Journal of Mathematical Extension
Vol. 17, No. 11, (2023) (5)1-26
URL: https://doi.org/10.30495/JME.2023.2671
ISSN: 1735-8299
Original Research Paper

Creation of New Univariate Distributions: A Novel Reduction Method From a Bivariate Distribution Family

F. Gurer*
Ankara University

M. Yilmaz
Ankara University

Abstract. Since some of the classical distributions fail to model data in statistics, necessity of having new distributions arises. Accordingly, studies about expanding well-known distribution families are increasing nowadays. In this paper, bivariate Ali-Mikhail-Haq copula family is reduced to univariate and under which conditions obtained distributions become a distribution function is investigated. Characteristics of these reduced distributions are reviewed, and parameter estimation is done with the help of maximum likelihood estimation method. The new distributions obtained by reducing bivariate or multivariate distributions to univariate are seen as more flexible than basic distributions. This flexibility leads us to think that use of this distribution for modelling different data sets and using them in various fields may be favourable. Therefore, our motivation for this article was to propose a method for reducing bivariate copulas to univariate, which has not been used in literature before. So that, wide range of use will be provided for modelling various data with this new method. The superiorities of the distributions used to model real data sets in the literature before and the newly proposed distributions are compared and evaluated in application.

Received: February 2023; Accepted: August 2023
*Corresponding Author

AMS Subject Classification: MSC code1; MSC code 2; more
Keywords and Phrases: Generating distribution, Ali-Mikhail-Haq distribution, copula, exponential distribution, data modelling.

1 Introduction

A lot of data sets are obtained when nature events are observed. Besides well-known distributions in literature, new distributions based on different theories (e.g compound distributions, mixed distributions, transmuted distributions) are recommended in recent years in order to model these data. New parameter or parameters are revealed when obtaining these distributions. This situation ensures essential distributions become more flexible and become successful in modelling some suitable data sets. Distributions which are based on mixed distribution theories are primarily addressed by [20]. Exponential geometric distribution, which is suggested by [1] became a guiding light to new compound distributions that are obtained in later years. Some recent studies in which new families of distribution is proposed as follows: A modified distribution with simple exchanging between Lindley distribution and exponential distribution without addition of a new parameter is proposed in [7]. In study [5], a new lifetime distribution is proposed by a combination of Rayleigh distribution and extended odd Weibull family to produce extended odd Weibull Rayleigh distribution. When it comes to [23], this study presented the Gompertz-generated family of distributions, with the object of improving a new extension of generalized extreme value distribution that was more flexible and comprehensive for extreme data. Recurrence relations are established for some moments of order statistics for different values of scale parameter and a new distribution is proposed in study [10] named half logistic-truncated exponential distribution. Transmuted distribution method, which is another distribution derivation method is frequently mentioned in recent years. This method, used first by [21], changed the structure of essential distribution via a specific formula and made data sets become better modelable. In this study, we will discuss the similarity of newly obtained distribution to transmuted distribution. [6], [17] and [25] can be cited as example transmuted distribution studies published in recent years.

The crucial purpose of this study is to propose a method for reducing bivariate copulas to univariate, which has not been used in literature until now. In addition, it is investigated whether new reduced distributions are modelable in real life data or not. In this paper, starting from copula theory which is related to bivariate dependency, suggestion of new distributions are achieved using a novel reduction method.

Only one random variable is used in order to calculate the probabilities involving same kind of events or discussion of probability models. In this kind of established probability models, univariate distributions are employed. Without robust background knowledge about the univariate distributions that will establish marginal (base) or conditional distributions, informations obtained about bivariate distributions cannot be completed. In some cases observing just one random variable value will not be functional. For example, when carrying out an experiment in order to get information about the individual's health condition, measuring only body weight of individual is not enough. Instead, it is important to get measurement value of body temperature, height, blood pressure etc. in addition to body weight. These different characteristics can be modelled in the same time with two or more different random variables. Multivariate distributions are used with probability models involving more than one random variable. In this paper, we just address bivariate continuous distribution.

Relationship between random variables is explained by the concept of dependence in statistical theory. Dependence poses a problem when doing inferences like point estimation, interval estimation for parameters of joint distribution of two random variables. Therefore, it is important to state the dependence clearly. Covariance is generally used in order to explain the dependence between random variables. However, covariance can only show the degree of dependence, while it cannot present functional structure of dependence mathematically. Revealing the structure of dependence mathematically can be possible by means of copula [9]. Copula, that is used for characterisation of dependence between random variables, denotes joint distribution function in terms of base distribution functions and measure of dependence. In conclusion, copula can be thought of as multivariate joint distribution function which involves univariate base distribution functions and measure of dependence (as parameter). Dependence parameter can affect the shape of dependence in two or more dimensional distributions. Based on this, bivariate Ali-Mikhail-Haq (AMH) copula distribution family that is firstly suggested by [4] is reviewed and reduced to univariate with the help of conditional distribution along the diagonal. Thus, how existing dependence parameter brings flexibility to univariate distribution is investigated.

In this paper, structures of two different distributions are explained primarily based on bivariate AMH distribution family. Subsequently, base distributon of these new distributions that has a feature of distribution function is exponentiated and characteristics are investigated. Parameter estimation is done with the estimation method of maximum likelihood (ML). A simulation study is performed to see the performance of numerical approximation of ML estimators.Since we know that when a distribution models data better in a specific field, it leads that distribution can be used in that field more effectively.The superiorities of the distributions used to modelling real data sets in the literature before and the newly proposed distributions are compared and discussed.

2 Construction of the New Generated Distribution Families

Joint distribution function of AMH distribution family is as follows,

$$H(x,y) = P\left(X \leq x, Y \leq y\right) = \frac{F(x)G(y)}{1-\theta(1-F(x))(1-G(y))}, -1 \leq \theta \leq 1. \tag{1}$$

Here, θ is the measure of dependence between X and Y. When $Y \leq t$ is given for this distribution function, conditioned probability of $X \leq t$ is found as:

$$H(t) = P\left(X \leq t | Y \leq t\right) = \frac{F(t)}{1-\theta(1-F(t))(1-G(t))}.$$

Expression after quadratic Taylor expansion of this conditioned distribution function is as follows:

$$H^{*}(t) = F(t)\left[1+\theta(1-F(t))(1-G(t)) + (\theta(1-F(t))(1-G(t)))^2\right] \tag{2}$$

F and G functions are base distributions and shown below:

$$H(t) = F(t)\left[1 + \theta\bar{F}^2(t) + \left(\theta\bar{F}^2(t)\right)^2\right]. \tag{3}$$

$\bar{F}(t) = 1 - F(t)$, $\bar{G}(t) = 1 - G(t)$ are survival functions. If equation (3) prove the characteristics of distribution function for $t \in R$ or not is checked below. Hence,

(i) $\lim_{t\to\infty} H(t) = 1$ ve $\lim_{t\to-\infty} H(t) = 0$. Since $F(t)$ is a distribution function and $\bar{F}(t) = 1 - F(t)$ is a survival function, (i) is clearly proved.

(ii) $H(t)$ is a non-decreasing function when $t \in R$, so if $\forall t_1 < t_2$, then $H(t_1) \leq H(t_2)$. Since the distribution is a continuous distribution, it is enough to show that $\frac{d}{dt}H(t) \geq 0$. Therefore, the sign of derivative will be checked under this transformation with doing $F(t) = u$ transformation for equation (3). Let $\phi(u) = u[1 + \theta(1-u)^2 + \theta^2(1-u)^4]$, then check the sign of derivative function for $\theta \leq 0$. If we reorganize equation (2) under this condition,

$$\phi^{'}(u) = \left[1 + \theta(1-u)^2\right]^2 - \theta(1-u)^2\left[1 + 4\theta u(1-u)\right] \tag{4}$$

is obtained. A lower bound for expression (4) is obtained as below:

$$\phi^{'}(u) \geq \left[1 + \theta(1-u)^2\right]^2 + \left[-\theta(1-u)^2(1+\theta)\right]. \tag{5}$$

Since both sums at the right side of inequality (5) is positive, the sign of derivative is also positive. When expression (5) is organised properly for $\theta > 0$, it is written as below:

$$\phi^{'}(u) = 1 + \theta(1-u)^2\left[1 - 4\theta u(1-u)\right] + \theta^2(1-u)^4 - 2\theta u(1-u).$$

A lower bound for derivative expression with the help of inequalities $1 - 4\theta u(1-u) \geq 1 - \theta$ and $-2\theta u(1-u) \geq -\frac{\theta}{2}$ is

$$\begin{aligned}
\phi^{'}(u) &\geq 1 + \theta(1-u)^2(1-\theta) + \theta^2(1-u)^4 - \frac{\theta}{2} \\
&= \left(1 - \frac{\theta}{2}\right) + \left(\theta(1-u)^2(1-\theta)\right) + \left(\theta^2(1-u)^4\right) \\
&\geq 0.
\end{aligned}$$

Then, since all three sums at the right side of inequality are positively signed, function ϕ is nondecreasing at u, for $\theta > 0$.Thus, function ϕ is nondecreasing at u for $\theta \in [-1, 1]$ so $\frac{d}{dt}H(t) \geq 0$ can be stated.

(iii) $H(t)$ is right continuous, so for $\varepsilon > 0$, $\lim_{\varepsilon \to 0} H(t + \varepsilon) = H(t)$.

Since $F(t)$ is a distribution function, this feature is also ensured. Because of that given features are ensured, expression given in (3) can be stated as distribution function. When we consider AMH distribution family at equation (1) to obtain a second different distribution and knowledge of conditioned distribution function in a similar way, second-order Taylor polynomial of probability $X \leq z$ when $Y \leq y$ is given is found as follows

$$\begin{aligned} P\left(X \leq z | Y \leq y\right) = F(z) \left[1 + \theta(1 - F(z))(1 - G(y))\right. \\ \left. + \left(\theta(1 - F(z))(1 - G(y))\right)^2\right] \end{aligned}$$

if we reconsider this conditioned distribution when $y \to -\infty$,

$$\lim_{y \to -\infty} P(X \leq z | Y \leq y) \cong F(z) \left[1 + \theta \bar{F}(z) + \theta^2 \bar{F}^2(z)\right]$$

is obtained. In order not to do mistake for distribution function which is obtained by (3), this function is shown as

$$H(z) = F(z) \left[1 + \theta \bar{F}(z) + \theta^2 \bar{F}^2(z)\right]. \tag{6}$$

If expression (6) shows the feature of distribution function or not under the condition of $z \in R$ is checked below. Hereunder,

(i) $\lim_{z \to \infty} H(z) = 1$ ve $\lim_{z \to -\infty} H(z) = 0$ is stated. Since $F(z)$ is a distribution function and $\bar{F}(z) = 1 - F(z)$ is a survival function, (i) is clearly ensured.

(ii) $H(z)$ is a non-decreasing function for $z \in R$, so for $\forall z_1 < z_2$, $H(z_1) \leq H(z_2)$. Since the distribution is a continuous, it is enough to show that $\frac{d}{dz}H(z) \geq 0$. Hence,$F(z) = v$ transformation will be done in expression (6) and the sign of derivative will be checked under this transformation. Derivative function is obtained as:

$$\phi'(v) = 1 + \theta - 2v\theta(1 + \theta(1 - v)) + \theta^2(1 - v)^2. \tag{7}$$

If we review the signs of terms of right side of the equation; $\theta(1 - v) \leq 0$ is obtained, and since $-1 \leq \theta \leq 1$, the term $-2v\theta(1 + \theta(1 - v))$ is defined as positive. Since other terms are also positive similarly, the sign of derivative function in equation (7) is stated as positive when $\theta \leq 0$. When expression (7) is properly organized for $\theta > 0$, it can be written as below:

$$\phi'(v) = \theta^2(1 - v)^2 + (1 - v\theta)^2 + (\theta - v\theta)^2 + \theta(1 - \theta).$$

Since $\theta(1 - \theta) > 0$ and $-1 \leq \theta \leq 1$, sums of the right side of the equation is positive and $\phi(v)$ function is a nondecreasing for $\theta > 0$ at v. Therefore, for $\theta \in [-1, 1]$, ϕ function is non-decreasing at v so $\frac{d}{dz}H(z) \geq 0$ is stated.

(iii) $H(z)$ is right continuous, so for $\varepsilon > 0$, $\lim_{\varepsilon \to 0} H(z + \varepsilon) = H(z)$.

It is known that $F(z)$ is a distribution function, so third feature is also ensured. Since all three given features are ensured, expression (6) can be stated as a distribution function. After reorganizing this obtained distribution function,

$$H(z) = (1 + \theta + \theta^2)F(z) - \theta F^2(z) - \theta^2 F(z)(2F(z) - F^2(z)) \tag{8}$$

is obtained. Let's show below that distribution obtained at (8) can be written as a transformed family given by [21]. When we know $u \in [0, 1]$, after transformation $F(z) = u$ is applied and necessary editings are done, it becomes

$$\varphi(u) = (1 + \theta + \theta^2)u - \theta u^2 - \theta^2 u(2u - u^2). \tag{9}$$

Additionally, this function can be written as

$$\begin{aligned}\varphi(u) &= u + u\theta + u\theta^2 - u^2\theta - 2u^2\theta^2 + u^3\theta^2 \\ &= u + u(1 - u)(\theta + \theta^2 - u\theta^2).\end{aligned}$$

Thus, as of $P(u) = (\theta+\theta^2-u\theta^2)$, under the knowledge of the polynomial transformation family that is suggested by [22], obtained (8) can also be regarded as a transformed family. Let function (8) is also a distribution function under the condition of no limitation for parameter θ. The condition of necessity of not being nondecreasing function under which θ parameter range is ensured is determined as below. Value of function (9) at the point of $u = 0$ and $u = 1$ is $\varphi(0) = 0$ and $\varphi(1) = 1$, respectively. There are no limitations for parameter at these points. If we'd like φ function to be nondecreasing in the range of $u \in [0, 1]$, $\varphi^{'} \geq 0$ should be ensured. Derivative function is as below:

$$\varphi^{'} = 3\theta^2u^2 - 4\theta^2u + \theta^2 - 2\theta u + 1 + \theta.$$

For $u = 0$, condition $\varphi^{'}(0) = 1 + \theta + \theta^2 \geq 0$ is ensured for $\forall\theta$, for $u = 1$ $\varphi^{'}(1) = 1 - \theta \geq 0$ condition is ensured when $\theta \leq 1$. For $u \in (0, 1)$, it should be $\varphi^{'} \geq 0$. If we look at the root of the equation for this purpose,

$$u_{1,2} = \frac{\theta + 2\theta^2 \pm \theta\sqrt{(\theta-1)(\theta+2)}}{3\theta^2}$$

is obtained. Here $\Delta = (\theta-1)(\theta+2)$. If $\Delta < 0$, real root cannot be found for u. It is an expected situation because $\varphi^{'} > 0$ or $\varphi^{'} < 0$ is ensured under this condition. Then, $\theta - 1 < 0$ in the previously found condition, in order to be as $\Delta < 0$, it should be $\theta + 2 \geq 0 \Rightarrow \theta \geq -2$. Additionally, $\varphi^{'}$ function is a convex function and it has a minimum point, which is $u^* = \frac{2}{3} + \frac{1}{3\theta}$. If we substitute this optimal point at the function:

$$\varphi^{'}(u^*) = \frac{-(\theta-1)(\theta+2)}{3\theta^2} \geq 0$$

is obtained. For this reason, for any $u \in (0, 1)$, it will be $\varphi^{'}(u) \geq \varphi^{'}(u^*) \geq 0$. Thus, for $-2 \leq \theta \leq 1$. Probability density function of distribution $H(t)$ is obtained as follows:

$$h(t) = f(t)\left[\left(1 + \theta\bar{F}^2(t) + \theta^2\bar{F}^4(t)\right) - 2\theta F(t)\bar{F}(t)\left(1 + 2\theta\bar{F}^2(t)\right)\right],$$

$t \in R, \theta \in [-1, 1]$. Probability density function of distribution $H(z)$ is

$$h(z) = f(z)\left[\left(1 + \theta\bar{F}(z) + \theta^2\bar{F}^2(z)\right) - \theta F(z)\left(1 + 2\theta\bar{F}(z)\right)\right],$$

$z \in R, \theta \in [-2, 1]$.

3 Statistical Properties of Amhbed Distributions

In this title, base distributions of $H(t)$ and $H(z)$ distributions exponentiated, AMHBED1 and AMHBED2 distributions are obtained and then the characteristics of these distributions are revealed. Reason of choosing exponential distribution as the basic distribution of the recommended distribution is when nature events are considered, exponential distributions have a wide range of area of use and it has a feature of memorylessness and simple statistical structure.

3.1 Moment generating function, expected value and variance

For *AMHBED1 distribution:* Moment generating function of AMHBED1 distribution can be written as

$$M_T(u) = \frac{1}{1-\alpha u} - \theta \frac{1}{1-\frac{\alpha}{2}u} + \theta \frac{1}{1-\frac{\alpha}{3}u} - \theta^2 \frac{1}{1-\frac{\alpha}{4}u} + \theta^2 \frac{1}{1-\frac{\alpha}{5}u} \ , \quad u < \frac{1}{\alpha}.$$

Let $w_1 = 1$, $w_2 = -\theta$, $w_3 = \theta$, $w_4 = -\theta^2$, $w_5 = \theta^2$ show coefficients and at the same time, $\frac{1}{1-\alpha u} = M_{Exp(\alpha)}(u)$, $\frac{1}{1-\frac{\alpha}{2}u} = M_{Exp(\frac{\alpha}{2})}(u)$, $\frac{1}{1-\frac{\alpha}{3}u} = M_{Exp(\frac{\alpha}{3})}(u)$, $\frac{1}{1-\frac{\alpha}{4}u} = M_{Exp(\frac{\alpha}{4})}(u)$ and $\frac{1}{1-\frac{\alpha}{5}u} = M_{Exp(\frac{\alpha}{5})}(u)$ is stated. In the light of such information, moment generating function can be expressed as linear combination. Therefore, moment generating function can be defined as

$$M_T(u) = \sum_{j=1}^{5} w_j M_{Exp(\frac{\alpha}{j})}(u).$$

Expected value is found as

$$E(T) = \frac{d}{du}\left[\sum_{j=1}^{5} w_j M_{Exp(\frac{\alpha}{j})}(u)\right]\Bigg|_{u=0} = \sum_{j=1}^{5} w_j \frac{\alpha}{j} = \alpha\left(1 - \frac{\theta}{6} - \frac{\theta^2}{20}\right). \tag{10}$$

If we look at expression (10) to see the relationship between obtained expected value expression and α, which is mean of exponential distribution,

$\theta \in [-1,0) \Rightarrow E(T) > \alpha$, $\theta = 0 \Rightarrow E(T) = \alpha$ and $\theta \in (0,1] \Rightarrow E(T) < \alpha$ is obtained. Second mass moment is $E(T^2) = 2\alpha^2\left(1 - \frac{5\theta}{36} - \frac{9\theta^2}{400}\right)$ and variance is:

$$\begin{aligned} V(T) &= E(T^2) - [E(T)]^2 \\ &= \alpha^2\left(1 + \frac{\theta}{18} + \frac{49\theta^2}{1800} - \frac{\theta^3}{60} - \frac{\theta^4}{400}\right). \end{aligned} \tag{11}$$

If we look at expression (11) in order to see the relationship between obtained variance and α^2, variance of exponential distribution; $\theta \in [-1,0) \Rightarrow V(T) < \alpha^2$, $\theta = 0 \Rightarrow V(T) = \alpha^2$ and $\theta \in (0,1] \Rightarrow V(T) > \alpha^2$ is obtained.
For *AMHBED2 distribution:* Moment generating function of AMHBED2 distribution can be defined as

$$M_Z(u) = (1-\theta)\frac{1}{1-\alpha u} + (\theta - \theta^2)\frac{1}{1-\frac{\alpha}{2}u} + \theta^2\frac{1}{1-\frac{\alpha}{3}u} \quad , \qquad u < \frac{1}{\alpha}.$$

Let $w_1 = (1-\theta)$, $w_2 = (\theta-\theta^2)$, $w_3 = \theta^2$ show coefficients and at the same time, $\frac{1}{1-\alpha u} = M_{Exp(\alpha)}(u)$, $\frac{1}{1-\frac{\alpha}{2}u} = M_{Exp(\frac{\alpha}{2})}(u)$ and $\frac{1}{1-\frac{\alpha}{3}u} = M_{Exp(\frac{\alpha}{3})}(u)$ moment generating function can be expressed as lineer combination in light of this information. Hence, moment generating function can be written as

$$M_Z(u) = \sum_{j=1}^{3} w_j M_{Exp(\frac{\alpha}{j})}(u).$$

Expected value is found as

$$E(Z) = \frac{d}{du}\left[\sum_{j=1}^{3} w_j M_{Exp(\frac{\alpha}{j})}(u)\right]\Bigg|_{u=0} = \sum_{j=1}^{3} w_j \frac{\alpha}{j} = \alpha\left(1 - \frac{\theta}{2} - \frac{\theta^2}{6}\right).$$

According to relationship between expected value expression and α, mean of exponential distribution, $\theta \in [-2,0) \Rightarrow E(Z) > \alpha$, $\theta = 0 \Rightarrow E(Z) = \alpha$ and $\theta \in (0,1] \Rightarrow E(Z) < \alpha$ is obtained. Second mass moment is $E(Z^2) = 2\alpha^2\left(1 - \frac{3\theta}{4} - \frac{5\theta^2}{36}\right)$ and variance is:

$$\begin{aligned} V(Z) &= E(Z^2) - [E(Z)]^2 \\ &= \alpha^2\left(1 - \frac{\theta}{2} - \frac{7\theta^2}{36} - \frac{\theta^3}{6} - \frac{\theta^4}{36}\right). \end{aligned}$$

If relationship between obtained this variance and α^2, variance of exponential distribtion is checked; $\theta \in [-2, 0) \Rightarrow V(Z) > \alpha^2$, $\theta = 0 \Rightarrow V(Z) = \alpha^2$ and $\theta \in (0, 1] \Rightarrow V(Z) < \alpha^2$ is concluded.

4 Parameter Estimations for AMHBED1 and AMHBED2 by ML Method

Likelihood function of obtained AMHBED1 distribution is found as

$$\begin{aligned} L(\alpha, \theta; \underline{t}) &= \prod_{i=1}^{n} h(t_i) \\ &= \frac{1}{\alpha^n} e^{-\frac{\sum_{i=1}^{n} t_i}{\alpha}} \prod_{i=1}^{n}\left[1 - 2\theta e^{\frac{-t_i}{\alpha}} + 3\theta e^{\frac{-2t_i}{\alpha}} - 4\theta^2 e^{\frac{-3t_i}{\alpha}} + 5\theta^2 e^{\frac{-4t_i}{\alpha}}\right] \end{aligned}$$

and the log-likelihood function is stated as

$$\begin{aligned} \log L(\alpha, \theta; \underline{t}) = &-n \log \alpha - \frac{\sum_{i=1}^{n} t_i}{\alpha} \\ &+ \sum_{i=1}^{n} \log\left(1 - 2\theta e^{\frac{-t_i}{\alpha}} + 3\theta e^{\frac{-2t_i}{\alpha}} - 4\theta^2 e^{\frac{-3t_i}{\alpha}} + 5\theta^2 e^{\frac{-4t_i}{\alpha}}\right). \end{aligned}$$

Equating partial derivatives with respect to the interested parameters to zero, (12) and (13) equations are obtained

$$\begin{aligned} \frac{\partial \log L(\alpha, \theta; \underline{t})}{\partial \alpha} = &-n\alpha + \sum_{i=1}^{n} t_i \\ &+ \sum_{i=1}^{n} \frac{t_i e^{\frac{-t_i}{\alpha}}\left(-2\theta + 6\theta e^{\frac{-t_i}{\alpha}} - 12\theta^2 e^{\frac{-2t_i}{\alpha}} + 20\theta^2 e^{\frac{-3t_i}{\alpha}}\right)}{1 - 2\theta e^{\frac{-t_i}{\alpha}} + 3\theta e^{\frac{-2t_i}{\alpha}} - 4\theta^2 e^{\frac{-3t_i}{\alpha}} + 5\theta^2 e^{\frac{-4t_i}{\alpha}}} = 0, \end{aligned} \tag{12}$$

$$\frac{\partial \log L(\alpha,\theta;\underline{\mathrm{t}})}{\partial\theta}=\sum_{i=1}^{n}\frac{e^{\frac{-t_i}{\alpha}}\left(-2+3e^{\frac{-t_i}{\alpha}}-8\theta e^{\frac{-2t_i}{\alpha}}+10\theta e^{\frac{-3t_i}{\alpha}}\right)}{1-2\theta e^{\frac{-t_i}{\alpha}}+3\theta e^{\frac{-2t_i}{\alpha}}-4\theta^2 e^{\frac{-3t_i}{\alpha}}+5\theta^2 e^{\frac{-4t_i}{\alpha}}}=0. \tag{13}$$

Likelihood function of obtained AMHBED2 distribution is found as,

$$\begin{aligned}L(\alpha,\theta;\underline{\mathrm{z}})&=\prod_{i=1}^{n}h(z_i)\\&=\frac{1}{\alpha^n}e^{-\frac{\sum\limits_{i=1}^{n}z_i}{\alpha}}\prod_{i=1}^{n}\left[1-\theta+2\theta e^{\frac{-z_i}{\alpha}}-2\theta^2 e^{\frac{-z_i}{\alpha}}+3\theta^2 e^{\frac{-2z_i}{\alpha}}\right]\end{aligned}$$

log-likelihood function is obtained as,

$$\begin{aligned}\log L(\alpha,\theta;\underline{\mathrm{z}})&=-n\log\alpha-\frac{\sum\limits_{i=1}^{n}z_i}{\alpha}\\&\quad+\sum_{i=1}^{n}\log\left(1-\theta+2\theta e^{\frac{-z_i}{\alpha}}-2\theta^2 e^{\frac{-z_i}{\alpha}}+3\theta^2 e^{\frac{-2z_i}{\alpha}}\right).\end{aligned}$$

By equating partial derivative functions to zero,

$$\begin{aligned}\frac{\partial \log L(\alpha,\theta;\underline{\mathrm{z}})}{\partial\alpha}&=\frac{-n}{\alpha}+\frac{\sum\limits_{i=1}^{n}z_i}{\alpha^2}\\&\quad+\frac{1}{\alpha^2}\sum_{i=1}^{n}\frac{z_i e^{\frac{-z_i}{\alpha}}\left(2\theta-2\theta^2+6\theta^2 e^{\frac{-z_i}{\alpha}}\right)}{1-\theta+2\theta e^{\frac{-z_i}{\alpha}}-2\theta^2 e^{\frac{-z_i}{\alpha}}+3\theta^2 e^{\frac{-2z_i}{\alpha}}}=0,\end{aligned} \tag{14}$$

$$\frac{\partial \log L(\alpha,\theta;\underline{\mathrm{z}})}{\partial\theta}=\sum_{i=1}^{n}\frac{-1+2e^{\frac{-z_i}{\alpha}}-4\theta e^{\frac{-Z_i}{\alpha}}+6\theta e^{\frac{-2z_i}{\alpha}}}{1-\theta+2\theta e^{\frac{-z_i}{\alpha}}-2\theta^2 e^{\frac{-z_i}{\alpha}}+3\theta^2 e^{\frac{-2z_i}{\alpha}}}=0. \tag{15}$$

(14) and (15) equations are concluded. Analytical solutions of ML estimators of parameters can not be obtained with these equations. Maximization problems of log-likelihood functions can be solved with an algorithm and ML estimations can be procured for these distributions.

The "mle" function, which finds the ML estimates by the convergence method based on the initial value determination strategy, is used in MATLAB [14] in order to obtain solutions. A structure containing optimization options are used for this function. This structure contains the constraints and decision variables that make up the objective function. Here, "GradObj" is marked that includes partial derivatives and which is one of the derivative based techniques [15].

4.1 Simulation study

In this section, Monte Carlo simulation study is conducted for parameter estimations of AMHBED1 and AMHBED2 distributions with the help of ML estimation method. Number of repetition is taken as $[|100,000/n|]$, inversely proportional with sample size to both minimize loss of information and prevent loss of time with considering [12] study. $[|.|]$ here expresses greatest integer function.

For AMHBED1 distribution, with parameter values $\alpha = 0.5, 1, 8$ and $\theta = 1, 0.7, 0.3, -0.3, -0.7, -1$, sample sizes are taken as $n = 30$ in small sample, $n = 50$ in middle sample and $n = 100$ for large sample. Results involve mean of parameter estimations, bias and root mean square error (RMSE) and given in Table 1,2. Similar simulation scenario is considered for AMHBED2 distribution and results are tabulated in Table 3,4. for $\alpha = 0.5, 1, 8$ and $\theta = 1, 0.7, 0.3, -1, -1.5, -2$ parameter values. MATLAB is used in simulation study.

Let T be an estimator of any θ parameter, bias and RMSE can be defined as below:

$$\begin{aligned} Bias_\theta(T) &= E_\theta(T) - \theta, \\ RMSE_\theta(T) &= \sqrt{E_\theta(T-\theta)^2} \\ &= \sqrt{Var_\theta(T) + Bias_\theta^2(T)}. \end{aligned}$$

Table 1: The means, biases and RMSEs for the ML estimators of the parameters of AMHBED1 distribution-I

n	Mean$\hat{\theta}$	Bias$\hat{\theta}$	RMSE$\hat{\theta}$	Mean$\hat{\alpha}$	Bias$\hat{\alpha}$	RMSE$\hat{\alpha}$
			$\theta = 1,$	$\alpha = 0.5$		
30	0.8477	-0.1523	0.2778	0.4781	-0.0219	0.1120
50	0.8907	-0.1093	0.1969	0.4836	-0.0164	0.0841
100	0.9274	-0.0726	0.1325	0.4904	-0.0096	0.0615
			$\theta = 1,$	$\alpha = 1$		
30	0.8476	-0.1524	0.2779	0.9557	-0.0443	0.2234
50	0.8906	-0.1094	0.1971	0.9666	-0.0334	0.1677
100	0.9273	-0.0727	0.1327	0.9803	-0.0197	0.1227
			$\theta = 1,$	$\alpha = 8$		
30	0.8465	-0.1535	0.2882	7.6128	-0.3872	1.7110
50	0.8834	-0.1166	0.2062	7.6820	-0.3180	1.3704
100	0.9206	-0.0794	0.1425	7.8004	-0.1996	0.9237

n	Mean$\hat{\theta}$	Bias$\hat{\theta}$	RMSE$\hat{\theta}$	Mean$\hat{\alpha}$	Bias$\hat{\alpha}$	RMSE$\hat{\alpha}$
			$\theta = 0.7,$	$\alpha = 0.5$		
30	0.5791	-0.1209	0.4346	0.4878	-0.0122	0.1093
50	0.6465	-0.0535	0.2987	0.4970	-0.0030	0.0870
100	0.6720	-0.0280	0.1909	0.4970	-0.0030	0.0623
			$\theta = 0.7,$	$\alpha = 1$		
30	0.5789	-0.1211	0.4346	0.9752	-0.0248	0.2181
50	0.6462	-0.0538	0.2988	0.9934	-0.0066	0.1735
100	0.6717	-0.0283	0.1910	0.9935	-0.0065	0.1242
			$\theta = 0.7,$	$\alpha = 8$		
30	0.5929	-0.1071	0.4001	7.8210	-0.1790	1.6818
50	0.6305	-0.0695	0.2929	7.8525	-0.1475	1.3628
100	0.6645	-0.0355	0.2052	7.9262	-0.0738	0.9321

n	Mean$\hat{\theta}$	Bias$\hat{\theta}$	RMSE$\hat{\theta}$	Mean$\hat{\alpha}$	Bias$\hat{\alpha}$	RMSE$\hat{\alpha}$
			$\theta = 0.3,$	$\alpha = 0.5$		
30	0.1482	-0.1518	0.5555	0.4982	-0.0018	0.1073
50	0.1779	-0.1221	0.4523	0.4943	-0.0057	0.0834
100	0.2588	-0.0412	0.2815	0.4997	-0.0003	0.0621
			$\theta = 0.3,$	$\alpha = 1$		
30	0.1479	-0.1521	0.5555	0.9960	-0.0040	0.2138
50	0.1776	-0.1224	0.4523	0.9882	-0.0118	0.1664
100	0.2584	-0.0416	0.2816	0.9991	-0.0009	0.1238
			$\theta = 0.3,$	$\alpha = 8$		
30	0.1469	-0.1531	0.5494	7.8899	-0.1101	1.6555
50	0.1916	-0.1084	0.4415	7.9027	-0.0973	1.2418
100	0.2399	-0.0601	0.3053	7.9356	-0.0644	0.9335

Table 2: The means, biases and RMSEs for the ML estimators of the parameters of AMHBED1 distribution-II

n	Mean$\hat{\theta}$	Bias$\hat{\theta}$	RMSE$\hat{\theta}$	Mean$\hat{\alpha}$	Bias$\hat{\alpha}$	RMSE$\hat{\alpha}$
			$\theta = -0.3,$	$\alpha = 0.5$		
30	-0.3524	-0.0524	0.5077	0.5038	0.0038	0.0964
50	-0.3690	-0.0690	0.4514	0.5001	0.0001	0.0723
100	-0.3605	-0.0605	0.3854	0.5014	0.0014	0.0520
			$\theta = -0.3,$	$\alpha = 1$		
30	-0.3531	-0.0531	0.5076	1.0070	0.0070	0.1920
50	-0.3698	-0.0698	0.4516	0.9997	-0.0003	0.1442
100	-0.3611	-0.0611	0.3853	1.0023	0.0023	0.1036
			$\theta = -0.3,$	$\alpha = 8$		
30	-0.3669	-0.0669	0.5076	7.9987	-0.0013	1.4548
50	-0.3547	-0.0547	0.4496	8.0228	0.0228	1.1316
100	-0.3795	-0.0795	0.3835	7.9651	-0.0349	0.8326

n	Mean$\hat{\theta}$	Bias$\hat{\theta}$	RMSE$\hat{\theta}$	Mean$\hat{\alpha}$	Bias$\hat{\alpha}$	RMSE$\hat{\alpha}$
			$\theta = -0.7,$	$\alpha = 0.5$		
30	-0.6204	0.0796	0.4340	0.5075	0.0075	0.0891
50	-0.6429	0.0571	0.3987	0.5068	0.0068	0.0681
100	-0.6590	0.0410	0.3363	0.5036	0.0036	0.0488
			$\theta = -0.7,$	$\alpha = 1$		
30	-0.6210	0.0790	0.4333	1.0144	0.0144	0.1776
50	-0.6434	0.0566	0.3982	1.0132	0.0132	0.1357
100	-0.6595	0.0405	0.3359	1.0067	0.0067	0.0974
			$\theta = -0.7,$	$\alpha = 8$		
30	-0.6345	0.0655	0.4298	8.1562	0.1562	1.3721
50	-0.6448	0.0552	0.3814	8.0802	0.0802	1.0788
100	-0.6440	0.0560	0.3397	8.0840	0.0840	0.8178

n	Mean$\hat{\theta}$	Bias$\hat{\theta}$	RMSE$\hat{\theta}$	Mean$\hat{\alpha}$	Bias$\hat{\alpha}$	RMSE$\hat{\alpha}$
			$\theta = -1,$	$\alpha = 0.5$		
30	-0.7406	0.2594	0.4679	0.5138	0.0138	0.0866
50	-0.7659	0.2341	0.4182	0.5137	0.0137	0.0698
100	-0.8228	0.1772	0.3306	0.5097	0.0097	0.0490
			$\theta = -1,$	$\alpha = 1$		
30	-0.7410	0.2590	0.4672	1.0272	0.0272	0.1725
50	-0.7671	0.2329	0.4164	1.0269	0.0269	0.1390
100	-0.8241	0.1759	0.3285	1.0188	0.0188	0.0976
			$\theta = -1,$	$\alpha = 8$		
30	-0.7345	0.2655	0.4702	8.2129	0.2129	1.3946
50	-0.7745	0.2255	0.4170	8.1964	0.1964	1.0576
100	-0.8224	0.1776	0.3317	8.1249	0.1249	0.7433

Table 3: The means, biases and RMSEs for the ML estimators of the parameters of AMHBED2 distribution-I

n	**Mean**$\hat{\theta}$	**Bias**$\hat{\theta}$	**RMSE**$\hat{\theta}$	**Mean**$\hat{\alpha}$	**Bias**$\hat{\alpha}$	**RMSE**$\hat{\alpha}$
			$\theta = 1,$	$\alpha = 0.5$		
30	0.9457	-0.0543	0.1472	0.4638	-0.0362	0.1206
50	0.9465	-0.0535	0.1451	0.4655	-0.0345	0.1065
100	0.9617	-0.0383	0.1230	0.4749	-0.0251	0.0836
			$\theta = 1,$	$\alpha = 1$		
30	0.9421	-0.0579	0.1493	0.9215	-0.0785	0.2481
50	0.9528	-0.0472	0.1383	0.9374	-0.0626	0.2056
100	0.9647	-0.0353	0.1203	0.9564	-0.0436	0.1622
			$\theta = 1,$	$\alpha = 8$		
30	0.9465	-0.0535	0.1491	7.4343	-0.5657	1.9454
50	0.9475	-0.0525	0.1455	7.4695	-0.5305	1.7237
100	0.9631	-0.0369	0.1277	7.6317	-0.3683	1.3526

n	**Mean**$\hat{\theta}$	**Bias**$\hat{\theta}$	**RMSE**$\hat{\theta}$	**Mean**$\hat{\alpha}$	**Bias**$\hat{\alpha}$	**RMSE**$\hat{\alpha}$
			$\theta = 0.7,$	$\alpha = 0.5$		
30	0.5887	-0.1113	0.4192	0.4945	-0.0055	0.1683
50	0.6110	-0.0890	0.3575	0.4995	-0.0005	0.1603
100	0.6403	-0.0597	0.2736	0.5044	0.0044	0.1433
			$\theta = 0.7,$	$\alpha = 1$		
30	0.5826	-0.1174	0.4293	0.9885	-0.0115	0.3380
50	0.6068	-0.0932	0.3542	0.9915	-0.0085	0.3141
100	0.6358	-0.0642	0.2605	0.9945	-0.0055	0.2786
			$\theta = 0.7,$	$\alpha = 8$		
30	0.5827	-0.1173	0.4226	7.8404	-0.1596	2.6481
50	0.6056	-0.0944	0.3582	7.9045	-0.0955	2.5203
100	0.6138	-0.0862	0.2813	7.7954	-0.2046	2.2655

n	**Mean**$\hat{\theta}$	**Bias**$\hat{\theta}$	**RMSE**$\hat{\theta}$	**Mean**$\hat{\alpha}$	**Bias**$\hat{\alpha}$	**RMSE**$\hat{\alpha}$
			$\theta = 0.3,$	$\alpha = 0.5$		
30	0.0578	-0.2422	0.6145	0.4912	-0.0088	0.1859
50	0.1416	-0.1584	0.5071	0.5037	0.0037	0.1673
100	0.2243	-0.0757	0.3713	0.5129	0.0129	0.1433
			$\theta = 0.3,$	$\alpha = 1$		
30	0.0399	-0.2601	0.6258	0.9736	-0.0264	0.3604
50	0.1136	-0.1864	0.5280	0.9889	-0.0111	0.3218
100	0.2248	-0.0752	0.3614	1.0240	0.0240	0.2816
			$\theta = 0.3,$	$\alpha = 8$		
30	0.0459	-0.2541	0.6126	7.6989	-0.3011	2.7145
50	0.1211	-0.1789	0.5001	7.7983	-0.2017	2.2648
100	0.1987	-0.1013	0.3612	7.9220	-0.0780	1.9178

Table 4: The means, biases and RMSEs for the ML estimators of the parameters of AMHBED2 distribution-II

n	Mean$\hat{\theta}$	Bias$\hat{\theta}$	RMSE$\hat{\theta}$	Mean$\hat{\alpha}$	Bias$\hat{\alpha}$	RMSE$\hat{\alpha}$
			$\theta = -1,$	$\alpha = 0.5$		
30	-0.9205	0.0795	0.4883	0.5214	0.0214	0.1207
50	-0.9136	0.0864	0.4268	0.5203	0.0203	0.0964
100	-0.9406	-0.0594	0.3180	0.5119	0.0119	0.0696
			$\theta = -1,$	$\alpha = 1$		
30	-0.8831	0.1169	0.5150	1.0575	0.0575	0.2543
50	-0.9146	0.0854	0.4165	1.0358	0.0358	0.1775
100	-0.9439	0.0561	0.3225	1.0234	0.0234	0.1191
			$\theta = -1,$	$\alpha = 8$		
30	-0.9117	0.0883	0.4902	8.3642	0.3642	1.7874
50	-0.9314	0.0686	0.4002	8.2371	0.2371	1.3166
100	-0.9232	0.0768	0.3385	8.1965	0.1965	1.0327

n	Mean$\hat{\theta}$	Bias$\hat{\theta}$	RMSE$\hat{\theta}$	Mean$\hat{\alpha}$	Bias$\hat{\alpha}$	RMSE$\hat{\alpha}$
			$\theta = -1.5,$	$\alpha = 0.5$		
30	-1.4599	0.0401	0.3554	0.5077	0.0077	0.0915
50	-1.4840	0.0160	0.2269	0.5019	0.0019	0.0607
100	-1.4992	0.0008	0.1408	0.5022	0.0022	0.0397
			$\theta = -1.5,$	$\alpha = 1$		
30	-1.4723	0.0277	0.3352	1.0078	0.0078	0.1724
50	-1.4858	0.0142	0.2414	1.0035	0.0035	0.1254
100	-1.4950	0.0050	0.1496	1.0003	0.0003	0.0842
			$\theta = -1.5,$	$\alpha = 8$		
30	-1.4614	0.0386	0.3609	8.1146	0.1146	1.3690
50	-1.4845	0.0155	0.2401	8.0195	0.0195	0.9543
100	-1.4855	0.0145	0.1463	8.0148	0.0148	0.6649

n	Mean$\hat{\theta}$	Bias$\hat{\theta}$	RMSE$\hat{\theta}$	Mean$\hat{\alpha}$	Bias$\hat{\alpha}$	RMSE$\hat{\alpha}$
			$\theta = -2,$	$\alpha = 0.5$		
30	-1.9678	0.0322	0.0821	0.5049	0.0049	0.0531
50	-1.9742	0.0258	0.0588	0.5028	0.0028	0.0402
100	-1.9779	0.0221	0.0410	0.5016	0.0016	0.0282
			$\theta = -2,$	$\alpha = 1$		
30	-1.9654	0.0346	0.0799	1.0116	0.0116	0.1099
50	-1.9695	0.0305	0.0639	1.0027	0.0027	0.0834
100	-1.9736	0.0264	0.0467	1.0035	0.0035	0.0606
			$\theta = -2,$	$\alpha = 8$		
30	-1.9691	0.0309	0.0762	8.1450	0.1450	0.8619
50	-1.9760	0.0240	0.0524	8.1008	0.1008	0.6601
100	-1.9793	0.0207	0.0402	8.1090	0.1090	0.4579

When Table 1, 2, 3, 4 are reviewed, it is clear that estimators work well for all sample sizes . Biases and RMSE values for both estimators decrease with the increase of sample size. ML estimators of α and θ are consistent estimators, in other words, it converges to the true value of parameter as sample size increases. ML estimators are used in Application section since performed simulation study shows that this estimator satisfies estimation procedures.

5 Application

In order to determine the area of use of proposed distributions, we benefit from real data sets in this part. We used K-S (Kolmogorov–Smirnov) goodness of fit test, Akaike information criterion(AIC), corrected Akaike information criterion (AICc) and Bayesian information criterion (BIC) to examine if the data sets fit for distribution and compare them with the other distributions.
K-S goodness of fit test is a reliable test which is well-known and commonly used [11]. Null and alternative hypothesis are defined as below mathematically.

$$H_0 : F(x) = F_0(x)$$
$$H_s : F(x) \neq F_0(x)$$

Distribution function of examined data set here is $F(x)$. $F_0(x)$ can be any distribution function which goodness of fit test is applied. K-S test statistics which is shown by D is written as:

$$D = \sup_x \left| \hat{F_n}(x) - F_0(x) \right| .$$

$\hat{F_n}(x)$ is empirical distribution function here. This equation is defined as the greatest of absolute difference between observed and expected values. Calculated value of D test statistics is compared with $d_{\alpha,n}$ critical value which is given by [19]. If $D > d_{\alpha,n}$, then null hypothesis is rejected. $\alpha = 0.05$ in this study. One of the disadvantages of the K–S test is that when the sample sizes are small, there are situations where the exact probability of a type I error might not be acceptably close to some desired level, because the test statistic, D, has a discrete distribution. Other one is that when prediction of parameters of distribution from sample is needed, K-S test doesn't give credible results [24]. However, if tabulated critical points are used in general, there is no harm in using this test statistic in this sense [2].

Let k be number of parameters, n be number of observations and L be likelihood function for AIC, AICcc and BIC this values are calculated with the help of $AIC = -2\log L + 2k$, $AICc = AIC + \frac{2k(k+1)}{n-k-1}$ and $BIC = -2\log L + k\log L$ equations. MATLAB is used for calculations in related data sets.

5.1 Data set 1

Data set given in Table 5 is river flood data of Wheaton River of Canada and involves 72 independent exceedances over given thresholds between years 1958-1984 in terms of m^3/s. This data set is primarily analysed by [8], and used in a lot of studies like [18], [3] in order to model different distributions.

Table 5: Wheaton River flood data (m^3/s)

1.70	2.20	14.40	1.10	0.40	20.60	5.30	0.70	13.00	12.00
9.30	1.40	18.70	8.50	25.50	11.60	14.10	22.10	1.10	2.50
14.40	1.70	37.60	0.60	2.20	39.00	0.30	15.00	11.00	7.30
22.90	1.70	0.10	1.10	0.60	9.00	1.70	7.00	20.10	0.40
14.10	9.90	10.40	10.70	30.00	3.60	5.60	30.80	13.30	4.20
25.50	3.40	11.90	21.50	27.60	36.40	2.70	64.00	1.50	2.50
27.40	1.00	27.10	20.20	16.80	5.30	9.70	27.50	2.50	27.00
1.90	2.80								

We fit AMHBED1, AMHBED2 and exponential distributions to abovementioned data. The MLE of the parameters, p value, the values of K-S statistic and AIC are given in the Table 6. A graphical impressions of the fitted models is displayed in Figure 1.

Table 6: MLEs of the model parameters and values of goodness of fit statistics for models (Flood data)

Model	Parameter Estimates	K-S	p value	AIC	AICc	BIC
AMHBED1	$\hat{\alpha} = 13.0532$ $\hat{\theta} = 0.5543$	0.1052	0.3768	504.3918	504.5657	508.9451
AMHBED2	$\hat{\alpha} = 9.2190$ $\hat{\theta} = -1.5056$	0.0749	0.7858	499.4530	499.6269	504.0063
Exponential Distribution	$\hat{\alpha} = 12.2042$	0.1422	0.0984	506.2559	506.3131	508.5326

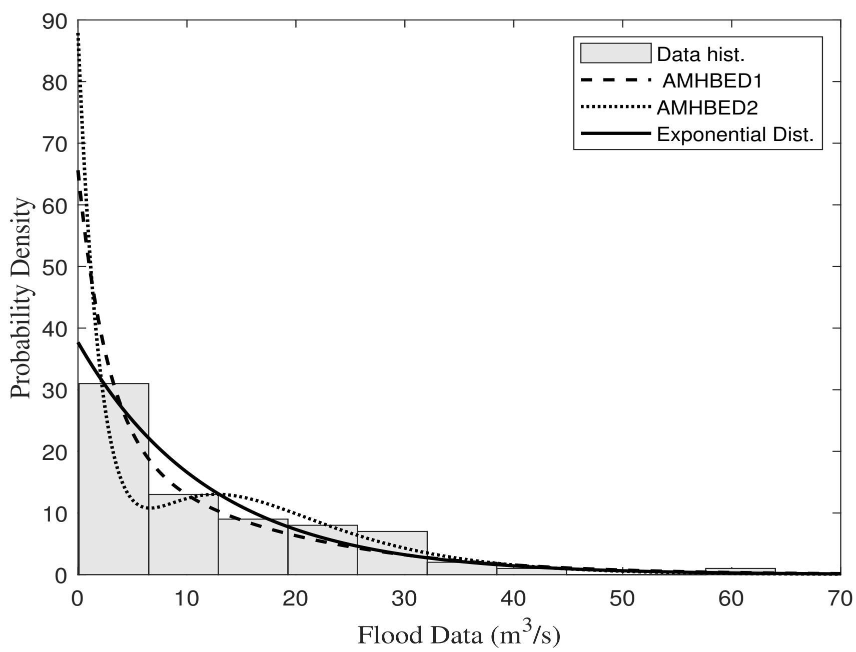

Figure 1: The histogram and the pdfs' of the fitted models for flood data.

It is clear from the Table 6 that based on K-S, AIC, AICc and BIC proposed AMHBED2 model provides a better fit than the other two models to this subject data set. In order to show the performance of AMHBED models for the flood data visually, the histogram is drawn. The relative histogram and the fitted pdf of the models are plotted in Figure 1.

5.2 Data set 2

The following data set were initially analysed by [16]. It consists of 58 recurrence times from insertion of the catheter to infection in terms of days for 58 kidney patients using portable dialysis equipment.

Table 7: Kidney data (day)

8	16	23	22	28	447	318	30	12	24	245	7
9	511	30	53	196	15	154	7	333	141	96	38
536	17	185	177	292	114	15	152	562	402	13	66
39	12	40	201	132	156	34	30	2	25	130	26
27	58	43	152	30	190	119	8	78	63		

Primarily, model fits of new distributions in literature is reached using kidney data set in Table 7. Distributions compared with AMHBED1 in Table 8 are weighted gamma-exponential (WGE) and weighted generalized exponential-exponential (WGEE) distributions proposed by [13]. Parameter estimates, AIC and BIC for WGE and WGEE distributions are included in that study (Table 1). K-S statistics and p values obtained from AMHBED1 and other modeled distributions are shown in Table 8. This data set couldn't be modeled with exponential ($p = 0.0455$) distribution and AMHBED2 (p=0.0191) distribution.

Table 8: MLEs of the model parameters and values of goodness of fit statistics for models (Kidney data)

Model	Parameter Estimates	K-S	p value	AIC	AICc	BIC
AMHBED1	$\hat{\alpha} = 145.8542$ $\hat{\theta} = 0.8626$	0.0951	0.6355	666.2486	666.4668	670.3695
WGE	$\hat{\alpha} = 19.0394$ $\hat{k} = 0.4376$ $\hat{\lambda} = 0.0052$	0.1246	0.3030	668.2001	668.6465	674.3814
WGEE	$\hat{\alpha} = 19.7047$ $\hat{\beta} = 0.3866$ $\hat{\lambda} = 0.0059$	0.1102	0.4494	667.5430	667.9874	673.7243

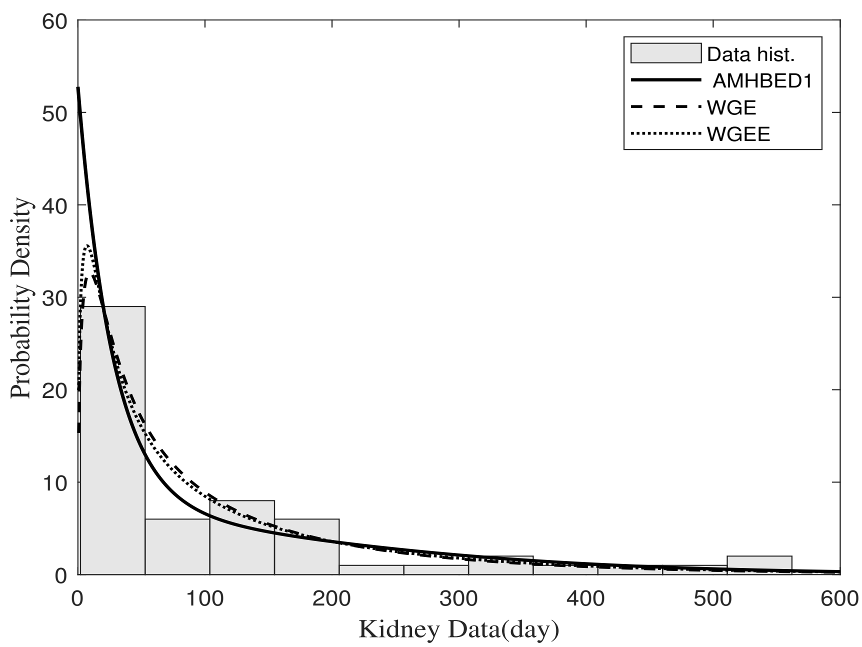

Figure 2: The histogram and the pdfs' of the fitted models for kidney data.

It is clear from the Table 8 that based on K-S, AIC, AICc and BIC proposed AMHBED1 model provides a better fit than the other WGE and WGEE models to this data set. In order to show the performance of this models for the kidney data visually, the histogram is drawn. The relative histogram and the fitted pdf of the models are plotted in Figure 2.

6 Conclusions

Three important results are accomplished in this study. Primarily, a bivariate distribution is reduced to univariate along the diagonal. In other saying, obtaining of new distributions with a new method is provided. Secondarily, interval extension is ensured under the condition that parameter of one of the recommended distribution is distribution function. This extension let the range of distribution parameter extend from $[-1;1]$ to $[-2;1]$. As a result, this extension provides more flexibility and convenience in the field of real data analysis. Lastly, it is seen that obtained distributions are more adequate to model the data sets previously used in literature. In conclusion, these two distributions, that are reduced to univariate starting from bivariate AMH distribution is presented as more convenient and more flexible for some fields like geography and medical compared to well-known classical and popular distributions. Additionally, new distributions can be reproduced by reducing of different bivariate distribution families to univariate with the help of this method. These reproduced distributions can be pathfinders to model data sets of different area of use.

References

[1] K. Adamidis and K.S Loukas, A lifetime distribution with decreasing failure rate. *Statistics and Probability Letters*, 39-1(1998), 35-42.

[2] A.P. Adhikari, *Exponentiality Test Using a Modified Lilliefors Test*, University of Northern Colorado, Colorado (2014).

[3] A. Akinsete, F. Famoye and C. Lee, The Beta-Pareto distribution, *Statistics: A journal of Theoretical and Applied Statistics*, 42-6(2008), 547-563.

[4] M.M. Ali, N.N. Mikhail and M.S. Haq, A class of bivariate distributions including the bivariate logistic, *J. Multivariate Analysis*, (1978), 405-412.

[5] H. M. Almongy, E. M. Almetwally , H. M. Aljohani, A.S. Alghamdi and E. H. Hafez, A new extended rayleigh distribution with applications of COVID-19 data, *Results in Physics*, 23(2021), 2211-3797.

[6] G.R. Aryal and C.P Tsokos, On the transmuted extreme value distribution with application, *Nonlinear Analysis: Theory, Methods and Applications*, 71-12(2009), 1401-1407.

[7] C. Chesneau, L. Tomy and J. Gillariose, A new modified Lindley distribution with properties and applications, *J. Statistics and Management Systems*, 24-7(2021), 1383-1403.

[8] V. Choulakian and M.A. Stephens, Goodness-of-fit tests for the generalized Pareto distribution, *Technometrics*, 43-4(2001), 478-484.

[9] N.I. Fisher, Copulas,*Encyclopedia of Statistical Sciences*, Updated Volume 1, S. Kotz, C. B. Read, D. L. Banks (eds.), John Wiley and Sons, New York, (1997), 159-164.

[10] A. Gul and M. Mohsin, Recurrence relations for moments of order statistics from half logistic-truncated exponential distribution, *Communications in Statistics-Theory and Methods*, 50-17(2021), 3889-3902.

[11] A.K. Gupta and T. Chen, Goodness-of-fit tests for the skew-normal-distribution, *Communications in Statistics-Simulation and Computation*, 30-4(2001), 907–930.

[12] Q. Islam, Q. and M.L. Tiku. Multiple linear regression model under nonnormality, *Communications in Statistics - Theory and Methods*, 33-10(2007), 2443-2467.

[13] A. Mahdavi, Two weighted distributions generated by exponential distribution, *J. Math. Ext.*, 9-1(2015), 1-12.

[14] The MathWorks Inc., *Statistics and Machine Learning Toolbox User's Guide (R2020a)*, Natick, Massachusetts, United States, 11-140, (2020).

[15] The MathWorks Inc., *Global Optimization Toolbox User's Guide (R2013a)*, Natick, Massachusetts, United States, 5-77, (2013).

[16] C. McGilchrist and C. Aisbett, Regression with frailty in survival analysis, *Biometrics*, 47-2(1991), 461-466.

[17] F. Merovci, Transmuted exponentiated exponential distribution, *Mathematical Sciences and Applications E-Notes*, 1-2(2013), 112-122.

[18] F.Merovci and L. Puka, Transmuted Pareto distribution,*ProbStat Forum*, 7(2014), 1-11.

[19] L.H. Miller, Table of percentage points of Kolmogorov statistics, *J. American Statistical Association*, 51-273(1956), 111-121.

[20] K. Pearson, Contributions to the mathematical theory of evolution, *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 185(1894), 71-110.

[21] W.T. Shaw and I.R. Buckley, The alchemy of probability distributions: Beyond gram-charlier and cornish-fisher expansions, and skew-normal or kurtotic-normal distributions, *IMA First Conference on Computational Finance*, Feb 7, (2007).

[22] W.T. Shaw and I.R. Buckley, The alchemy of probability distributions: beyond Gram-Charlier expansions, and a skew-kurtotic-normal distribution from a rank transmutation map,*IMA First Conference on Computational Finance*, Jan 5, (2009).

[23] E. Tanprayoon, U. Tonggumnead and S. Aryuyuen, A new extension of generalized extreme value distribution: Extreme value analysis and return level estimation of the rainfall data, *Trends in Sciences*, 20-1(2023).

[24] R. Wilcox, Kolmogorov–smirnov test, *Encyclopedia of Biostatistics*, John Wiley and Sons Ltd, (2005).

[25] M. Yılmaz A new distribution family constructed by fractional polynomial rank transmutation, *J. Iranian Statistical Society*, 17-2(2018), 141-164.

Feyza Gürer
PhD of Statistics
Department of Statistics
Ankara University
Ankara, Turkey
E-mail: fgunay@ankara.edu.tr

Mehmet Yılmaz
Professor of Statistics
Department of Statistics
Ankara University
Ankara, Turkey
E-mail: mehmetyilmaz@ankara.edu.tr