {U_k}^f$ for every $k \geq 0$. We will show that ${U_i}^f {U_j}^f \subseteq {U_{i+j}}^f$ for all $i, j \geq 1$. Let $xy \in {U_i}^f {U_j}^f$, where $x \in {U_i}^f$ and $y \in {U_j}^f$. By [6, 2.2], we have

$$\begin{aligned} \widetilde{xy} = \widetilde{x}\widetilde{y} \in Clos_{\widetilde{R}}(\widetilde{f^{(i)}})Clos_{\widetilde{R}}(\widetilde{f^{(j)}}) &= (\widetilde{I}_i)_w(\widetilde{I}_j)_w \\ &\subseteq (\widetilde{I_{i+j}})_w \\ &= Clos_{\widetilde{R}}(\widetilde{f^{(i+j)}}). \end{aligned}$$

Now by Proposition 2.3(a), we have $xy \in {U_{i+j}}^f$. □

Definition 2.5. Let $f = \{I_n\}_{n\geq 0}$ be a filtration on R and M be an $R-$module. Then $\{Clos_R(f^{(k)}, M)\}_{k\geq 0}$ is a filtration on R by Theorem 2.4. This filtration is called the weak integral closure of a filtration $f = \{I_n\}_{n\geq 0}$ relative to M and is denoted by $f^{-(M)}$. By Proposition 2.3(a), we have $\widetilde{f^{-(M)}} = (\widetilde{f})^-$.

The following theorem shows that $f \to f^{-(M)}$ is a closure operation and this is a semi-prime operation when R is a Noetherian ring.

Theorem 2.6. *Let $f = \{I_n\}_{n\geq 0}$ and $g = \{J_n\}_{n\geq 0}$ be filtrations on R and M be an $R-$module. Then we have the following statements.*

(a) $f \leq f^{-} \leq f^{-(M)}$.

(b) *If* $f \leq g$, *then* $f^{-(M)} \leq g^{-(M)}$.

(c) $(f^{-(M)})^{-(M)} = f^{-(M)}$.

(d) *If R is a Noetherian ring, then* $f^{-(M)}g^{-(M)} \leq (fg)^{-(M)}$.

Proof. (a) and (b) are clear.

(c) By (a) and (b), we have $f^{-(M)} \leq (f^{-(M)})^{-(M)}$.

Now let $x \in Clos_R((f^{-(M)})^{(k)}, M)$. Then $\widetilde{x} \in Clos_{\widetilde{R}}((\widetilde{f^{-(M)})^{(k)}})$ by Proposition 2.3(a). But since $\widetilde{f^{-(M)}} = \{Clos_R(f^{(k)}, M)/(0:_R M)\}_{k\geq 0}$ we can see

$$Clos_{\widetilde{R}}((\widetilde{f^{-(M)})^{(k)}}) = (Clos_R(f^{(k)}, M)/(0:_R M))_w.$$

By Proposition 2.3(b), we have $Clos_R(f^{(k)}, M)/(0:_R M) = Clos_{\widetilde{R}}(\widetilde{f^{(k)}})$. Also, we know $Clos_{\widetilde{R}}(\widetilde{f^{(k)}}) = (\widetilde{I_k})_w$. Then

$$Clos_{\widetilde{R}}((\widetilde{f^{-(M)})^{(k)}}) = ((\widetilde{I_k})_w)_w.$$

By [6, 2.4(3)], we have $((\widetilde{I_k})_w)_w = (\widetilde{I_k})_w$. Thus

$$Clos_{\widetilde{R}}((\widetilde{f^{-(M)})^{(k)}}) = (\widetilde{I_k})_w = Clos_{\widetilde{R}}(\widetilde{f^{(k)}}).$$

So, if $x \in Clos_R((f^{-(M)})^{(k)}, M)$ then $\widetilde{x} \in Clos_{\widetilde{R}}(\widetilde{f^{(k)}})$. Thus by Proposition 2.3(a), we have if $x \in Clos_R((f^{-(M)})^{(k)}, M)$ then $x \in Clos_R(f^{(k)}, M)$. This shows $(f^{-(M)})^{-(M)} \leq f^{-(M)}$ and so $(f^{-(M)})^{-(M)} = f^{-(M)}$.

(d) Let $x \in Clos_R(f^{(k)}, M)$ and $y \in Clos_R(g^{(k)}, M)$. Then $\widetilde{x} \in Clos_{\widetilde{R}}(\widetilde{f^{(k)}})$ and $\widetilde{y} \in Clos_{\widetilde{R}}(\widetilde{g^{(k)}})$. Thus

$$\widetilde{x}\widetilde{y} \in Clos_{\widetilde{R}}(\widetilde{f^{(k)}})Clos_{\widetilde{R}}(\widetilde{g^{(k)}}) = (\widetilde{I_k})_w(\widetilde{J_k})_w.$$

Since R is a Noetherian ring by [6, 2.4(4)], we have

$$(\widetilde{I_k})_w(\widetilde{J_k})_w \subseteq (\widetilde{I_k}\widetilde{J_k})_w.$$

Now since $(\widetilde{I_k}\widetilde{J_k})_w = (\widetilde{I_kJ_k})_w$, we have

$$\widetilde{x}\widetilde{y} \in (\widetilde{I_kJ_k})_w = Clos_{\widetilde{R}}(\widetilde{(fg)^{(k)}}).$$

So $xy \in Clos_R((fg)^{(k)}, M)$ by Proposition 2.3(a). This shows $f^{-(M)}g^{-(M)} \leq (fg)^{-(M)}$. $\square$

Theorem 2.7. (See [2, 2.5].) *Let $f = \{I_n\}_{n\geq 0}$ be a filtration of ideals on R and $0 \to M' \to M \to M'' \to 0$ be an exact sequence of $R-$modules. Then*

$$Clos_R(f, M) = Clos_R(f, M') \cap Clos_R(f, M'').$$

Remark 2.8. Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of $R-$modules and $f = \{I_n\}_{n\geq 0}$ be a filtration of ideals on R. By Theorem 2.7, we have

$$Clos_R(f^{(k)}, M) = Clos_R(f^{(k)}, M') \cap Clos_R(f^{(k)}, M'')$$

for every $k \geq 0$. This shows

$$f^{-(M)} = f^{-(M')} \cap f^{-(M'')}.$$

Remark 2.9. Let $\phi : R \to S$ be a ring homomorphism and $f = \{I_n\}_{n\geq 0}$ be a filtration of ideals on S. Then $\{\phi^{-1}(I_n)\}_{n\geq 0}$ is a filtration of ideals on R. This filtration is denoted by $\phi^{-1}(f)$.

Theorem 2.10. *Let $\phi : R \to S$ be a ring epimorphism and $f = \{I_n\}_{n\geq 0}$ be a filtration of ideals on S. Then for every $S-$module M we have*

$$Clos_S(f, M) = \phi(Clos_R(\phi^{-1}(f), M)).$$

Proof. First we note, M can be an $R-$module if we define

$$r\alpha = \phi(r)\alpha \qquad \forall r \in R, \forall \alpha \in M.$$

Let $y = \phi(x) \in Clos_S(f, M)$. Since $y \in Clos_S(f, M)$, there exists a positive integer m such that

$$y^m + s_1y^{m-1} + \cdots + s_m \in (0 :_S M),$$

where $s_i \in I_i$ for every $1 \leq i \leq m$. Let $s_i = \phi(r_i)$ for every $1 \leq i \leq m$. Then $r_i \in \phi^{-1}(I_i)$ for every $1 \leq i \leq m$ and we have

$$x^m + r_1 x^{m-1} + \cdots + r_m \in (0:_R M).$$

This shows $x \in Clos_R(\phi^{-1}(f), M)$ and so $y = \phi(x) \in \phi(Clos_R(\phi^{-1}(f), M))$. Then $Clos_S(f, M) \subseteq \phi(Clos_R(\phi^{-1}(f), M))$.

For inverse inclusion, let $z \in Clos_R(\phi^{-1}(f), M)$. Then there exists a positive integer m such that

$$z^m + r_1 z^{m-1} + \cdots + r_m \in (0:_R M),$$

where $r_i \in \phi^{-1}(I_i)$ for every $1 \leq i \leq m$. Then

$$\phi(z)^m + \phi(r_1)\phi(z)^{m-1} + \cdots + \phi(r_m) \in (0:_S M),$$

where $\phi(r_i) \in I_i$ for every $1 \leq i \leq m$. This shows $\phi(z) \in Clos_S(f, M)$. Thus $\phi(Clos_R(\phi^{-1}(f), M)) \subseteq Clos_S(f, M)$ and this completes the proof. □

The following remark shows that Theorem 2.10, cannot be true when the homomorphism ϕ is not epic.

Remark 2.11. Let F be a filed and $F[t]$ be the polynomial ring in one indeterminate t. For every $\lambda_0 + \lambda_1 t + \cdots + \lambda_n t^n \in F[t]$ and for every $\alpha \in F$, we consider the following scalar multiplication defined by

$$(\lambda_0 + \lambda_1 t + \cdots + \lambda_n t^n).\alpha = (\lambda_0 + \lambda_1 + \cdots + \lambda_n)\alpha.$$

It is easy to see that, F is an $F[t]-$module. Let the map $\phi : F \to F[t]$ be defined by $\phi(\alpha) = \alpha$ for every $\alpha \in F$. We know ϕ is a homomorphism but ϕ is not epic. Let $I = (t)$ be the principal ideal of $F[t]$ generated by t. Then $f = \{I^n\}_{n\geq 0}$ is the I-adic filtration on $F[t]$. Now it is easy to see that $Clos_F(\phi^{-1}(f), F) = 0$ and so $\phi(Clos_F(\phi^{-1}(f), F)) = 0$. Now we note

$$(0:_{F[t]} F) = \{\lambda_0 + \lambda_1 t + \cdots + \lambda_n t^n \in F[t] : \lambda_0 + \lambda_1 + \cdots + \lambda_n = 0\}.$$

If $a_1 = -t \in I$, $a_2 = t^2 \in I^2$ and $a_3 = -t^3 \in I^3$ then

$$1^3 + a_1 1^2 + a_2 1 + a_3 \in (0:_{F[t]} F).$$

This shows $1 \in Clos_{F[t]}(f, F)$ and so $Clos_{F[t]}(f, F) = F[t]$.

3 Main Results

Definition 3.1. (See [6, 3.1(2)].) Let $f = \{I_n\}_{n\geq 0}$ be a filtration on a Noetherian ring R. Then every element of

$$A^{-}(f) = \{P : P \in Ass_R(R/Clos_R(f^{(k)}))\ for\ some\ k \geq 1\}$$

are called the asymptotic prime divisors of f.

Definition 3.2. Let $f = \{I_n\}_{n\geq 0}$ be a filtration on a Noetherian ring R and M be an $R-$module. Then every element of

$$A^{-}(f, M) = \{P : P \in Ass_R(R/Clos_R(f^{(k)}, M))\ for\ some\ k \geq 1\}$$

are called the asymptotic prime divisors of f relative to M.

Remark 3.3. Let $f = \{I_n\}_{n\geq 0}$ be a Noetherian filtration of ideals on a Noetherian ring R and let M be an $R-$module. Then $A^{-}(f, M)$ is a finite set.

Proof. It is clear by [6, 3.3(2)]. Because $P \in A^{-}(f, M)$ if and only if $\widetilde{P} \in A^{-}(\widetilde{f})$. □

Theorem 3.4. *Let $f = \{I_n\}_{n\geq 0}$ be a filtration of ideals on a Noetherian ring R and $0 \to M' \to M \to M'' \to 0$ be an exact sequence of $R-$modules. Then*

(a) $A^{-}(f, M) \subseteq A^{-}(f, M') \cup A^{-}(f, M'')$;

(b) the inclusion $A^{-}(f, M') \subseteq A^{-}(f, M)$ holds, if M' is one of the following type of non-zero module:

(i) M' is a projective module;

(ii) M' is a divisible module;

(iii) M' is an injective module over an integral domain;

(iv) M' is a finitely generated torsion free module.

Proof. (a) Let $P \in A^{-}(f, M)$. Then there exists an $x \in R$ such that $P = Ann_R(x + Clos_R(f^{(k)}, M))$ for some $k \geq 1$. Then for every $a \in P$, we have $ax \in Clos_R(f^{(k)}, M)$. But by Theorem 2.7, we have

$$Clos_R(f^{(k)}, M) = Clos_R(f^{(k)}, M') \cap Clos_R(f^{(k)}, M'')\ \forall k \geq 0.$$

This shows $ax \in Clos_R(f^{(k)}, M')$ and $ax \in Clos_R(f^{(k)}, M'')$. Then $a \in Ann_R(x+Clos_R(f^{(k)}, M'))$ and $a \in Ann_R(x+Clos_R(f^{(k)}, M''))$. So $P \subseteq Ann_R(x + Clos_R(f^{(k)}, M'))$ and $P \subseteq Ann_R(x + Clos_R(f^{(k)}, M''))$. Now we will show

$$Ann_R(x + Clos_R(f^{(k)}, M')) \subseteq P \quad \text{or} \quad Ann_R(x + Clos_R(f^{(k)}, M'')) \subseteq P.$$

Let $r_1 \in Ann_R(x+Clos_R(f^{(k)}, M'))$ and $r_2 \in Ann_R(x+Clos_R(f^{(k)}, M''))$ such that $r_1, r_2 \notin P$. This shows

$$(r_1r_2)x \in Clos_R(f^{(k)}, M') \cap Clos_R(f^{(k)}, M'').$$

Thus by Theorem 2.7, we have $(r_1r_2)x \in Clos_R(f^{(k)}, M)$ and so $r_1r_2 \in P$. Since P is a prime ideal, this is a contradiction. Then $Ann_R(x + Clos_R(f^{(k)}, M')) \subseteq P$ or $Ann_R(x + Clos_R(f^{(k)}, M'')) \subseteq P$. Then

$$P = Ann_R(x + Clos_R(f^{(k)}, M')) \quad \text{or} \quad P = Ann_R(x + Clos_R(f^{(k)}, M'')).$$

and this completes the proof.

(b) Let $P \in A^-(f, M')$. Then there exists an $x \in R$ such that $P = Ann_R(x + Clos_R(f^{(k)}, M'))$. Let $r \in Ann_R(x + Clos_R(f^{(k)}, M))$. But by Theorem 2.7, we have

$$rx \in Clos_R(f^{(k)}, M) \subseteq Clos_R(f^{(k)}, M').$$

Then $r \in P$ and so $Ann_R(x + Clos_R(f^{(k)}, M)) \subseteq P$. Now let $a \in P$. Then $ax \in Clos_R(f^{(k)}, M')$. We know from [2, 2.4], $Clos_R(f^{(k)}, M') = Clos_R(f^{(k)})$ if the non-zero module M' has one of the conditions (i) to (iv). Since $Clos_R(f^{(k)}) \subseteq Clos_R(f^{(k)}, M)$, we have $a \in Ann_R(x + Clos_R(f^{(k)}, M))$. Then $P = Ann_R(x + Clos_R(f^{(k)}, M))$ and so $P \in A^-(f, M)$. □

In the rest of this section, we assume that R and S are Noetherian rings.

Theorem 3.5. *Let $\phi : R \to S$ be a ring epimorphism and $f = \{I_n\}_{n\geq 0}$ be a filtration of ideals on S. Let M be an $S-$module.*

(a) If $P \in A^-(f, M)$, then $\phi^{-1}(P) \in A^-(\phi^{-1}(f), M)$.

(b) If $Q \in A^-(\phi^{-1}(f), M)$, then $\phi(Q) \in A^-(f, M)$.

Proof. (a) First we note that, if $P \in Spec(S)$, then $\phi^{-1}(P) \in Spec(R)$. Also we note that $Ker(\phi) \subseteq Clos_R(\phi^{-1}(f^{(k)}), M)$ for every $k \geq 0$. Let $P \in A^-(f, M)$. Then for some $k \geq 1$, there exists $\overline{y} = y + Clos_S(f^{(k)}, M) \in S/Clos_S(f^{(k)}, M)$ such that $P = Ann_S(\overline{y})$. Let $y = \phi(x)$ and $\overline{x} = x + Clos_R(\phi^{-1}(f^{(k)}), M) \in R/Clos_R(\phi^{-1}(f^{(k)}), M)$. If we show $\phi^{-1}(P) = Ann_R(\overline{x})$ then $\phi^{-1}(P) \in A^-(\phi^{-1}(f), M)$.

Let $a \in \phi^{-1}(P)$. Thus $\phi(a) \in P$ and so $\phi(a)y \in Clos_S(f^{(k)}, M)$. Now by Theorem 2.10, we have

$$\begin{aligned} ax \in \phi^{-1}(Clos_S(f^{(k)}, M)) &= \phi^{-1}(\phi(Clos_R(\phi^{-1}(f^{(k)}), M))) \\ &= Clos_R(\phi^{-1}(f^{(k)}), M) + ker(\phi) \\ &= Clos_R(\phi^{-1}(f^{(k)}), M). \end{aligned}$$

Hence $a \in Ann_R(\overline{x})$ and so $\phi^{-1}(P) \subseteq Ann_R(\overline{x})$. Now let $b \in Ann_R(\overline{x})$. Then $bx \in Clos_R(\phi^{-1}(f^{(k)}), M)$. By Theorem 2.10, we can see

$$\phi(b)y = \phi(bx) \in \phi(Clos_R(\phi^{-1}(f^{(k)}), M)) = Clos_S(f^{(k)}, M).$$

Thus $\phi(b) \in Ann_S(\overline{y}) = P$ and so $b \in \phi^{-1}(P)$. This shows $Ann_R(\overline{x}) \subseteq \phi^{-1}(P)$. Then $\phi^{-1}(P) = Ann_R(\overline{x})$ and so $\phi^{-1}(P) \in A^-(\phi^{-1}(f), M)$.

(b) Let $Q \in A^-(\phi^{-1}(f), M)$. Then there exists an $x \in R$ such that $Q = Ann_R(x + Clos_R(\phi^{-1}(f^{(k)}), M))$ for some $k \geq 1$. Since ϕ is an epimorphism and

$$Ker(\phi) \subseteq Clos_R(\phi^{-1}(f^{(k)}), M) \subseteq Ann_R(x+Clos_R(\phi^{-1}(f^{(k)}), M)) = Q,$$

we can see $\phi(Q) \neq S$ and $\phi(Q) \in Spec(S)$. Now we will prove $\phi(Q) \in A^-(f, M)$. It is enough to show that $\phi(Q) = Ann_S(\phi(x)+Clos_S(f^{(k)}, M))$. Let $a \in Q$. Then $ax \in Clos_R(\phi^{-1}(f^{(k)}), M)$ and so by Theorem 2.10,

$$\phi(a)\phi(x) = \phi(ax) \in \phi(Clos_R(\phi^{-1}(f^{(k)}), M)) = Clos_S(f^{(k)}, M).$$

Hence $\phi(a) \in Ann_S(\phi(x) + Clos_S(f^{(k)}, M))$ and this shows $\phi(Q) \subseteq Ann_S(\phi(x)+Clos_S(f^{(k)}, M))$. Now let $d \in Ann_S(\phi(x)+Clos_S(f^{(k)}, M))$. Since ϕ is an epimorphism, there exists a $c \in R$ such that $d = \phi(c)$. Since

$$\phi(cx) = \phi(c)\phi(x) \in Clos_S(f^{(k)}, M) = \phi(Clos_R(\phi^{-1}(f^{(k)}), M))$$

we have

$$cx \in Clos_R(\phi^{-1}(f^{(k)}), M) + Ker(\phi).$$

Since $Ker(\phi) \subseteq Clos_R(\phi^{-1}(f^{(k)}), M)$,

$$c \in Ann_R(x + Clos_R(\phi^{-1}(f^{(k)}), M)) = Q.$$

Then $d = \phi(c) \in \phi(Q)$. This shows $Ann_S(\phi(x)+Clos_S(f^{(k)}, M)) \subseteq \phi(Q)$ and so the proof is completed. □

Remark 3.6. Let $\phi : R \to S$ be a ring homorphism. For every ideal I of R the ideal generated by $\phi(I)$ is denoted by I^e. Also for every ideal J of S the ideal $\phi^{-1}(J)$ is denoted by J^c. The ideals I^e and J^c are respectively called the extension of I under ϕ and the contraction of J under ϕ.

Corollary 3.7. *Let $\phi : R \to S$ be a ring epimorphism and $f = \{I_n\}_{n\geq 0}$ be a filtration of ideals on S. Let M be an $S-$module. Then*

$$A^-(\phi^{-1}(f), M) = \{P^c : P \in A^-(f, M)\}.$$

Proof. Let $Q \in A^-(\phi^{-1}(f), M)$. Then there exists an $x \in R$ such that $Q = Ann_R(x + Clos_R(\phi^{-1}(f^{(k)}), M))$ for some $k \geq 1$. Since ϕ is an epimorphism, $Q^e = \phi(Q)$. We know $\phi^{-1}(\phi(Q)) = Q + Ker(\phi)$. This shows $Q^{ec} = Q + Ker(\phi)$. But

$$Ker(\phi) \subseteq Clos_R(\phi^{-1}(f^{(k)}), M) \subseteq Q,$$

and so $Q^{ec} = Q$. By Theorem 3.5, $Q^e = \phi(Q) \in A^-(f, M)$ and so $Q \in \{P^c : P \in A^-(f, M)\}$. So far, we have proved that

$$A^-(\phi^{-1}(f), M) \subseteq \{P^c : P \in A^-(f, M)\}.$$

The converse inclusion immediately follows from Theorem 3.5. Thus we have

$$A^-(\phi^{-1}(f), M) = \{P^c : P \in A^-(f, M)\}.$$

□

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