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An Inertial Type Subgradient Extragradient Method for Common Fixed Point and Variational Inequalities Problems in Real Banach Space

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Abstract. In this paper, we introduce a new inertial type algorithm with self - adaptive step - size technique for approximating a common element of the set of solutions of pseudomonotone variational inequality problem and the set of common fixed point of a finite family of generic generalized nonspreading mappings in uniformly smooth and 2 - uniformly convex real Banach space. Furthermore, we prove a strong convergence theorem of our algorithm without prior knowledge of the Lipschitz constant of the operator under some mild assumptions. We also give numerical examples to illustrate the performance of our algorithm. Our result generalize and improve many existing results in the literature.

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1 Introduction

Let E be a real Banach space and E^* be its dual space. Let C be a nonempty, closed and convex subset of E, and $A : C \to E^*$ be a mapping. The problem of finding a point $x^* \in C$ such that

$$\langle Ax^*, y - x^* \rangle \geq 0, \quad \forall\, y \in C, \tag{1}$$

is called a *variational inequality problem.* The set of solutions of variational inequality problem (1) is denoted by $VI(C, A)$. The study of variational inequality problem originates from solving minimization problems involving infinite-dimensional functions and calculus of variation. As an analytical application of mechanics to the solution of partial differential equations in infinite-dimensional spaces (see, for example, [44, 36] and references therein). Hartman and Stampacchia [18] initiated the systematic study of the theory of variational inequality problem in 1966. Later in 1967 Lions and Stampacchia [30] studied the existence and uniqueness of the solution. Since then, the theory of variational inequality problem has received much attention due to its wide applications in various areas of pure and applied sciences, such as optimal control, image recovery, resource allocations, networking, transportation, signal processing and so on (see, for example, [31, 23, 14, 5] and references therein). The constraints can clearly be expressed as variational inequality problems and (or) as fixed point problems. Consequently, the problem of finding common elements of the set of solutions of variational inequality problems and the set of fixed points of nonlinear operators has become an interesting area of research for many researchers working in the area of nonlinear operator theory (see, for example, [8, 33, 34, 19] and the references contained in them).

Many researchers in their quest to find solutions of variational inequality problems have proposed and analyzed various iterative approximation methods (see for example, [22, 9, 48]). A number of results on iterative methods proposed for approximating solutions of variational inequality problems are studied such that the operator A was often considered to be inverse strongly monotone (see, for instance [17, 28] and references therein). In order to relax the inverse strongly monotone condition imposed on the operator A, Korpelevich [26] proposed the following extragradient method in a finite dimensional Euclidean space $\mathbb{R}^n$:

$$\begin{cases} x_1 = x \in C \\ y_n = P_C(x_n - \lambda A(x_n)), \\ x_{n+1} = P_C(x_n - \lambda A(y_n)) \quad \forall n \geq 0, \end{cases} \tag{2}$$

where $\lambda \in (0, \frac{1}{L})$, A is monotone and Lipschitz and P_C is the metric projection onto C. They proved that the sequence $\{x_n\}$ generated by algorithm (2) converges weakly to a solution of problem (1). However, it is notice that the extragradient method require the computation at each step of the iteration process two projections onto a closed and convex subset C of H. This might affect the efficiency of the extragradient method if the feasible set is not simple enough which might also increase the computational cost. In order to overcome this drawback, Several modifications of the extragradient method were proposed (see, for example [11, 10, 25, 20, 50, 49] and references therein) for solving variational inequality problem (1). In particular, Tseng [50] proposed the following Tseng's extragradient method

$$\begin{cases} x_1 = x \in C \\ y_n = P_C(x_n - \lambda A(x_n)), \\ x_{n+1} = y_n - \lambda(A(y_n) - A(x_n)) \quad \forall n \geq 0, \end{cases} \tag{3}$$

where $\lambda \in (0, \frac{1}{L})$, A is monotone and Lipschitz and P_C is the metric projection onto C. They proved that the sequence $\{x_n\}$ generated by algorithm (3) converges weakly to a solution of problem (1) in a real Hilbert space. Another modification of the extragradient method was proposed by Censor et al. [11] as follows:

$$\begin{cases} x_0 \in H, \\ y_n = P_C(x_n - \lambda A(x_n)), \\ T_n = \{z \in H : \langle x_n - \lambda A(x_n) - y_n, z - y_n \rangle \leq 0\}, \\ x_{n+1} = P_{T_n}(x_n - \lambda A(y_n)), \quad \forall \ n \geq 0. \end{cases} \tag{4}$$

They modified the extragradient method (2) by replacing the second projection onto a closed and convex subset C with a projection onto the

half space T_n. Algorithm (4) is therefore called subgradient extragradient method. Observe that, the set T_n is a half space, thus algorithm (4) is simpler to implement than algorithm (2). They proved that the sequence $\{x_n\}$ generated by algorithm (4) converges weakly to a solution of problem (1) in a real Hilbert space under some mild assumptions.
Recently, Kraikaew and Saejung [27] in order to obtain strong convergence, combined the subgradient extragradient method (4) with Halpern method and thus proposed the following iterative algorithm:

$$\begin{cases} x_0 \in H, \\ y_n = P_C(x_n - \lambda A(x_n)), \\ T_n = \{z \in H : \langle x_n - \lambda A(x_n) - y_n, z - y_n \rangle \leq 0\}, \\ z_n = \alpha_n x_0 + (1-\alpha_n) P_{T_n}(x_n - \lambda A(y_n)), \\ x_{n+1} = \beta_n x_n + (1-\beta_n) S z_n, \qquad \forall \ n \geq 0, \end{cases} \tag{5}$$

where $\beta_n \subset [a, b] \in (0, 1)$, for some $a, b \in (0, 1)$ and $\{\alpha_n\}$ is a sequence in $[0, 1]$ satisfying $\underset{n\to\infty}{lim} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. They proved that the sequence $\{x_n\}$ generated by algorithm (5) converges strongly to a point $x^* \in VI(C, A) \cap F(S)$ in a real Hilbert space under some mild assumptions.
Rezapour and Zakeri [41] see also [42] studied the problem of finding a common element of the set of fixed points of some nonlinear mappings, the set of solutions to a variational inclusion problems and the set of solutions of some generalized equilibrium problem in a real Hilbert space. Still in the setting of Hilbrt space, Rezapour et al. [43] studied an extragradient methods for solving split feasibility problems, generalized equilibrium problems, and fixed point problems involving some nonlinear operators.
Chidume et al. [13] proposed the following Krasnoselskii type algorithm in a uniformly smooth, 2 - uniformly convex real Banach space for approximating common element of solutions of a variational inequality problem and common fixed point of a countable family of relatively

nonexpansive mappings as:

$$\begin{cases} x_0 = x \in E, \\ y_n = \Pi_C J^{-1}(Jx_n - \lambda A(x_n)), \\ T_n = \{z \in E : \langle Jx_n - \lambda A(x_n) - Jy_n, z - y_n \rangle \leq 0\}, \\ t_n = \Pi_{T_n} J^{-1}(Jx_n - \lambda A(y_n)), \\ z_n = J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)Jt_n), \\ x_{n+1} = J^{-1}(\lambda Jx_n + (1-\lambda)J(Sz_n)), \quad \forall \ n \geq 0, \end{cases} \tag{6}$$

where A is monotone, $\lambda \in (0,1)$ such that $\lambda < \frac{\alpha}{K}$, α is a constant, K is Lipschitz constant and $\{\alpha_n\}$ is a sequence in $[0,1]$ satisfying $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. They proved that the sequence $\{x_n\}$ generated by algorithm (6) converges strongly to a point $x^* = \Pi_{F(S)\bigcap VI(C,A)} x_0$ under some mild assumptions.

Observe that all the methods mentioned above require a prior knowledge of the Lipschitz constant of the operator A as input parameter which is very difficult to estimate when solving some practical problems.

Ma [32] introduced a new subgradient extragradient method with a self adaptive step size for solving monotone variational inequality problems in Banach space without prior knowledge of Lipschitz constant of the operator. They proved that the sequence $\{x_n\}$ generated by the proposed algorithm converges strongly to a point $\hat{x} = \Pi_{VI(C,A)} Jx_0$ under some mild assumptions.

Algorithm A

(Step 0) Take $\lambda_0 > 0, x_0 \in E$ be a given starting point, $\mu \in (0,1)$.

(Step 1) Given the current iterate x_n, compute

$y_n = \Pi_C(Jx_n - \lambda_n A(x_n))$,

If $x_n = y_n$, then stop: x_n is a solution. Otherwise, go to step 2.

(Step 2) Construct the set $T_n = \{w \in E : \langle Jx_n - \lambda_n A(x_n) - Jy_n, w - y_n\rangle \leq 0\}$, and compute

$z_n = \Pi_{T_n}(Jx_n - \lambda_n A(y_n)), \qquad x_{n+1} = J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)Jz_n),$

(Step 3) compute

$$\lambda_{n+1} = \begin{cases} \min\{\frac{\mu(||x_n - y_n||^2 + ||y_n - z_n||^2)}{2\langle A(x_n) - A(y_n), z_n - y_n\rangle}, \lambda_n\}, \ if \ \langle A(x_n) - A(y_n), z_n - y_n\rangle > 0, \\ \lambda_n, \qquad otherwise, \end{cases}$$

Set $n := n + 1$ and return to step 1.
However, in 1964 Polyak [40] introduced the technique of inertial extrapolation process as a means of speeding up the rate of convergence of iterative methods.
Many researchers have proposed and analyzed a large number of inertial type iterative schemes (see, for example [16, 48, 36] and references therein).
Very recently, Tan and Qin [47] introduced the following Viscosity type inertial subgradient extragradient algorithm for solving monotone variational inequality problem in real Hilbert spaces. They proved that the sequence $\{x_n\}$ generated by the proposed algorithm converges strongly to an element $x^* = P_{VI(C,A)}f(x^*)$ under some mild assumptions.
Algorithm B
Initialization: Set $\tau > 0, \lambda_1 > 0, \mu \in (0,1)$. Choose a nonnegative real sequence $\{\alpha_n\}$ such that $\sum_{n=1}^{\infty} \alpha_n < +\infty$. Let $x_0, x_1 \in H$ be arbitrarily chosen.
Iterative steps: Calculate x_{n+1} as follows:

$$\begin{cases} u_n = x_n + \tau_n(x_n - x_{n-1}), \\ y_n = P_C(u_n - \lambda_n A(u_n)), \\ T_n = \{z \in H : \langle u_n - \lambda_n A(u_n) - y_n, z - y_n\rangle \leq 0\}, \\ z_n = P_{T_n}(u_n - \lambda_n A(y_n)), \\ x_{n+1} = \psi_n f(x_n) + (1 - \psi_n) z_n, \quad \forall \ n \geq 1, \end{cases} \tag{7}$$

$$\tau_n = \begin{cases} \min\{\frac{\theta_n}{||x_n - x_{n-1}||}, \tau\} & , if \quad x_n \neq x_{n-1}, \\ \tau, & otherwise, \end{cases} \tag{8}$$

$$\lambda_{n+1} = \begin{cases} \min\{\frac{\mu||u_n - y_n||}{||Au_n - Ay_n||}, \lambda_n + \alpha_n\} & , if \quad Au_n - Ay_n \neq 0, \\ \lambda_n + \alpha_n, & otherwise, \end{cases} \tag{9}$$

where $\{\tau_n\}$ and $\{\lambda_n\}$ are updated by (8) and (9) respectively.
On the other hand, Vuong [51] used the extragradient method (2) to solve pseudomonotone variational inequalities problems in Hilbert spaces, and thus, proved that the sequence $\{x_n\}$ generated by algorithm (2) converges weakly to a solution of problem (1).

Motivated by the above works, in this paper, we introduce a new inertial type subgradient extragradient algorithm with self adaptive step size technique for approximating common element in the set of solutions of pseudomonotone variational inequality problem and the set of common fixed point of a finite family of generic generalized nonspreading mappings in uniformly smooth and 2 - uniformly convex Banach space. Furthermore, we prove a strong convergence of our algorithm to a solution of the stated problem without prior knowledge of the Lipschitz constant of the operator under some mild assumptions. Our result generalize and extend many existing results in the literature. We give some numerical examples in order to illustrates the performance of our proposed algorithm. Our result generalize and extend many existing results in the literature such as those in [27, 13, 41, 42, 43].

2 Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and $\mathbb{R}$ the set of real numbers. Let E be a real Banach space with norm $||.||$, and let E^* be the dual space of E, and $S_E = \{x \in E : ||x|| = 1\}$ is the unit sphere of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^* \rangle$. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus δ of convexity of E is the function $\delta_E : (0, 2] \longrightarrow [0, 1]$ defined by

$$\delta_E(\epsilon) = inf\{1 - \frac{||x + y||}{2} : ||x|| \leq 1, ||y|| \leq 1, ||x - y|| \geq \epsilon\}.$$

E is uniformly convex if and only if $\delta_E(\epsilon) > 0$ for every $\epsilon \in (0, 2]$. The space E is said to be 2-uniformly convex if there exists a constant $c > 0$ such that $\delta_E(\epsilon) \geq c\epsilon^2$ for every $\epsilon \in (0, 2]$. Observe that every 2-uniformly convex Banach space is uniformly convex.
A Banach space E is called smooth if the limit

$$\lim_{t \to 0} \frac{||x + ty|| - ||x||}{t}$$

exists for all $x, y \in S_E$ and for any $\lambda \in (0, 1)$, if $||\lambda x + (1 - \lambda)y|| < 1$ for all $x, y \in S_E$ with $x \neq y$, then E is said to be strictly convex.

The modulus of smoothness of E is the function $\rho_E : [0,\infty) \longrightarrow [0,\infty)$ defined by

$$\rho_E(\tau) = \sup\{\frac{||x+\tau y|| - ||x-\tau y||}{2} - 1 : x, y \in S_E\}.$$

E is called uniformly smooth if the $\lim\limits_{\tau\to 0}\frac{\rho_E(\tau)}{\tau} = 0$; q - uniformly smooth if there exists a positive constant C_q such that $\rho_E(\tau) \leq C_q(\tau)^q$ for any $\tau > 0$.

Observe that every q - uniformly smooth Banach space is uniformly smooth. Also, every uniformly convex Banach space is strictly convex and reflexive. Typical examples of such spaces, (see, for example Chidume [12], pp. 34, 54) are L_p, l_p and W_p^m which are q - uniformly smooth for $1 \leq q < 2$; 2 - uniformly smooth and uniformly convex (see, for instance [52]). The normalized duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^*\rangle = ||x||.||x^*||, ||x^*|| = ||x||\}$$

for all $x \in E$.

Remark 2.1. *Observe that the normalized duality mapping J has the following basic properties (see, for more details [12]):*

(T1) If E is smooth Banach space, then J is single - valued mapping from E into E^*;

(T2) If E is strictly convex Banach space, then J is one to one;

(T3) If E is uniformly smooth Banach space, then J is uniformly norm to norm continuous on each bounded subset of E;

(T4) If E is reflexive Banach space, then J is surjective;

(T5) If E is reflexive, smooth and strictly convex Banach space with dual E^* and $J^* : E^* \longrightarrow E$ is the normalized duality mapping in E^*, then $J^* = J^{-1}$;

(T6) If E is reflexive, smooth and strictly convex Banach space, then the normalized duality mapping J is single - valued, one to one and onto.

A Banach space E is uniformly smooth if and only if E^* is uniformly convex.

Let E be a reflexive, smooth and strictly convex Banach space and C be a nonempty, closed and convex subset of E (see, for more details [3]).
A mapping $\phi : E \times E \longrightarrow [0, \infty)$ defined by

$$\phi(x, y) = ||x||^2 - 2\langle x, Jy\rangle + ||y||^2, \quad \forall \; x, y \in E, \tag{10}$$

Observe that in a Hilbert space H, $\phi(x, y) = ||x - y||^2, \quad \forall \; x, y \in H$.
Obviously, the functional ϕ satisfies the following properties (see, for more details [3]).

$$(||x|| - ||y||)^2 \quad \leq \quad \phi(x, y) \leq (||x|| + ||y||)^2, \quad \forall \; x, y \in E; \tag{11}$$

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy\rangle, \quad \forall \; x, y, z \in E; \tag{12}$$

$$\phi(x, y) = \langle x, Jx-Jy\rangle + \langle y-x, Jy\rangle \leq ||x||||Jx-Jy|| + ||y-x||||y||, \forall x, y \in E; \tag{13}$$

$$\phi(z, J^{-1}(\alpha Jx + (1 - \alpha)Jy)) \leq \alpha\phi(z, x) + (1 - \alpha)\phi(z, y), \quad \forall \; x, y \in E, \tag{14}$$

and $\alpha \in (0, 1)$.

Remark 2.2. *Let E be a strictly convex Banach space, then for all $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$ (see, for example [38]).*

Define a functional $V : E \times E^* \longrightarrow [0, \infty)$ (see for example [3]) by

$$V(x, x^*) = ||x||^2 - 2\langle x, x^*\rangle + ||y||^2, \;\; \forall \, x \in E, \quad and \;\; x^* \in E^*. \tag{15}$$

The following relation is easily verified,

$$V(x, x^*) = \phi(x, J^{-1}(x^*)), \;\; \forall \, x \in E, \quad and \;\; x^* \in E^*. \tag{16}$$

Observe that the mapping g defined by fixing $x \in E$, and $g(x^*) = V(x, x^*)$ for all $x^* \in E^*$ is a continuous, convex function from E^* into $\mathbb{R}$.

Lemma 2.3. *[3] Let E be a strictly convex, reflexive and smooth Banach space, and let V be as defined in (15). Then*

$$V(x,x^*)+2\langle J^{-1}x^*-x,y^*\rangle \leq V(x,x^*+y^*) \;\; \forall\, x\in E, \;\; and \;\; x^*,y^*\in E^*. \tag{17}$$

Let E be a reflexive, strictly convex and smooth Banach space and C be a nonempty, closed and convex subset of E.
It is shown that, see Alber [3] for each $x \in E$, there exists a unique element $k\in C$ (written as $\Pi_C x$) such that

$$\phi(k,x)=\inf_{y\in C}\phi(y,x).$$

The mapping $\Pi_C : E \longrightarrow C$ defined by $\Pi_C x = k$, is called generalized projection (see, for example [3]).
Note, if E is a Hilbert space H, then Π_C is a metric projection of H onto C.

Lemma 2.4. *(see for more details [2],[3]) Let E be a smooth, reflexive and strictly convex Banach space and C be a nonempty, closed and convex subset of E. Then the following inequalities hold:*

$$\phi(x,\Pi_C y)+\phi(\Pi_C y,y)\leq \phi(x,y), \quad \forall \;\; x\in C \;\; and \;\; y\in E; \tag{18}$$

$$If \quad x\in E \quad and \quad z\in C, \quad then \qquad z=\Pi_C x \Longleftrightarrow \langle z-y, Jx-Jz\rangle \geq 0, \tag{19}$$

for all $y\in C$.

Lemma 2.5. *[39] Let E be a uniformly smooth Banach space and $r>0$. Then, there exists a continuous, strictly increasing and convex function $g:[0,2r]\longrightarrow[0,\infty)$ with $g(0)=0$ and*

$$\phi(u,J^{-1}(tJx+(1-t)Jy))\leq t\phi(u,x)+(1-t)\phi(u,y)-t(1-t)g(||Jx-Jy||), \tag{20}$$

$\forall \;\; t\in[0,1], \;\; u\in E \;\; and \;\; x,y\in B_r(0)$ where $B_r(0):=\{h\in E: ||h||\leq r\}$.

Lemma 2.6. *[38] Let E be a uniformly convex and smooth Banach space and $\{\mu_n\}$ and $\{\lambda_n\}$ be two sequences in E. If $\lim\limits_{n\to\infty}\phi(\mu_n,\lambda_n)=0$ and either $\{\mu_n\}$ or $\{\lambda_n\}$ is bounded, then $\lim\limits_{n\to\infty}||\mu_n-\lambda_n||=0$.*

Lemma 2.7. *[6, 52] Let E be a 2 - uniformly convex Banach space. Then, there exists $\tau > 0$ such that*

$$\frac{1}{\tau}||x-y||^2 \leq \phi(x,y), \quad \forall \ x,y \in E. \tag{21}$$

Remark 2.8. *Without loss of generality, we may assume $\tau \in (0,1)$.*

Definition 2.9. *Let $T : C \to C$ be a mapping.*

1. A point $x \in C$ is called a fixed point of T if $Tx = x$, where $F(T) := \{x \in C : Tx = x\}$ is the set of fixed point of T.

2. A point $x \in C$ is said to be asymptotic fixed point of T, if there exists sequence $\{x_n\} \subseteq C$ such that $x_n \rightharpoonup x$ and $\underset{n\to\infty}{lim}||x_n - Tx_n|| = 0$. We denote the set of all asymptotic fixed point of T by $\hat{F}(T)$.

3. T is said to be quasi - ϕ - nonexpansive if $F(T) \neq \emptyset$, and
$$\phi(p,Tx) \leq \phi(p,x), \quad \forall \ x \in C \ \ and \ \ p \in F(T). \tag{22}$$

4. T is called nonspreading [24] if for all $x, y \in C$ and
$$\phi(Tx,Ty) + \phi(Ty,Tx) \leq \phi(Tx,y) + \phi(Ty,x). \tag{23}$$

5. T is called generalized nonspreading [4] for all $x, y \in C$ if there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that
$$\begin{aligned}\alpha\phi(Tx,Ty) &+ (1-\alpha)\phi(x,Ty) + \gamma\{\phi(Ty,Tx) - \phi(Ty,x)\}\\ &\leq \beta\phi(Tx,y) + (1-\beta)\phi(x,y) + \delta\{\phi(y,Tx) - \phi(y,x)\}.\end{aligned}$$

6. T is called generic $(\alpha, \beta, \gamma, \delta, \varepsilon, \xi)$ generalized nonspreading [46] if for all $x, y \in C$ the following inequalities holds: (i) $(\alpha+\beta+\gamma+\delta) \geq 0$; (ii) $(\alpha + \beta) > 0$; and (iii)
$$\begin{aligned}\alpha\phi(Tx,Ty) &+ \beta\phi(x,Ty) + \gamma\phi(Tx,y) + \delta\phi(x,y)\\ &\leq \varepsilon\{\phi(Ty,Tx) - \phi(Ty,x)\} + \xi\{\phi(y,Tx) - \phi(y,x)\}.\end{aligned} \tag{24}$$

Remark 2.10. *Observe that, (i) if $\alpha+\beta=-\gamma-\delta=1$, then generic $(\alpha,\beta,\gamma,\delta,\varepsilon,\xi)-$ generalized nonspreading mapping reduces to generalized nonspreading; (ii) if $\alpha=1,\beta=\delta=\xi=0$ and $\gamma=\varepsilon=-1$ then generic $(\alpha,\beta,\gamma,\delta,\varepsilon,\xi)-$ generalized nonspreading mapping reduces to nonspreading mapping; (iii) if $\alpha=1,\beta=\gamma=\xi=\varepsilon=0,\delta=-1$ then generic $(\alpha,\beta,\gamma,\delta,\varepsilon,\xi)-$ generalized nonspreading mapping reduces to nonexpansive mapping.*

7. T is called demiclosed at zero if for any sequence $\{x_n\}\subset C$ with $x_n\rightharpoonup x\in C$ and
$$||x_n-Tx_n||\longrightarrow 0 \quad as \quad n\longrightarrow\infty, \quad then \quad Tx=x.$$

Definition 2.11. *Let $A:C\to E^*$ be a mapping. Then A is said to be*

1. monotone if the following inequality hold
$$\langle Ax-Ay,x-y\rangle\geq 0, \quad \forall\, x,y\in C.$$
2. pseudomonotone if
$$\langle A(x),y-x\rangle\geq 0\Rightarrow\langle A(y),y-x\rangle\geq 0, \quad \forall\, x,y\in C.$$
3. Lipschitz continuous if there exists a constant $L>0$ such that
$$||Ax-Ay||\leq L||x-y||, \quad \forall\, x,y\in C.$$
4. weakly sequentially continuous if for any $\{x_n\}\subset C$ such that $x_n\rightharpoonup x$ implies $Ax_n\rightharpoonup Ax$.

Definition 2.12. [37, 29] *Let $A:C\longrightarrow E^*$ be an operator. The Minty variational inequality problem $(MVIP)$ consist of finding a point $x^*\in C$ such that*
$$\langle A(x^*),y-x^*\rangle\geq 0, \quad \forall\, y\in C. \tag{25}$$
The set of solutions of (25) is denoted by $M(C,A)$. Some existing results for the $(MVIP)$ have been presented in [29]. Also, the assumption that $M(C,A)\neq\emptyset$ has been used in solving the variational inequality problem $VI(C,A)$ in finite dimensional spaces (see, for example [45]).

Lemma 2.13. *[37] Consider the variational inequality problem* $VI(C, A)$. *Suppose the mapping* $h : [0, 1] \longrightarrow E^*$ *defined by* $h(t) = A(tx + (1 - t)y)$ *and* $t \in [0, 1]$ *is continuous for all* $x, y \in C$ *(i.e, h is hemicontinuous), then* $M(C, A) \subset VI(C, A)$. *Moreover, if A is pseudomonotone, then* $VI(C, A)$ *is closed, convex and* $VI(C, A) = M(C, A)$.

Lemma 2.14. *[53] If* $\{b_n\}$ *is a sequence of nonnegative real numbers satisfying the following inequality:*

$$b_{n+1} \leq (1 - \psi_n)b_n + \psi_n\sigma_n + \gamma_n, \quad n \geq 0,$$

where (i) $\{\psi_n\} \subset [0, 1]$, $\sum_{n=1}^{\infty} \psi_n = \infty$; *(ii)* $\limsup \sigma_n \leq 0$; *(iii)* $\gamma_n \geq 0$ *and* $\sum_{n=1}^{\infty} \gamma_n < \infty$, *then,* $b_n \longrightarrow 0$ *as* $n \longrightarrow \infty$.

Lemma 2.15. *[35] Let* $\{b_n\}$ *be a sequence of real numbers such that there exists a subsequence* $\{b_{n_i}\}$ *of* $\{b_n\}$ *such that* $b_{n_i} < b_{n_i+1}$ *for all* $i \in \mathbb{N}$. *Then, there exists a nondecreasing sequence* $\{m_k\} \subset \mathbb{N}$ *such that* $m_k \longrightarrow \infty$ *and the following properties are satisfied for all* $k \in \mathbb{N}$*;*

$$b_{m_k} \leq b_{m_k+1} \quad \text{and} \quad b_k \leq b_{m_k+1},$$
$$In \;\; fact, \quad m_k = \max\{j \leq k : b_j < b_{j+1}\}.$$

Lemma 2.16. *[46] Let E be a strictly convex Banach space with a uniformly Gâteaux differentiable norm, let C be a nonempty, closed and convex subset of E and let T be a generic generalized nonspreading mapping of C into itself such that* $F(T)$ *is nonempty. Then* $\hat{F}(T) = F(T)$.

Lemma 2.17. *[38] Let E be a smooth and strictly convex Banach space, let C be a nonempty, closed and convex subset of E and let T be a quasi - nonexpansive mapping from C into itself such that* $F(T)$ *is nonempty. Assume that*

$$\phi(u, Ty) \leq \phi(u, y),$$

for all $u \in F(T)$ *and* $y \in C$. *Then* $F(T)$ *is closed and convex.*

Lemma 2.18. *[46] Let E be smooth and strictly convex Banach space and C be a nonempty, closed and convex subset of E. Let* $T : C \longrightarrow C$ *be generic* $(\alpha, \beta, \gamma, \delta, \varepsilon, \xi)$ *- generalized nonspreading mapping. Assume* $F(T) \neq \emptyset$, *then T is quasi - nonexpansive and hence* $F(T)$ *is closed and*

convex.
We give the proof of Lemma 2.18 for the sake of completeness.
Proof. *Since $T : C \longrightarrow C$ is generic $(\alpha, \beta, \gamma, \delta, \varepsilon, \xi)$ - generalized non-spreading mapping with $F(T) \neq \emptyset$, then for any $y \in C$, let $p \in F(T)$ and replace x with p in equation (24) of definition (2.9), we obtain*

$$\begin{aligned}\alpha\phi(Tx, Ty) + \beta\phi(x, Ty) + \gamma\phi(Tx, y) + \delta\phi(x, y) \\ \leq \varepsilon\{\phi(Ty, Tx) - \phi(Ty, x)\} + \xi\{\phi(y, Tx) - \phi(y, x)\}\end{aligned}$$

$$\begin{aligned}\alpha\phi(p, Ty) + \beta\phi(p, Ty) + \gamma\phi(p, y) + \delta\phi(p, y) \\ \leq \varepsilon\{\phi(Ty, p) - \phi(Ty, p)\} + \xi\{\phi(y, p) - \phi(y, p)\}\end{aligned}$$

$$\alpha\phi(p, Ty) + \beta\phi(p, Ty) + \gamma\phi(p, y) + \delta\phi(p, y) \leq 0.$$

This implies that

$$(\alpha + \beta)\phi(p, Ty) + (\gamma + \delta)\phi(p, y) \leq 0.$$

Thus

$$(\alpha + \beta)\phi(p, Ty) \leq -(\gamma + \delta)\phi(p, y).$$

Using conditions (i) and (ii) of definition (2.9) (6), we have

$$\begin{aligned}\phi(p, Ty) &\leq -\frac{(\gamma + \delta)}{(\alpha + \beta)}\phi(p, y) \\ &\leq \phi(p, y)\end{aligned}$$

$$\phi(p, Ty) \leq \phi(p, y). \tag{26}$$

Hence, T is quasi - nonexpansive and by Lemma 2.17, we have that $F(T)$ is closed and convex.

3 Main Results

In this section, we first establish an important Lemma and then prove a strong convergence theorem for finding a common element of the set of solutions of pseudomonotone variational inequality problem and common fixed point of a finite family of generic generalized nonspreading mappings in uniformly smooth and 2 - uniformly convex real Banach space. Furthermore, to obtain strong convergence of our algorithm, we make the following assumptions.
Assumption A

(A1) Let E be uniformly smooth and 2 - uniformly convex real Banach space and C be nonempty, closed and convex subset of E.

(A2) $A : E \longrightarrow E^*$ is pseudomonotone, L - Lipschitz continuous and weakly sequentially continuous on E.

(A3) For each $i \in \{1, 2, ..., M\}$, $\{T_i\}$ be a finite family of generic $(\alpha, \beta, \gamma, \delta, \varepsilon, \xi)$ - generalized nonspreading mapping of E into itself such that $\hat{F}(T_i) = F(T_i)$. Assume $\Omega = F(T_M \circ T_{M-1} \circ, ..., \circ T_1) = \bigcap_{i=1}^{M} F(T_i) \neq \emptyset$.

(A4) The solution set $\Gamma = VI(C, A) \bigcap \bigcap_{i=1}^{M} F(T_i) \neq \emptyset$.

Assumption B**,** we assume that the control sequences satisfy:

(B1) $\{\tau_n\}$ is a positive sequence such that $\lim\limits_{n\to\infty} \frac{\tau_n}{\alpha_n} = 0$.

(B2) $\{\alpha_n\} \subset (0, 1)$ satisfies $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

(B3) $\beta_n \in (a, b)$, where $0 < a < b < 1$.

Algorithm J
Initialization: Take $\lambda_1 > 0, \mu, \theta \in (0, 1)$. Select initial data $x_0, x_1 \in E$.
Given x_{n-1}, x_n and θ_n for each $n \geq 1$, choose θ_n such that $\theta_n \in [0, \hat{\theta}_n]$.

Iterative steps: Calculate x_{n+1} and λ_{n+1} as follows:

$$\begin{cases} u_n = J^{-1}(Jx_n + \theta_n(Jx_{n-1} - Jx_n)), \\ y_n = \Pi_C J^{-1}(Ju_n - \lambda_n A(u_n)), \\ S_n = \{z \in E : \langle Ju_n - \lambda_n A(u_n) - Jy_n, z - y_n\rangle \leq 0\}, \\ z_n = \Pi_{S_n} J^{-1}(Ju_n - \lambda_n A(y_n)), \\ w_n = J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)Jz_n), \\ x_{n+1} = J^{-1}(\beta_n Jz_n + (1-\beta_n)J(Tw_n)), \quad \forall \ n \geq 1, \end{cases} \tag{27}$$

where $T = T_M \circ T_{M-1}\circ, ..., \circ T_1$, $\hat{\theta}_n$ and λ_{n+1} are updated by (28) and (29) respectively.

$$\hat{\theta}_n = \begin{cases} \min\{\frac{\tau_n}{||x_n - x_{n-1}||}, \theta\}, & if \quad x_n \neq x_{n-1}, \\ \theta, & otherwise \end{cases} \tag{28}$$

$$\lambda_{n+1} = \begin{cases} \min\{\frac{\mu(||u_n - y_n||^2 + ||y_n - z_n||^2)}{2\langle A(u_n) - A(y_n), z_n - y_n\rangle}, \lambda_n\}, & if \ \langle A(u_n) - A(y_n), z_n - y_n\rangle > 0, \\ \lambda_n, & otherwise \end{cases} \tag{29}$$

In order to prove the strong convergence result of algorithm (27), we first prove the following lemma which plays an important role in the proof of the main result.

Lemma 3.1. *Suppose that $\{u_n\}$, $\{y_n\}$, $\{z_n\}$, $\{\lambda_n\}$ are sequences generated by algorithm (27) and assumptions (A1) - (A4) hold, then*

1. *If $u_n = y_n$ for some $n \geq 1$, then $u_n \in VI(C, A)$.*
2. *The sequence $\{\lambda_n\}$ generated by (29) is a nonincreasing sequence and $\lim\limits_{n\to\infty} \lambda_n = \lambda \geq \frac{\mu}{L}$.*

Proof. (1) *Suppose that $u_n = y_n$ for some $n \geq 1$. Then from algorithm (27), we have*

$$u_n = \Pi_C J^{-1}(Ju_n - \lambda_n A(u_n)).$$

Thus, $u_n \in C$. Using the definition of $\{y_n\}$ in algorithm (27) and the property of generalized projection Π_C onto C in equation (19) of Lemma 2.4, we have

$$\langle Ju_n - \lambda_n A(u_n) - Ju_n, u_n - y\rangle \geq 0, \quad \forall \, y \in C.$$

Thus,

$$\langle -\lambda_n A(u_n), u_n - y\rangle = \lambda_n \langle A(u_n), y - u_n\rangle \geq 0, \quad \forall\, y \in C.$$

Since $\lambda_n \geq 0$, *we obtain that* $\langle A(u_n), y - u_n\rangle \geq 0$. *Hence,* $u_n \in VI(C, A)$.
(**2**) *It follows from* (29) *that* $\lambda_{n+1} \leq \lambda_n$, *for all* $n \in \mathbb{N}$. *Furthermore, Since* A *is a Lipschitz continuous mapping with positive constant* L, *in a case where* $\langle A(u_n) - A(y_n), z_n - y_n\rangle > 0$, *we obtain*

$$\begin{aligned}\lambda_{n+1} = \frac{\mu(||u_n - y_n||^2 + ||y_n - z_n||^2)}{2\langle A(u_n) - A(y_n), z_n - y_n\rangle} &\geq \frac{2\mu||u_n - y_n||||y_n - z_n||}{2||A(u_n) - A(y_n)||||z_n - y_n||} \\ &\geq \frac{\mu||u_n - y_n||}{L||u_n - y_n||} \\ &= \frac{\mu}{L}.\end{aligned}$$

Since $\{\lambda_n\}$ *is a nonincreasing sequence which is bounded below by* $\min\{\frac{\mu}{L}, \lambda_1\}$, *we conclude that*

$$\lim_{n\to\infty} \lambda_n = \lambda \geq \frac{\mu}{L}.$$

Remark 3.2. *From definition* (28), *we have that*

$$\lim_{n\to\infty} \theta_n(\phi(x^*, x_{n-1}) - \phi(x^*, x_n)) = 0.$$

Proof. We have that $\theta_n||x_n - x_{n-1}|| \leq \tau_n$ for each $n \geq 1$, which together with $\lim_{n\to\infty} \frac{\tau_n}{\alpha_n} = 0$ implies

$$\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}||x_n - x_{n-1}|| \leq \lim_{n\to\infty} \frac{\tau_n}{\alpha_n} = 0. \tag{30}$$

Hence,

$$\begin{aligned}\phi(x^*, x_{n-1}) - \phi(x^*, x_n) &= ||x^*||^2 - 2\langle x^*, Jx_{n-1}\rangle + ||x_{n-1}||^2 \\ &\quad -(||x^*||^2 - 2\langle x^*, Jx_n\rangle + ||x_n||^2) \\ &= ||x_{n-1}||^2 - ||x_n||^2 + 2\langle x^*, Jx_n - Jx_{n-1}\rangle \\ &\leq ||x_{n-1} - x_n||(||x_n|| + ||x_{n-1}||) \\ &\quad +2||x^*||||Jx_{n-1} - Jx_n||.\end{aligned} \tag{31}$$

Since E is uniformly smooth, then J is norm to norm uniformly continuous on bounded subset of E, we obtain from (30) that

$$\lim_{n\to\infty} \alpha_n.\frac{\theta_n}{\alpha_n}||Jx_n - Jx_{n-1}|| = 0. \tag{32}$$

Thus,

$$\lim_{n\to\infty} \alpha_n(\frac{\theta_n}{\alpha_n}||x_{n-1}-x_n||(||x_n||+||x_{n-1}||)+2\frac{\theta_n}{\alpha_n}||x^*||||Jx_{n-1}-Jx_n||) = 0.$$

Hence,

$$\lim_{n\to\infty} \theta_n(\phi(x^*, x_{n-1}) - \phi(x^*, x_n)) = 0. \tag{33}$$

Theorem 3.3. *Suppose that assumptions* $(A1) - (A4)$ *hold, and the sequence* $\{\alpha_n\} \subset (0, 1)$ *satisfy* $\beta_n \in (a, b)$, *where* $0 < a < b < 1$, $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$. *Let* $\{x_n\}$ *be the sequence generated by algorithm* (27). *Then* $\{x_n\}$ *converges strongly to a solution* $\bar{x} = \Pi_{VI(C,A)\,\bigcap\,\bigcap_{i=1}^{M} F(T_i)} x_0$.

Proof. The proof is divided into two steps.
Step 1: Let $\{x_n\}$ be the sequence generated by (27). Then $\{x_n\}$ is bounded.
Let $x^* \in VI(C, A) \cap \bigcap_{i=1}^{M} F(T_i)$. Now, Observe that $y_n \in C$, then we have $\langle A(x^*), y_n - x^*\rangle \geq 0$, for all $n \in \mathbb{N}$.
Since A is pseudomonotone, we have $\langle A(y_n), y_n - x^*\rangle \geq 0$, for all $n \in \mathbb{N}$.
Then

$$0 \leq \langle A(y_n), y_n - x^* + z_n - z_n\rangle = \langle A(y_n), y_n - z_n\rangle - \langle A(y_n), x^* - z_n\rangle$$

which implies that

$$\langle A(y_n), y_n - z_n\rangle \geq \langle A(y_n), x^* - z_n\rangle, \qquad \forall\ n \in \mathbb{N}. \tag{34}$$

From the definition of S_n in algorithm (27) and the fact that $z_n \in S_n$, we have that

$$\langle Ju_n - \lambda_n A(u_n) - Jy_n, z_n - y_n\rangle \leq 0.$$

Thus,

$$\langle Ju_n - \lambda_n A(y_n) - Jy_n, z_n - y_n\rangle$$

$$
\begin{aligned}
&= \langle Ju_n - \lambda_n A(u_n) - Jy_n, z_n - y_n\rangle + \lambda_n\langle A(u_n) - A(y_n), z_n - y_n\rangle \\
&\leq \lambda_n\langle A(u_n) - A(y_n), z_n - y_n\rangle \qquad (35)
\end{aligned}
$$

Applying Lemma 2.4, we have

$$
\begin{aligned}
\phi(x^*, z_n) &= \phi(x^*, \Pi_{S_n} J^{-1}(Ju_n - \lambda_n A(y_n))) \\
&\leq \phi(x^*, J^{-1}(Ju_n - \lambda_n A(y_n))) - \phi(z_n, J^{-1}(Ju_n - \lambda_n A(y_n))) \\
&= ||x^*||^2 - 2\langle x^*, J(J^{-1}(Ju_n - \lambda_n A(y_n)))\rangle \\
&\quad + ||J^{-1}(Ju_n - \lambda_n A(y_n))||^2 - ||z_n||^2 \\
&\quad + 2\langle z_n, J(J^{-1}(Ju_n - \lambda_n A(y_n)))\rangle - ||J^{-1}(Ju_n - \lambda_n A(y_n))||^2 \\
&= ||x^*||^2 - 2\langle x^*, Ju_n\rangle + ||u_n||^2 - (||z_n||^2 - 2\langle z_n, Ju_n\rangle + ||u_n||^2) \\
&\quad + 2\lambda_n\langle A(y_n), x^* - z_n\rangle \\
&= \phi(x^*, u_n) - \phi(z_n, u_n) + 2\lambda_n\langle A(y_n), x^* - z_n\rangle
\end{aligned}
$$

Thus, from (12), (34), (35) and (29), we obtain

$$
\begin{aligned}
\phi(x^*, z_n) &\leq \phi(x^*, u_n) - \phi(z_n, u_n) + 2\lambda_n\langle A(y_n), y_n - z_n\rangle \\
&= \phi(x^*, u_n) - [\phi(u_n, y_n) + \phi(y_n, z_n) + 2\langle Ju_n - Jy_n, y_n - z_n\rangle] \\
&\quad + 2\lambda_n\langle A(y_n), y_n - z_n\rangle \\
&= \phi(x^*, u_n) - \phi(u_n, y_n) - \phi(y_n, z_n) \\
&\quad - 2\langle Ju_n - \lambda_n A(y_n) - Jy_n, y_n - z_n\rangle \\
&\leq \phi(x^*, u_n) - \phi(u_n, y_n) - \phi(y_n, z_n) \\
&\quad + 2\lambda_n\langle A(u_n) - A(y_n), z_n - y_n\rangle \\
&\leq \phi(x^*, u_n) - \phi(u_n, y_n) - \phi(y_n, z_n) \\
&\quad + \frac{\lambda_n\mu}{\lambda_{n+1}}(||u_n - y_n||^2 + ||y_n - z_n||^2) \qquad (36)
\end{aligned}
$$

From (36) and by Lemma 3.1 (2) and Lemma 2.7, we have

$$
\begin{aligned}
\phi(x^*, z_n) &\leq \phi(x^*, u_n) - \phi(u_n, y_n) - \phi(y_n, z_n) \\
&\quad + \frac{\lambda_n\mu}{\lambda_{n+1}}(\tau\phi(u_n, y_n) + \tau\phi(y_n, z_n)) \\
&= \phi(x^*, u_n) - [(1 - \frac{\lambda_n\mu\tau}{\lambda_{n+1}})((\phi(u_n, y_n) + \phi(y_n, z_n))] \\
&\leq \phi(x^*, u_n) - (1 - \mu\tau)(\phi(u_n, y_n) + \phi(y_n, z_n)) \qquad (37) \\
&\leq \phi(x^*, u_n) \qquad (38)
\end{aligned}
$$

Using the definition of $\{u_n\}$ in algorithm (27), we obtain

$$\begin{aligned}\phi(x^*, u_n) &= \phi(x^*, J^{-1}((1-\theta_n)Jx_n + \theta_n Jx_{n-1})) \\ &= ||x^*||^2 - 2\langle x^*, J(J^{-1}((1-\theta_n)Jx_n + \theta_n Jx_{n-1}))\rangle \\ &\quad +||J^{-1}((1-\theta_n)Jx_n + \theta_n Jx_{n-1})||^2 \\ &\leq (1-\theta_n)\phi(x^*, x_n) + \theta_n\phi(x^*, x_{n-1}) \end{aligned} \tag{39}$$

Applying the definition of $\{w_n\}$ in algorithm (27), we have

$$\begin{aligned}\phi(x^*, w_n) &= \phi(x^*, J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)Jz_n)) \\ &= ||x^*||^2 - 2\langle x^*, J(J^{-1}\alpha_n Jx_0 + (1-\alpha_n)Jz_n)\rangle \\ &\quad +||J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)Jz_n)||^2 \\ &\leq \alpha_n\phi(x^*, x_0) + (1-\alpha_n)\phi(x^*, z_n) \end{aligned} \tag{40}$$

Thus, using the definition of $\{x_{n+1}\}$ in algorithm (27) and Lemma 2.18, we have

$$\begin{aligned}\phi(x^*, x_{n+1}) &= \phi(x^*, J^{-1}(\beta_n Jz_n + (1-\beta_n)J(Tw_n))) \\ &= ||x^*||^2 - 2\langle x^*, J(J^{-1}\beta_n Jz_n + (1-\beta_n)J(Tw_n))\rangle \\ &\quad +||J^{-1}(\beta_n Jz_n + (1-\beta_n)J(Tw_n))||^2 \\ &\leq ||x^*||^2 - 2\beta_n\langle x^*, Jz_n\rangle - 2(1-\beta_n)\langle x^*, J(Tw_n)\rangle \\ &\quad +\beta_n||Jz_n||^2 + (1-\beta_n)||J(Tw_n)||^2 \\ &\leq \beta_n\phi(x^*, z_n) + (1-\beta_n)\phi(x^*, Tw_n) \\ &= \beta_n\phi(x^*, z_n) + (1-\beta_n)\phi(x^*, T_M \circ T_{M-1}\circ, ..., \circ T_1(w_n)) \\ &\leq \beta_n\phi(x^*, z_n) + (1-\beta_n)\phi(x^*, T_{M-1}\circ, ..., \circ T_1(w_n)) \\ &\leq \beta_n\phi(x^*, z_n) + (1-\beta_n)\phi(x^*, w_n) \end{aligned} \tag{41}$$

Substituting (40) and (39) into (41), we have

$$\begin{aligned}\phi(x^*, x_{n+1}) &\leq \beta_n\phi(x^*, z_n) + (1-\beta_n)[\alpha_n\phi(x^*, x_0) + (1-\alpha_n)\phi(x^*, z_n)] \\ &= \beta_n\phi(x^*, z_n) + \alpha_n(1-\beta_n)\phi(x^*, x_0) \\ &\quad +(1-\beta_n)(1-\alpha_n)\phi(x^*, z_n) \\ &= (\beta_n + (1-\beta_n)(1-\alpha_n))\phi(x^*, z_n) + \alpha_n(1-\beta_n)\phi(x^*, x_0) \\ &= (1-(1-\beta_n)\alpha_n)\phi(x^*, z_n) + (1-\beta_n)\alpha_n\phi(x^*, x_0) \\ &= (1-(1-\beta_n)\alpha_n)\phi(x^*, u_n) + (1-\beta_n)\alpha_n\phi(x^*, x_0) \end{aligned}$$

$$
\begin{aligned}
\phi(x^*, x_{n+1}) &\le (1-(1-\beta_n)\alpha_n)[(1-\theta_n)\phi(x^*, x_n) + \theta_n \phi(x^*, x_{n-1})] \\
&\quad +(1-\beta_n)\alpha_n \phi(x^*, x_0) \\
&\le \max\{\phi(x^*, x_0), \max\{\phi(x^*, x_n), \phi(x^*, x_{n-1})\}\} \\
&\vdots \\
&\le \max\{\phi(x^*, x_0), \max\{\phi(x^*, x_1), \phi(x^*, x_0)\}\} \qquad (42)
\end{aligned}
$$

Hence, $\{\phi(x^*, x_n)\}$ is bounded. Since $\frac{1}{\tau}||x_n - x^*||^2 \le \phi(x^*, x_n)$, we have that $\{x_n\}$ is bounded. Consequently, $\{u_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{w_n\}$ are also bounded. $\square$

Step 2

We now prove that the sequence $\{x_n\}$ generated by algorithm (27) converges strongly to a point $\bar{x} = \Pi_{VI(C,A) \,\bigcap\, \bigcap_{i=1}^M F(T_i)} x_0$.

Proof. Let $\bar{x} = \Pi_{VI(C,A) \,\bigcap\, \bigcap_{i=1}^M F(T_i)} x_0$. From equation (19) of Lemma 2.4, we have

$$
\langle Jx_0 - J\bar{x}, z - \bar{x}\rangle \le 0, \qquad \forall \ z \in VI(C, A).
$$

From step 1, we have that, there exists $N_0 \ge 0$, such that for all $n \ge N_0$,

$$
\phi(x^*, z_n) \le \phi(x^*, u_n) \ \ \textit{and} \ \ \phi(x^*, u_n) \le (1-\theta_n)\phi(x^*, x_n) + \theta_n \phi(x^*, x_{n-1}),
$$

and the following sequences $\{x_n\}$, $\{u_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{w_n\}$ are bounded. Furthermore, we estimate $\phi(x^*, x_{n+1})$ using inequality (37) and Lemma 2.18 for every $n \ge N_0$.

$$
\begin{aligned}
\phi(x^*, x_{n+1}) &= \phi(x^*, J^{-1}(\beta_n J z_n + (1-\beta_n)J(Tw_n))) \\
&\le \beta_n \phi(x^*, z_n) + (1-\beta_n)\phi(x^*, Tw_n) \\
&= \beta_n \phi(x^*, z_n) + (1-\beta_n)\phi(x^*, T_M \circ T_{M-1}\circ, ..., \circ T_1(w_n)) \\
&\le \beta_n \phi(x^*, z_n) + (1-\beta_n)\phi(x^*, T_{M-1}\circ, ..., \circ T_1(w_n)) \\
&\le \beta_n \phi(x^*, z_n) + (1-\beta_n)\phi(x^*, w_n) \\
&\le \beta_n \phi(x^*, z_n) + (1-\beta_n)[\alpha_n \phi(x^*, x_0) + (1-\alpha_n)\phi(x^*, z_n)] \\
&= \beta_n \phi(x^*, z_n) + (1-\beta_n)(1-\alpha_n)\phi(x^*, z_n) \\
&\quad +\alpha_n(1-\beta_n)\phi(x^*, x_0) \\
&= (\beta_n + (1-\beta_n)(1-\alpha_n))\phi(x^*, z_n) + \alpha_n(1-\beta_n)\phi(x^*, x_0)
\end{aligned}
$$

$$\begin{aligned}
\phi(x^*,x_{n+1}) &\leq (\beta_n+(1-\beta_n)(1-\alpha_n))(\phi(x^*,u_n)\\
&\quad -(1-\mu\tau)(\phi(u_n,y_n)+\phi(y_n,z_n)))+\alpha_n(1-\beta_n)\phi(x^*,x_0)\\
&= (1-(1-\beta_n)\alpha_n)(\phi(x^*,u_n)\\
&\quad -(1-\mu\tau)(\phi(u_n,y_n)+\phi(y_n,z_n))+\alpha_n(1-\beta_n)\phi(x^*,x_0)\\
&= \phi(x^*,u_n)-(1-\mu\tau)(\phi(u_n,y_n)\\
&\quad +\phi(y_n,z_n))+\alpha_n(1-\beta_n)\phi(x^*,x_0)-(1-\beta_n)\alpha_n(\phi(x^*,u_n)\\
&\quad -(1-\mu\tau)(\phi(u_n,y_n)+\phi(y_n,z_n))\\
&= \phi(x^*,u_n)-(1-\mu\tau)(\phi(u_n,y_n)\\
&\quad +\phi(y_n,z_n))+\alpha_n(1-\beta_n)\phi(x^*,x_0)-(1-\beta_n)\alpha_n\phi(x^*,u_n)\\
&\quad +(1-\beta_n)\alpha_n(1-\mu\tau)(\phi(u_n,y_n)+\phi(y_n,z_n))\\
&= (1-(1-\beta_n)\alpha_n)\phi(x^*,u_n)\\
&\quad -(1-\mu\tau)(\phi(u_n,y_n)+\phi(y_n,z_n))+\alpha_n(1-\beta_n)\phi(x^*,x_0)\\
&\quad +\alpha_n(1-\beta_n)(1-\mu\tau)(\phi(u_n,y_n)+\phi(y_n,z_n))\\
&= (1-(1-\beta_n)\alpha_n)\phi(x^*,u_n)\\
&\quad -(1-\mu\tau)(\phi(u_n,y_n)+\phi(y_n,z_n))+\alpha_n(1-\beta_n)\phi(x^*,x_0)\\
&\quad +\alpha_n\sigma_n(\phi(u_n,y_n)+\phi(y_n,z_n))\\
&= (1-(1-\beta_n)\alpha_n)[(1-\theta_n)\phi(x^*,x_n)+\theta_n\phi(x^*,x_{n-1})]\\
&\quad -(1-\mu\tau)(\phi(u_n,y_n)+\phi(y_n,z_n))\\
&\quad +\alpha_n(1-\beta_n)\phi(x^*,x_0)+\alpha_n\sigma_n(\phi(u_n,y_n)+\phi(y_n,z_n))\\
&= (1-\theta_n)\phi(x^*,x_n)+\theta_n\phi(x^*,x_{n-1})\\
&\quad -(1-\beta_n)\alpha_n[(1-\theta_n)\phi(x^*,x_n)+\theta_n\phi(x^*,x_{n-1})]\\
&\quad -(1-\mu\tau)(\phi(u_n,y_n)+\phi(y_n,z_n))+\alpha_n(1-\beta_n)\phi(x^*,x_0)\\
&\quad +\alpha_n\sigma_n(\phi(u_n,y_n)+\phi(y_n,z_n))\\
&\leq (1-\theta_n)\phi(x^*,x_n)+\theta_n\phi(x^*,x_{n-1})\\
&\quad -(1-\mu\tau)(\phi(u_n,y_n)+\phi(y_n,z_n))+\alpha_n(1-\beta_n)\phi(x^*,x_0)\\
&\quad +\alpha_n\sigma_n(\phi(u_n,y_n)+\phi(y_n,z_n)) \qquad (43)
\end{aligned}$$

where $\sigma_n=(1-\beta_n)(1-\mu\tau)$.
The remaining part of the proof will be divided into two cases.
Case I. Suppose that there exists $N_1\in\mathbb{N}$ $(N_1\geq N_0)$ such that $\{\phi(x^*,x_n)\}_{n=N_1}^{\infty}$ is nonincreasing. Since the sequence $\{\phi(x^*,x_n)\}_{n=1}^{\infty}$ is bounded then it

converges for all $n > N_1 \geq N_0$. That is,

$$\lim_{n\to\infty}(\phi(x^*, x_n) - \phi(x^*, x_{n+1})) = 0. \tag{44}$$

This implies from (43) that

$$\begin{aligned}(1-\mu\tau)&(\phi(u_n, y_n) + \phi(y_n, z_n))\\ &\leq \theta_n(\phi(x^*, x_{n-1}) - \phi(x^*, x_n)) + \phi(x^*, x_n) - \phi(x^*, x_{n+1})\\ &+\alpha_n\sigma_n(\phi(u_n, y_n) + \phi(y_n, z_n)) + \alpha_n(1-\beta_n)\phi(x^*, x_0).\end{aligned} \tag{45}$$

Using (44), equation (33) of Remark 3.2 and condition $(B2)$ together with the boundedness of $\{u_n\}, \{y_n\}$ and $\{z_n\}$, we have from (45) that

$$\begin{aligned}(1-\mu\tau)&(\phi(u_n, y_n) + \phi(y_n, z_n))\\ &\leq \theta_n(\phi(x^*, x_{n-1}) - \phi(x^*, x_n)) + \phi(x^*, x_n) - \phi(x^*, x_{n+1})\\ &+\alpha_n\sigma_n(\phi(u_n, y_n) + \phi(y_n, z_n))\\ &+\alpha_n(1-\beta_n)\phi(x^*, x_0) \longrightarrow 0, \ \ as \ \ n \longrightarrow \infty.\end{aligned}$$

Hence,

$$\lim_{n\to\infty}\phi(u_n, y_n) = \lim_{n\to\infty}\phi(y_n, z_n) = 0.$$

Thus, from Lemma 2.6, we have that

$$\lim_{n\to\infty}||u_n - y_n|| = \lim_{n\to\infty}||y_n - z_n|| = 0. \tag{46}$$

Using Lemma 2.5, Lemma 2.18 and the definitions of $\{x_{n+1}\}, \{u_n\}, \{w_n\}$, we obtain

$$\begin{aligned}\phi(x^*, x_{n+1}) &= \phi(x^*, J^{-1}(\beta_n J z_n + (1-\beta_n)J(Tw_n)))\\ &\leq \beta_n\phi(x^*, z_n) + (1-\beta_n)\phi(x^*, Tw_n)\\ &\quad -\beta_n(1-\beta_n)g(||Jz_n - JTw_n||)\\ &= \beta_n\phi(x^*, z_n) + (1-\beta_n)\phi(x^*, T_M \circ T_{M-1}\circ, ..., \circ T_1(w_n))\\ &\quad -\beta_n(1-\beta_n)g(||Jz_n - JTw_n||)\\ &\leq \beta_n\phi(x^*, z_n) + (1-\beta_n)\phi(x^*, T_{M-1}\circ, ..., \circ T_1(w_n))\\ &\quad -\beta_n(1-\beta_n)g(||Jz_n - JTw_n||)\\ &\leq \beta_n\phi(x^*, z_n) + (1-\beta_n)\phi(x^*, w_n)\\ &\quad -\beta_n(1-\beta_n)g(||Jz_n - JTw_n||)\end{aligned}$$

$$
\begin{aligned}
\phi(x^*, x_{n+1}) &\leq \beta_n\phi(x^*, z_n) + (1-\beta_n)[\alpha_n\phi(x^*, x_0) + (1-\alpha_n)\phi(x^*, z_n)] \\
&\quad -\beta_n(1-\beta_n)g(||Jz_n - JTw_n||) \\
&= (\beta_n + (1-\beta_n)(1-\alpha_n))\phi(x^*, z_n) + (1-\beta_n)\alpha_n\phi(x^*, x_0) \\
&\quad -\beta_n(1-\beta_n)g(||Jz_n - JTw_n||) \\
&= (1-(1-\beta_n)\alpha_n)\phi(x^*, z_n) + (1-\beta_n)\alpha_n\phi(x^*, x_0) \\
&\quad -\beta_n(1-\beta_n)g(||Jz_n - JTw_n||) \\
&\leq (1-(1-\beta_n)\alpha_n)\phi(x^*, u_n) + (1-\beta_n)\alpha_n\phi(x^*, x_0) \\
&\quad -\beta_n(1-\beta_n)g(||Jz_n - JTw_n||) \\
&= (1-(1-\beta_n)\alpha_n)[(1-\theta_n)\phi(x^*, x_n) + \theta_n\phi(x^*, x_{n-1})] \\
&\quad +(1-\beta_n)\alpha_n\phi(x^*, x_0) - \beta_n(1-\beta_n)g(||Jz_n - JTw_n||) \\
&= (1-\theta_n)\phi(x^*, x_n) + \theta_n\phi(x^*, x_{n-1}) \\
&\quad -(1-\beta_n)\alpha_n[(1-\theta_n)\phi(x^*, x_n) + \theta_n\phi(x^*, x_{n-1})] \\
&\quad +(1-\beta_n)\alpha_n\phi(x^*, x_0) - \beta_n(1-\beta_n)g(||Jz_n - JTw_n||) \\
&\leq (1-\theta_n)\phi(x^*, x_n) + \theta_n\phi(x^*, x_{n-1}) + (1-\beta_n)\alpha_n\phi(x^*, x_0) \\
&\quad -\beta_n(1-\beta_n)g(||Jz_n - JTw_n||). \qquad (47)
\end{aligned}
$$

This implies from (47) that

$$
\begin{aligned}
0 &\leq \beta_n(1-\beta_n)g(||Jz_n - JTw_n||) \leq \phi(x^*, x_n) - \phi(x^*, x_{n+1}) \\
&\quad +\theta_n(\phi(x^*, x_{n-1}) - \phi(x^*, x_n)) + (1-\beta_n)\alpha_n\phi(x^*, x_0). \qquad (48)
\end{aligned}
$$

From (44), equation (33) of Remark 3.2 together with condition $(B2)$, we have from (48) that

$$
\begin{aligned}
&\beta_n(1-\beta_n)g(||Jz_n - JTw_n||) \\
&\quad \leq \phi(x^*, x_n) - \phi(x^*, x_{n+1}) + \theta_n(\phi(x^*, x_{n-1}) - \phi(x^*, x_n)) \\
&\quad +(1-\beta_n)\alpha_n\phi(x^*, x_0) \longrightarrow 0, \ \ as \ \ n \longrightarrow \infty.
\end{aligned}
$$

Thus, using the property of g in Lemma 2.5,we have

$$
\lim_{n\to\infty} ||Jz_n - JTw_n|| = 0. \qquad (49)
$$

Since E^* is uniformly smooth, then J^{-1} is uniformly norm to norm continuous on bounded subsets. Hence, we have from (49) that

$$
\lim_{n\to\infty} ||z_n - Tw_n|| = 0. \qquad (50)
$$

Using the definition of $\{x_{n+1}\}$ in algorithm (27), we have

$$Jx_{n+1} = \beta_n Jz_n + (1-\beta_n)J(Tw_n)$$

$$Jx_{n+1} - Jz_n = (\beta_n Jz_n + (1-\beta_n)J(Tw_n)) - Jz_n$$

$$\begin{aligned} ||Jx_{n+1} - Jz_n|| &= ||\beta_n Jz_n + (1-\beta_n)J(Tw_n) \\ &\quad -[\beta_n Jz_n + (1-\beta_n)Jz_n]|| \\ &= ||(1-\beta_n)(J(Tw_n) - Jz_n)|| \\ &= (1-\beta_n)||J(Tw_n) - Jz_n||. \end{aligned}$$

Thus from this and (49), we have

$$\lim_{n\to\infty} ||Jx_{n+1} - Jz_n|| = 0. \tag{51}$$

Since E^* is uniformly smooth, we have from (51) that

$$\lim_{n\to\infty} ||x_{n+1} - z_n|| = 0. \tag{52}$$

From the definition of $\{w_n\}$ in algorithm (27), we have

$$Jw_n - Jz_n = (\alpha_n Jx_0 + (1-\alpha_n)Jz_n) - Jz_n$$

$$||Jw_n - Jz_n|| = \alpha_n ||Jx_0 - Jz_n||$$

Using condition $(B2)$, we obtain

$$\lim_{n\to\infty} ||Jw_n - Jz_n|| = 0. \tag{53}$$

Since E^* is uniformly smooth, we have from (53) that

$$\lim_{n\to\infty} ||w_n - z_n|| = 0. \tag{54}$$

Thus, we have from (54) and (50) that

$$\begin{aligned} ||Tw_n - w_n|| &= ||Tw_n - z_n + z_n - w_n|| \\ &\leq ||Tw_n - z_n|| + ||z_n - w_n|| \longrightarrow 0, \ as \ n \longrightarrow \infty. \end{aligned}$$

Hence,

$$\lim_{n\to\infty} ||Tw_n - w_n|| = 0. \tag{55}$$

From the definition of $\{u_n\}$ in algorithm (27) and equation (32) of Remark 3.2, we obtain

$$\begin{aligned} ||Ju_n - Jx_n|| &= ||\theta_n(Jx_{n-1} - Jx_n)|| \\ &= \alpha_n.\frac{\theta_n}{\alpha_n}||Jx_{n-1} - Jx_n|| \longrightarrow 0, \quad as \;\; n \longrightarrow \infty. \end{aligned}$$

Hence,

$$\lim_{n\to\infty} ||Ju_n - Jx_n|| = 0. \tag{56}$$

Since E^* is uniformly smooth, then J^{-1} is uniformly norm to norm continuous on bounded subsets of E^*, we have

$$\lim_{n\to\infty} ||u_n - x_n|| = 0. \tag{57}$$

Furthermore, we have from (57) and (46) that

$$\begin{aligned} ||y_n - x_n|| &= ||y_n - u_n + u_n - x_n|| \\ &\leq ||y_n - u_n|| + ||u_n - x_n|| \longrightarrow 0, \quad as \;\; n \longrightarrow \infty. \end{aligned}$$

Thus,

$$\lim_{n\to\infty} ||y_n - x_n|| = 0. \tag{58}$$

From (58), (52) and (46), we obtain

$$\begin{aligned} ||x_{n+1} - x_n|| &= ||x_{n+1} - z_n + z_n - y_n + y_n - x_n|| \\ &\leq ||x_{n+1} - z_n|| + ||z_n - y_n|| + ||y_n - x_n|| \longrightarrow 0, \end{aligned}$$

as $n \longrightarrow \infty$. Hence,

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0. \tag{59}$$

Furthermore, from (59), (52) and (54), we have

$$\begin{aligned} ||x_n - w_n|| &= ||x_n - x_{n+1} + x_{n+1} - z_n + z_n - w_n|| \\ &\leq ||x_n - x_{n+1}|| + ||x_{n+1} - z_n|| + ||z_n - w_n|| \longrightarrow 0, \end{aligned}$$

as $n \longrightarrow \infty$. Thus,

$$\lim_{n\to\infty} ||x_n - w_n|| = 0. \tag{60}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \rightharpoonup u^*$, which implies that $w_{n_k} \rightharpoonup u^*$ as $k \longrightarrow \infty$. Since $\lim_{k\to\infty} ||Tw_{n_k} - w_{n_k}|| = 0$, by Lemma 2.16 it follows that $u^* \in \bigcap_{i=1}^{M} F(T_i)$. Next, we show that $u^* \in VI(C, A)$.
We have $\{u_{n_k}\}$ converges weakly to $u^* \in C$ since $||x_{n_k} - u_{n_k}|| \longrightarrow 0$ *as* $k \longrightarrow \infty$, then $y_{n_k} \rightharpoonup u^*$ since $||y_{n_k} - u_{n_k}|| \longrightarrow 0$ *as* $k \longrightarrow \infty$. From the definition of $y_{n_k} = \Pi_C J^{-1}(Ju_{n_k} - \lambda_{n_k} A(u_{n_k}))$, we have from equation (19) of Lemma 2.4 that for all $z \in C$,

$$\langle Ju_{n_k} - \lambda_{n_k} A(u_{n_k}) - Jy_{n_k}, z - y_{n_k}\rangle \leq 0.$$

This implies that

$$\langle Ju_{n_k} - Jy_{n_k}, z - y_{n_k}\rangle \leq \lambda_{n_k}\langle A(u_{n_k}), z - y_{n_k}\rangle.$$

Then for all $z \in C$, we have

$$\frac{1}{\lambda_{n_k}}\langle Ju_{n_k} - Jy_{n_k}, z - y_{n_k}\rangle + \langle A(u_{n_k}), y_{n_k} - u_{n_k}\rangle \leq \langle A(u_{n_k}), z - u_{n_k}\rangle \tag{61}$$

Fixing $z \in C$ and letting $k \longrightarrow +\infty$ in (61) also remembering that $||y_{n_k} - u_{n_k}|| \longrightarrow 0$ *as* $k \longrightarrow \infty$ together with the fact that $\liminf_{n\to\infty} \lambda_{n_k} > 0$, we have

$$\liminf_{n\to\infty} \langle A(u_{n_k}), z - u_{n_k}\rangle \geq 0. \tag{62}$$

Let $\{\varepsilon_k\}$ be a decreasing nonnegative sequence such that $\lim_{n\to\infty} \varepsilon_k = 0$. For each ε_k, we denote the smallest positive integer N_k such that for all $k \geq N_k$,

$$\langle A(u_{n_k}), z - u_{n_k}\rangle + \varepsilon_k \geq 0 \tag{63}$$

Furthermore, as $\{\varepsilon_k\}$ is decreasing, it is easy to see that the sequence $\{N_k\}$ is increasing. Thus, if there exists a subsequence $\{u_{n_{k_j}}\} \subset \{u_{n_k}\}$, such that for each $j \geq 1, A(u_{n_{k_j}}) \neq 0$, and setting

$$s_{n_{k_j}} = \frac{A(u_{n_{k_j}})}{||A(u_{n_{k_j}})||^2},$$

we have $\langle A(u_{n_{k_j}}), s_{n_{k_j}}\rangle = 1$ for each $j \geq 1$. It follows from (63) that for each $j \geq 1$

$$\langle A(u_{n_{k_j}}), z + \varepsilon_k s_{n_{k_j}} - u_{n_{k_j}}\rangle \geq 0. \tag{64}$$

Thus, since A is pseudomonotone, we obtain from (64) that

$$\langle A(z + \varepsilon_k s_{n_{k_j}}), z + \varepsilon_k s_{n_{k_j}} - u_{n_{k_j}}\rangle \geq 0. \tag{65}$$

Since $\{u_{n_k}\}$ converges weakly to $u^* \in C$, and A is weakly sequentially continuous, we have that $A(u_{n_k})$ converges weakly to $A(u^*)$. If $A(u^*) = 0$, then $u^* \in VI(C, A)$. Suppose that $A(u^*) \neq 0$. Then, by sequential weak lower semicontinuity of the norm, we have the following

$$0 < ||A(u^*)|| \leq \liminf_{k\to\infty} ||A(u_{n_k})||.$$

Since $\{u_{n_{k_i}}\} \subset \{u_{n_k}\}$ and $\varepsilon_k \longrightarrow 0$ as $k \longrightarrow \infty$, we obtain

$$0 \leq \limsup_{k\to\infty} ||\varepsilon_k s_{n_k}|| = \limsup_{k\to\infty} \left(\frac{\varepsilon_k}{||A(u_{n_k})||}\right) \leq \frac{\limsup_{k\to\infty} \varepsilon_k}{\liminf_{k\to\infty} ||A(u_{n_k})||} \leq \frac{0}{||A(u^*)||} = 0$$

Taking limit as $j \longrightarrow \infty$ in (65), we obtain

$$\langle A(z), z - u^*\rangle \geq 0.$$

Thus, it follows from Lemma 2.13 that $u^* \in VI(C, A)$. Furthermore, from (55) and (60) we have that $u^* \in F(T_i)$ for all $i \in \{1, 2, ..., M\}$, thus $u^* \in \bigcap_{i=1}^{M} F(T_i)$. Hence, $u^* \in \Gamma$.

Next, we show that $\{x_n\}$ converges strongly to a point $\bar{x} = \Pi_\Gamma x_0$. Since

$\{x_n\}$ is bounded, then, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \rightharpoonup u^*$ and

$$\begin{aligned}\limsup_{n\to\infty}\langle x_n-\bar{x}, Jx_0-J\bar{x}\rangle &= \lim_{k\to\infty}\langle x_{n_k}-\bar{x}, Jx_0-J\bar{x}\rangle\\ &= \langle u^*-\bar{x}, Jx_0-J\bar{x}\rangle. \end{aligned} \tag{66}$$

Thus, from equation (19) of Lemma 2.4 and (66), we have

$$\limsup_{n\to\infty}\langle x_n-\bar{x}, Jx_0-J\bar{x}\rangle = \langle u^*-\bar{x}, Jx_0-J\bar{x}\rangle \leq 0 \tag{67}$$

Hence, it follows from (67) that

$$\limsup_{n\to\infty}\langle w_n-\bar{x}, Jx_0-J\bar{x}\rangle \leq 0 \tag{68}$$

Furthermore, from the definition of $\phi(\bar{x}, x_{n+1})$ in algorithm (27), Lemma 2.18 and Lemma 2.3, we obtain

$$\begin{aligned}\phi(\bar{x}, x_{n+1}) &= \phi(\bar{x}, J^{-1}(\beta_n Jz_n+(1-\beta_n)J(Tw_n)))\\ &\leq \beta_n\phi(\bar{x}, z_n)+(1-\beta_n)\phi(\bar{x}, Tw_n)\\ &\leq \beta_n\phi(\bar{x}, z_n)+(1-\beta_n)\phi(\bar{x}, T_M\circ T_{M-1}\circ, ..., \circ T_1(w_n))\\ &\leq \beta_n\phi(\bar{x}, z_n)+(1-\beta_n)\phi(\bar{x}, T_{M-1}\circ, ..., \circ T_1(w_n))\\ &\leq \beta_n\phi(\bar{x}, z_n)+(1-\beta_n)\phi(\bar{x}, w_n)\\ &\leq \beta_n\phi(\bar{x}, z_n)+(1-\beta_n)[\phi(\bar{x}, J^{-1}(\alpha_n Jx_0+(1-\alpha_n)Jz_n))]\\ &= \beta_n\phi(\bar{x}, z_n)+(1-\beta_n)[V(\bar{x}, \alpha_n Jx_0+(1-\alpha_n)Jz_n)]\\ &\leq \beta_n\phi(\bar{x}, z_n)\\ &\quad +(1-\beta_n)[V(\bar{x}, \alpha_n Jx_0+(1-\alpha_n)Jz_n-\alpha_n(Jx_0-J\bar{x}))\\ &\quad -2\langle J^{-1}(\alpha_n Jx_0+(1-\alpha_n)Jz_n))-\bar{x}, -\alpha_n(Jx_0-J\bar{x})\rangle]\\ &= \beta_n\phi(\bar{x}, z_n)+(1-\beta_n)[V(\bar{x}, \alpha_n J\bar{x}+(1-\alpha_n)Jz_n)\\ &\quad +2\alpha_n\langle w_n-\bar{x}, Jx_0-J\bar{x}\rangle]\\ &= \beta_n\phi(\bar{x}, z_n)+(1-\beta_n)[\alpha_n\phi(\bar{x}, \bar{x})+(1-\alpha_n)\phi(\bar{x}, z_n)\\ &\quad +2\alpha_n\langle w_n-\bar{x}, Jx_0-J\bar{x}\rangle]\\ &\leq \beta_n\phi(\bar{x}, z_n)+(1-\beta_n)[(1-\alpha_n)\phi(\bar{x}, z_n)\\ &\quad +2\alpha_n\langle w_n-\bar{x}, Jx_0-J\bar{x}\rangle]\end{aligned}$$

$$
\begin{aligned}
\phi(\bar{x}, x_{n+1}) &\leq (\beta_n + (1-\beta_n)(1-\alpha_n))\phi(\bar{x}, z_n) \\
&\quad +2(1-\beta_n)\alpha_n\langle w_n - \bar{x}, Jx_0 - J\bar{x}\rangle \\
&\leq (\beta_n + (1-\beta_n)(1-\alpha_n))\phi(\bar{x}, u_n) \\
&\quad +2(1-\beta_n)\alpha_n\langle w_n - \bar{x}, Jx_0 - J\bar{x}\rangle \\
&= (1-(1-\beta_n)\alpha_n)[(1-\theta_n)\phi(\bar{x}, x_n) + \theta_n\phi(\bar{x}, x_{n-1})] \\
&\quad +2(1-\beta_n)\alpha_n\langle w_n - \bar{x}, Jx_0 - J\bar{x}\rangle \\
&= (1-\theta_n)\phi(\bar{x}, x_n) + \theta_n\phi(\bar{x}, x_{n-1}) \\
&\quad +2(1-\beta_n)\alpha_n\langle w_n - \bar{x}, Jx_0 - J\bar{x}\rangle \\
&\quad -(1-\beta_n)\alpha_n[(1-\theta_n)\phi(\bar{x}, x_n) + \theta_n\phi(\bar{x}, x_{n-1})] \\
&= \phi(\bar{x}, x_n) + \theta_n(\phi(\bar{x}, x_{n-1}) - \phi(\bar{x}, x_n)) \\
&\quad -\alpha_n[(1-\theta_n)\phi(\bar{x}, x_n) + \theta_n\phi(\bar{x}, x_{n-1})] \\
&\quad +2(1-\beta_n)\alpha_n\langle w_n - \bar{x}, Jx_0 - J\bar{x}\rangle \\
&\quad +\alpha_n\beta_n[(1-\theta_n)\phi(\bar{x}, x_n) + \theta_n\phi(\bar{x}, x_{n-1})] \\
&= (1-(1-\beta_n)\alpha_n)\phi(\bar{x}, x_n) + \theta_n(\phi(\bar{x}, x_{n-1}) - \phi(\bar{x}, x_n)) \\
&\quad -\alpha_n(1-\beta_n)[\theta_n(\phi(\bar{x}, x_{n-1}) - \phi(\bar{x}, x_n))] \\
&\quad +2(1-\beta_n)\alpha_n\langle w_n - \bar{x}, Jx_0 - J\bar{x}\rangle. \qquad (69)
\end{aligned}
$$

Setting $\psi_n = (1-\beta_n)\alpha_n$, $\sigma_n = 2\langle w_n - \bar{x}, Jx_0 - J\bar{x}\rangle$ and $\gamma_n = [1-\alpha_n(1-\beta_n)]\theta_n(\phi(\bar{x}, x_{n-1}) - \phi(\bar{x}, x_n))$.
Now, applying Lemma 2.14, (68), (69), equation (33) of Remark 3.2 and from condition $(B2)$, we obtain

$$\lim_{n\to\infty} \phi(\bar{x}, x_n) = 0.$$

Thus, from Lemma 2.6, we have

$$\lim_{n\to\infty} ||\bar{x} - x_n|| = 0. \qquad (70)$$

Hence, $x_n \longrightarrow \bar{x}$ where $\bar{x} = \Pi_\Gamma x_0$.
Case II. Suppose that the sequence $\{\phi(p, x_n)\}_{n=1}^{\infty}$ is not a nonincreasing sequence. Then, let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that

$$\phi(p, x_{n_k}) < \phi(p, x_{n_k+1}), \quad for \quad all \quad k \in \mathbb{N}.$$

Then, using Lemma 2.15, there exists a nondecreasing sequence $\{m_s\} \subseteq \mathbb{N}$ such that $m_s \longrightarrow \infty$ as $s \longrightarrow \infty$, and,

$$\phi(p, x_{m_s}) \leq \phi(p, x_{m_s+1}) \quad \textit{and} \quad \phi(p, x_s) \leq \phi(p, x_{m_s+1}).$$

Since $\{\phi(p, x_{m_s})\}$ is bounded, then $\lim_{s\to\infty} \phi(p, x_{m_s})$ exist.
Therefore, using the same approach as in case (I), we have the following

$$(i) \underset{s\to\infty}{lim} ||x_{m_s} - w_{m_s}|| = 0, (ii) \underset{s\to\infty}{lim} ||u_{m_s} - y_{m_s}|| = 0, (iii) \underset{s\to\infty}{lim} ||z_{m_s} - y_{m_s}|| = 0$$

$$(iv) \underset{s\to\infty}{lim} ||x_{m_s+1} - x_{m_s}|| = 0.$$

Now, following the same steps as in the proof of case (I), we obtain

$$\limsup_{s\to\infty} \langle w_{m_s+1} - \bar{x}, Jx_0 - J\bar{x} \rangle = \limsup_{s\to\infty} \langle w_{m_s} - \bar{x}, Jx_0 - J\bar{x} \rangle \leq 0. \quad (71)$$

Furthermore, from (69) for all $m_s \geq N_0$, we have

$$\begin{aligned} \phi(\bar{x}, x_{m_s+1}) \quad \leq \quad & (1 - (1 - \beta_{m_s})\alpha_{m_s})\phi(\bar{x}, x_{m_s}) \\ & + \theta_{m_s}(\phi(\bar{x}, x_{m_s-1}) - \phi(\bar{x}, x_{m_s})) \\ & - \alpha_{m_s}(1 - \beta_{m_s})[\theta_{m_s}(\phi(\bar{x}, x_{m_s-1}) - \phi(\bar{x}, x_{m_s}))] \\ & + 2(1 - \beta_{m_s})\alpha_{m_s}\langle w_{m_s} - \bar{x}, Jx_0 - J\bar{x} \rangle \\ < \quad & (1 - (1 - \beta_{m_s})\alpha_{m_s})\phi(\bar{x}, x_{m_s+1}) \\ & + \theta_{m_s}(\phi(\bar{x}, x_{m_s-1}) - \phi(\bar{x}, x_{m_s})) \\ & - \alpha_{m_s}(1 - \beta_{m_s})[\theta_{m_s}(\phi(\bar{x}, x_{m_s-1}) - \phi(\bar{x}, x_{m_s}))] \\ & + 2(1 - \beta_{m_s})\alpha_{m_s}\langle w_{m_s} - \bar{x}, Jx_0 - J\bar{x} \rangle \end{aligned}$$

Hence,

$$\begin{aligned} & (1 - \beta_{m_s})\alpha_{m_s}\phi(\bar{x}, x_{m_s+1}) \\ & \quad < [1 - (1 - \beta_{m_s})\alpha_{m_s}]\theta_{m_s}(\phi(\bar{x}, x_{m_s-1}) - \phi(\bar{x}, x_{m_s})) \\ & \quad + 2(1 - \beta_{m_s})\alpha_{m_s}\langle w_{m_s} - \bar{x}, Jx_0 - J\bar{x} \rangle \end{aligned}$$

Thus,

$$\begin{aligned} & (1 - \beta_{m_s})\alpha_{m_s}\phi(\bar{x}, x_{m_s+1}) \\ & \quad < 2(1 - \beta_{m_s})\alpha_{m_s}\langle w_{m_s} - \bar{x}, Jx_0 - J\bar{x} \rangle \end{aligned} \quad (72)$$

Since $0 < (1-\beta_{m_s})\alpha_{m_s} < 1$ for all $s \geq 0$ and $\phi(\bar{x}, x_{m_s}) \leq \phi(\bar{x}, x_{m_s+1})$, we have

$$\phi(\bar{x}, x_{m_s}) < \phi(\bar{x}, x_{m_s+1}) < 2\langle w_{m_s} - \bar{x}, Jx_0 - J\bar{x}\rangle.$$

This implies

$$\limsup_{s\to\infty} \phi(\bar{x}, x_{m_s}) < \limsup_{s\to\infty} 2\ \langle w_{m_s} - \bar{x}, Jx_0 - J\bar{x}\rangle \leq 0.$$

Thus,

$$\limsup_{s\to\infty} \phi(\bar{x}, x_{m_s}) = 0,$$

which by Lemma 2.6, we have

$$\mathop{lim}_{s\to\infty} ||\bar{x} - x_{m_s}|| = 0.$$

However, we know that $\phi(\bar{x}, x_s) \leq \phi(\bar{x}, x_{m_s+1})$ for all $s \in \mathbb{N}$, hence, $\mathop{lim}_{s\to\infty} \phi(\bar{x}, x_s) = 0$, which by Lemma 2.6, we have

$$\mathop{lim}_{s\to\infty} ||\bar{x} - x_s|| = 0.$$

Hence, $x_s \longrightarrow \bar{x}$ where $\bar{x} = \Pi_\Gamma x_0$. □

Corollary 3.4. *Let E be uniformly smooth and 2 - uniformly convex Banach space, $A : E \longrightarrow E^*$ be a monotone and Lipschitz continuous operator and $\{T_i\}_{i=1}^M$ be a finite family of generic $(\alpha, \beta, \gamma, \delta, \varepsilon, \xi)$ - generalized nonspreading mappings of E into itself. Let $\{u_n\}$, $\{y_n\}$, $\{w_n\}$ and $\{z_n\}$ be sequences generated by algorithm (27) and $\{\alpha_n\} \subset (0, 1)$ satisfy $\beta_n \in (a, b)$ where $0 < a < b < 1$, $\mathop{lim}_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^\infty \alpha_n = \infty$ be sequences satisfying assumptions $(A1) - (A4)$ of algorithm (27). Suppose $\Gamma = VI(C, A) \cap \bigcap_{i=1}^M F(T_i) \neq \emptyset$. Then, the sequence $\{x_n\}$ generated by algorithm (27) converges strongly to a solution $\bar{x} = \Pi_\Gamma x_0$.*

Proof. *Observe that in this case the weak sequential continuity of A in assumption (A2) of algorithm (27) has to be droped since it follows from the monotonicity of A and (61) that*

$$\begin{aligned} \frac{1}{\lambda_{n_k}}\langle Ju_{n_k} - Jy_{n_k}, z - y_{n_k}\rangle + \langle A(u_{n_k}), y_{n_k} - u_{n_k}\rangle \\ \leq \langle A(u_{n_k}), z - u_{n_k}\rangle \\ \leq\ \ \langle A(z), z - u_{n_k}\rangle. \end{aligned} \tag{73}$$

Furthermore, passing limit as $k \longrightarrow \infty$ *in inequality (73) and applying the fact that* $||u_{n_k} - y_{n_k}|| \longrightarrow 0, \;\; as \;\; k \longrightarrow \infty$, *we obtain*

$$\langle A(z), z - u^* \rangle \geq 0, \;\; \forall \;\; z \in C.$$

Hence, it follows from Theorem (3.3) that the sequence $\{x_n\}$ *converges strongly to a solution* $\bar{x} = \Pi_\Gamma x_0$.

Corollary 3.5. *Let* H *be a real Hilbert space,* $A : H \longrightarrow H$ *be pseudomonotone and Lipschitz continuous operator, and* $\{T_i\}_{i=1}^M$ *be a finite family of normally generalized hybrid mappings of* H *into itself. Let* $\{u_n\}$, $\{y_n\}$, $\{w_n\}$ *and* $\{z_n\}$ *be sequences generated by algorithm* (27) *and* $\beta_n \in (a, b)$ *where* $0 < a < b < 1$, $\{\alpha_n\} \subset (0, 1)$ *satisfy* $\lim\limits_{n \to \infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$ *be sequences satisfying assumptions* $(A1) - (A4)$ *of algorithm* (27). *Suppose* $\Gamma = VI(C, A) \bigcap \bigcap_{i=1}^M F(T_i) \neq \emptyset$. *Then, the sequence* $\{x_n\}$ *generated by algorithm* (27) *converges strongly to a solution* $\bar{x} = P_\Gamma x_0$.

Proof. *By Remark 2.10, the generic* $(\alpha, \beta, \gamma, \delta, \varepsilon, \xi)$ *- generalized nonspreading mappings reduces to normally generalized hybrid mapping in Hilbert space i.e, there exists* $\alpha_1, \beta_1, \gamma_1, \delta_1 \in \mathbb{R}$ *such that*

$$\alpha_1||Tx - Ty||^2 + \beta_1, ||x - Ty||^2 + \gamma_1||Tx - y||^2 + \delta_1||x - y||^2 \leq 0, \;\; \forall \; x, y \in C,$$

where $\alpha_1 = \alpha - \epsilon, \beta_1 = \beta + \epsilon, \gamma_1 = \gamma - \zeta$ *and* $\delta_1 = \delta + \zeta$ *satisfying* $\alpha_1 + \beta_1 = \alpha + \beta > 0$ *and* $\alpha_1 + \beta_1 + \gamma_1 + \delta_1 = \alpha + \beta + \gamma + \delta \geq 0$. *Thus by Theorem* (3.3), *we have that the sequence* $\{x_n\}$ *converges strongly to a solution* $\bar{x} = P_\Gamma x_0$.

4 Numerical Examples

In this section, we intend to demonstrate the efficiency of our Algorithm 3.1 with the aid of numerical experiments. Furthermore, we compare our iterative method with the methods of Ma [32] (Alg. A), Chidume et al. [13] (Alg. (6)) and Kraikaew and Saejung [27] (Alg. (5)).

Example 4.1. Let $E = L_2[0, 1]$ and $C = \{x \in L_2[0, 1] : \langle a, x \rangle \leq b\}$, where $a = t^2 + 1$ and $b = 1$, with norm $||x|| = \sqrt{\int_0^1 |x(t)|^2 dt}$ and inner product $\langle x, y\rangle = \int_0^t x(t)y(t)dt$, for all $x, y \in L_2([0,1])$, $t \in [0,1]$. Define metric projection P_C as follows:

$$P_C(x) = \begin{cases} x, & \text{if } x \in C \\ \frac{b-\langle a,x\rangle}{||a||_{L_2}} a + x, & \text{otherwise.} \end{cases} \tag{74}$$

Let $A : L_2[0,1] \to L_2[0,1]$ be defined by $A(x(t)) = e^{-||x||} \int_0^t x(s)ds$, for all $x \in L_2[0,1]$, $t, s \in [0,1]$, then, A is pseudomonotone and uniformly continuous mapping (see [49]) and let $T(x(t)) = \int_0^t x(s)ds$, for all $x \in L_2[0,1]$, $t \in [0,1]$, then T is nonexpansive mapping which is also generalized nonspreading mapping. For the control parameters, we choose as follows: Algorithm 27: $\alpha_n = \frac{1}{n^{25}}$, $\beta_n = \frac{n}{2(n+1)}(1-\alpha_n)$, $\tau_n = \frac{\alpha_n}{n^{1.1}}$ and $\theta_n = \hat{\theta}_n$; Algorithm A: $\alpha_n = \frac{1}{100n}$; Algorithm (6): $\alpha_n = \frac{1}{2^n}$, $\lambda = 0.5$; Algorithm (5): $\alpha_n = \frac{1}{n+1}$, $\beta_n = \frac{1}{3}(1-\alpha_n)$. We define the sequence $\text{TOL}_n := ||x_{n+1} - x_n||^2$ and apply the stopping criterion $\text{TOL}_n < \varepsilon$ for the iterative processes because the solution to the problem is unknown. ε is the predetermined error. Here, the terminating condition is set to $\varepsilon = 10^{-5}$. For the numerical experiments illustrated in Figure 1 and Table 1 below, we take into consideration the resulting cases.

Case 1: $x_0 = 2t$ and $x_1 = t^2$.

Case 2: $x_0 = t^3 + 3t$ and $x_1 = 4t^5 + 2t^3 + t$.

Case 3: $x_0 = 6t^6 + 3t^3 + t$ and $x_1 = t$.

Case 4: $x_0 = 8t^4 + 2t^2$ and $x_1 = 1/3t^3 + t^2$.

Table 1: Comparison of Alg. 27, Alg. A, Alg. (6) and Alg. (5).

Cases		**Alg. 27**	**Alg. A**	**Alg. (6)**	**Alg. (5)**
1	Iter.	30	76	67	54
	CPU (time)	12.6322	13.1864	17.5749	17.2129
2	Iter.	26	105	68	103
	CPU (time)	6.6802	11.8637	10.7646	26.2987
3	Iter.	30	130	68	126
	CPU (time)	7.5148	17.2607	10.3812	41.8932
4	Iter.	29	135	69	136
	CPU (time)	7.2560	18.4226	10.6401	53.3921

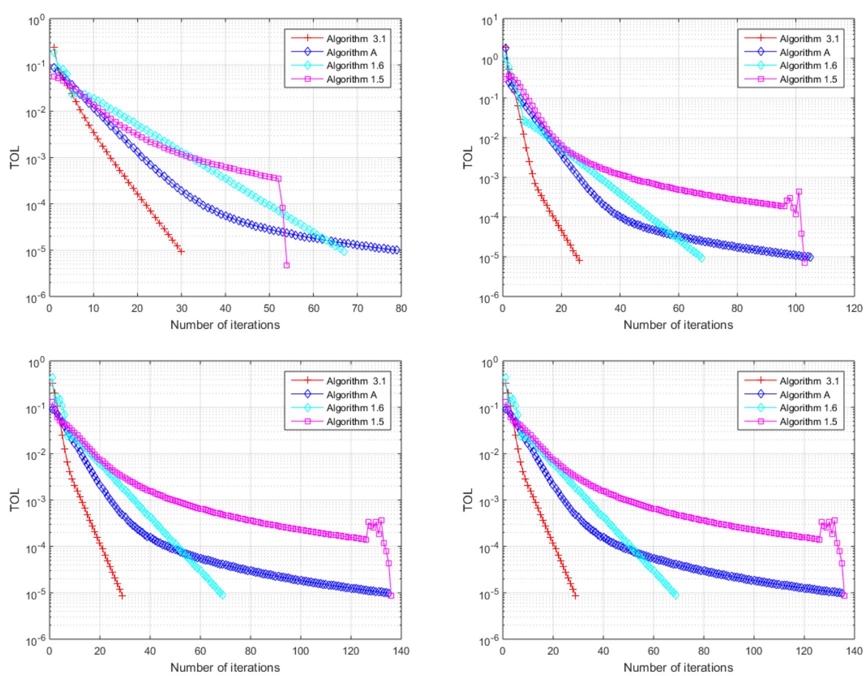

Figure 1: (**Top Left**): Case 1; (**Top Right**): Case 2; (**Bottom Left**):Case 3; (**Bottom Right**): Case 4, the error plotting of comparison of Alg. 27, Alg. A, ALg. (6) and Alg. (5) for Example 4.1.

Example 4.2. Let $E = \mathbb{R}^N$. Define $A : \mathbb{R}^N \to \mathbb{R}^N$ by $A(x) = Mx + q$, where the matrix M is formed as: $M = V \sum V'$, where $V = I - \frac{2vv'}{\|v\|^2}$ and $\sum = \text{diag}(\sigma_{11}, \sigma_{12}, \cdots, \sigma_{1N})$ are the householder and the diagonal matrix, and

$$\sigma_{1j} = \cos\frac{j\pi}{N+1} + 1 + \frac{\cos\frac{\pi}{N+1} + 1 - \widehat{C}(\cos\frac{N\pi}{N+1} + 1)}{\widehat{C} - 1}, \quad j = 1, 2, \cdots, N,$$

with $\widehat{C}$ been the present condition number of M ([19], Example 5.2). In the numerical computation, we choose $\widehat{C} = 10^4$, $q = 0$ and uniformly take the vector $v \in \mathbb{R}^N$ in $(-1, 1)$. Thus, A is pseudomonotone and Lipschitz continuous with $K = ||M||$ (see [19]). By setting $C = \{x \in \mathbb{R}^N : ||x|| \leq 1\}$, Matlab is used to efficiently compute the projection onto C. Moreover, we examine various instances of the problems dimension. That is, $N = 20, 30, 40, 60$, with starting points $x_1 = (1, 1, \ldots, 1)'$ and $x_0 = (0, 0, \ldots, 0)'$. In this example, we take the stopping criterion to be $\varepsilon = 10^{-5}$ and obtain the numerical results shown in Table 2 and Figure 2.

Table 2: Numerical results for Example 4.2 with $\varepsilon = 10^{-5}$.

N		**Alg. 27**	**Alg. A**	**Alg. (6)**	**Alg. (5)**
5	Iter.	832	8125	12053	1188
	CPU	0.2196	0.9656	4.1120	0.3920
7	Iter.	1436	15696	10807	1532
	CPU	0.2188	1.3669	3.3585	0.3680
11	Iter.	1860	17377	32919	1964
	CPU	0.1195	0.9919	10.2074	0.5366
13	Iter.	2152	25691	22295	2903
	CPU	0.2483	2.1576	8.2650	0.9547

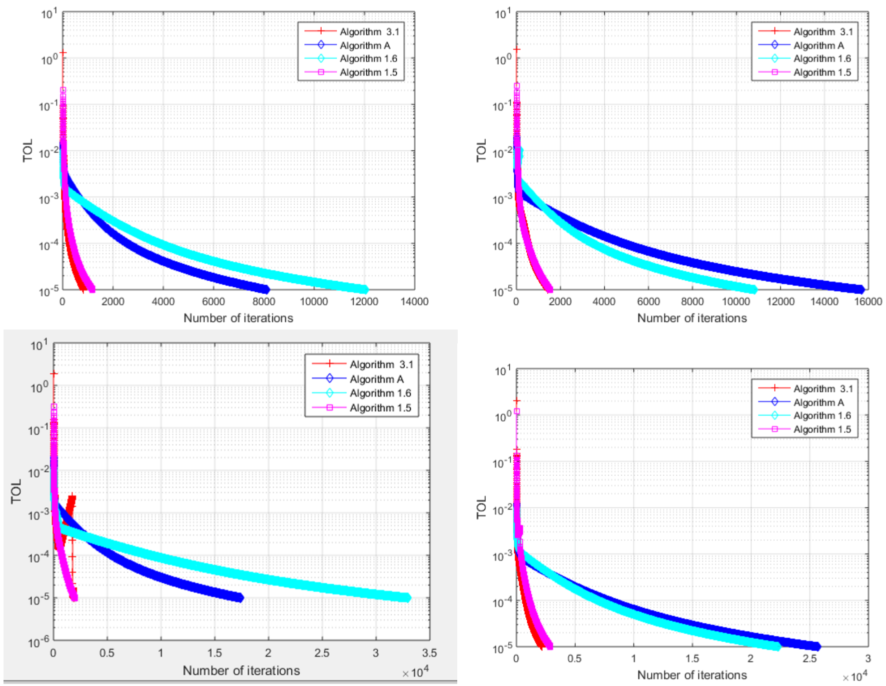

Figure 2: The behavior of TOL_n with $\varepsilon = 10^{-5}$ for Example 4.2: (**Top Left**): $N = 20$; (**Top Right**): $N = 30$; (**Bottom Left**): $N = 40$; (**Bottom Right**): $N = 60$.

5 Conclusion

This paper introduced a new inertial subgradient extragradient algorithm with self adaptive step size for approximating common element of the set of solutions of pseudomonotone variational inequality problem and common fixed point of a finite family of generic generalized nonspreading mappings in uniformly smooth and 2 - uniformly convex Banach space. Furthermore, we proved a strong convergence theorem of our algorithm to a solution of the stated problem without prior knowledge of the Lipschitz constant of the operator under some mild assumptions. We presented some numerical examples in order to illustrates the performance of our proposed algorithm. Our result generalize and improve many existing results in the literature. For instance, our result is presented in a more general setting of Banach space than Hilbert space, there by extending the recent results such as those in [27, 41, 42, 43], again the operators considered here, family of generic generalized non-spreading mappings are more general than those considered in many recent results such as those in [13, 41]. It is going to be of interest for further studies in this direction to consider dropping the condition of Uniform smoothness and 2-uniform convexity of the space also for a more general operators than those considered here.

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