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Two Weighted Distributions Generated by Exponential Distribution

A. Mahdavi
Vali-e-Asr University of Rafsanjan

Abstract. In this paper, we propose new classes of weighted distributions by incorporating exponential distribution in Azzalini's method. Resulting weighted models generated by exponential distribution are: the weighted gamma-exponential model and the weighted generalized exponenatial-exponential model. The hazard rate function of these distributions has different shapes including increasing, decreasing and unimodal. The moment properties of the proposed distributions are studied. Maximum likelihood estimators (MLEs) of the unknown parameters cannot be obtained in explicit forms and they have to be obtained by solving some numerical methods. Two data sets have been analyzed for illustrative purposes, which show that the proposed models can be used quite effectively in analyzing real data.

AMS Subject Classification: 60E05; 62E10
Keywords and Phrases: Exponential distribution, gamma distribution, generalized exponential distribution, hazard rate function, moment generating function

1. Introduction

Weighted distributions are used to adjust the probabilities of the events and provide an approach to dealing with model specification and data

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interpretation problems. To see more applications of statistical distributions reader is referred to [1,4,9]. Azzalini [3] introduces a shape parameter to a normal distribution by using a weighted function. Afterwards extensive works on introducing shape parameters for other symmetric distributions have been defined and several properties and their inference procedures have been discussed by several authors, see for example [2,5,10]. Recently some authors effort to implement Azzalini's idea for skewed distributions. Gupta and Kundu [6] introduce the new class of weighted exponential (WE) distribution by implementing Azzalini's method to the exponential distribution as follows: a random variable is said to have a weighted exponential distribution, denoted by $WE(\alpha, \lambda)$ if its probability density function (PDF) is given by

$$f_X(x) = \frac{\alpha+1}{\alpha}\lambda e^{-\lambda x}(1-e^{-\lambda\alpha x}), \quad x>0, \alpha>0, \lambda>0. \tag{1}$$

Here α and λ are the shape and scale parameters, respectively. It is observed that the proposed WE distribution has several interesting properties and it can be used quite effectively to analyze skewed data.

Lemma 1.1. *Let U and V be two non-negative continuous independent random variables with PDFs f and g and CDFs (cumulative distribution functions) F and G , respectively. Then for any $\alpha > 0$, the PDF of random variable $X = U$ if $V < \alpha U$ is*

$$f_X(x) = \frac{f(x)G(\alpha x)}{w}, \quad x>0. \tag{2}$$

where $w = \int_0^\infty f(x)G(\alpha x)dx$ and $0 < w < \infty$.

The above general result is useful to construct weighted non-negative models based on Azzalini's idea. For example, the WE distribution is obtained when U and V follows an exponential distributions with mean $1/\lambda$.

The aim of this paper is to introduce weighted distributions based on exponential distribution by taking g as a PDF of exponential distribution with mean $1/\lambda$. In this paper we consider f as the PDF of gamma and generalized exponential distributions. Consequently the following weighted models generated by exponential distribution: the weighted gamma-exponential model (Section 2) and the weighted generalized exponential-exponential model (Section 3). We study moment properties of each of these models and provide graphical illustrations.

The motivation of this study is to introduce two new statistical distributions which extend the gamma, generalized exponential and WE distributions. Their hazard rate functions have different shapes including increasing, decreasing and unimodal. Thus, they can be used to provide a good fit for the real data than well-known distributions.

The rest of the paper is organized as follows: in Section 2 we introduce weighted gamma-exponential distribution. Section 3 presents weighted generalized exponential-exponential distribution. In Section 4 we present two real data analysis results for illustrative purposes. Finally, Section 5 offers some concluding remarks.

2. Weighted Gamma-Exponential Model

In this section, we introduce the definition of weighted gamma distribution based on exponential distribution by taking U as a $gamma(k, \lambda)$ with density function:

$$f(u) = \frac{\lambda^k}{\Gamma(k)} u^{k-1} e^{-\lambda u}, \quad u > 0. \tag{3}$$

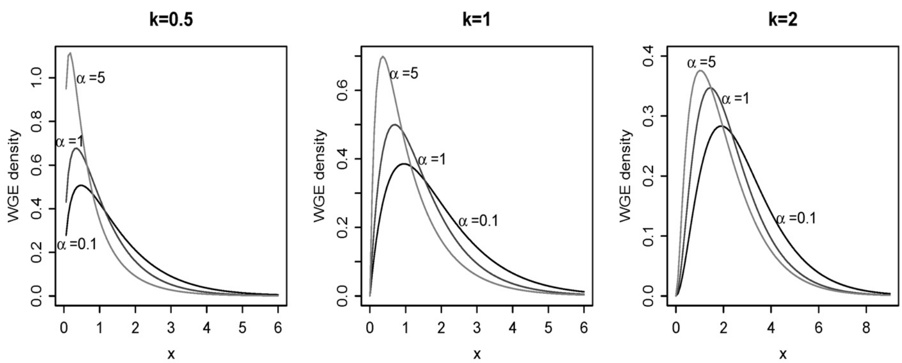

Figure 1. Plots of the WGE density function for fixed scale parameter $\lambda = 1$ and some selected shape parameters.

Then (2) generates a weighted gamma distribution based on exponential distribution, denoted by $WGE(\alpha, k, \lambda)$, with two shape parameters $k > 0, \alpha > 0$ and scale parameter λ, as follows

$$f_X(x) = \frac{\lambda^k x^{k-1} e^{-\lambda x}(1 - e^{-\alpha\lambda x})}{\Gamma(k)[1-(1+\alpha)^{-k}]}, \quad for\ x > 0.$$

and 0 otherwise.

Remark 2.1. *When $k = 1$, then $WGE(\alpha, k = 1, \lambda) = WE(\alpha, \lambda)$. The CDF of WGE distribution is given by*

$$F(x) = \frac{(1+\alpha)^k \gamma(k, \lambda x) - \gamma(k, \lambda(1+\alpha)x)}{(1+\alpha)^k - 1}, \quad x > 0,$$

where $\gamma(a, x) = \int_0^x e^{-t} t^{a-1} dt$ is the incomplete gamma function. Also, the survival reliability function $S(x)$ and the Hazard rate function (HRF), $h(x)$, for WGE distribution are in the following forms

$$S(x) = \frac{(1+\alpha)^k[1-\gamma(k, \lambda x)] + \gamma(k, \lambda(1+\alpha)x) - 1}{(1+\alpha)^k - 1}, \quad x > 0,$$

$$h(x) = \frac{(1+\alpha)^k \lambda^k x^{k-1} e^{-\lambda x}(1 - e^{-\alpha\lambda x})}{\Gamma(k)\{(1+\alpha)^k[1-\gamma(k, \lambda x)] + \gamma(k, \lambda(1+\alpha)x) - 1\}}, \quad x > 0.$$

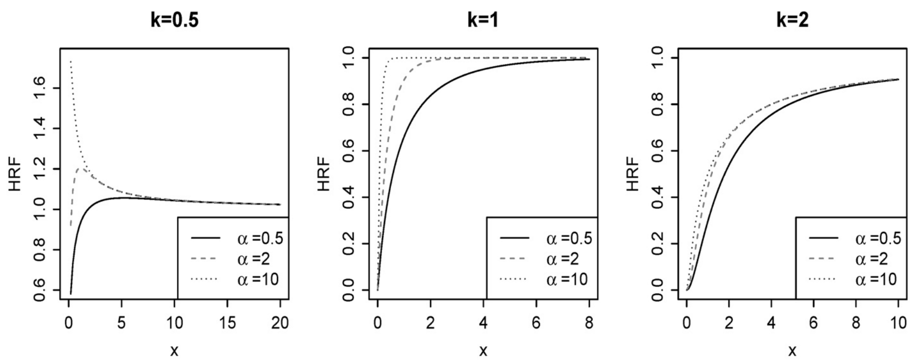

Figure 2. Plots of the WGE hazard rate function for fixed scale parameter $\lambda = 1$ and some selected shape parameters.

Figure 1 shows some of the different shapes of $WGE(\alpha, k, \lambda)$ for selected values of the shape parameters and fixed scale parameter $\lambda = 1$. It is an unimodal density function for various values of the shape parameters. It is easy to show that if $\alpha \to 0$, then WGE converge to $gamma(k+1, \lambda)$ and if $\alpha \to \infty$ then it converge to $gamma(k, \lambda)$. The graphes of the WGE Hazard rate function are provided in Figure 2. The HRF of the WGE distribution can be unimodal, decreasing or increasing depending on the values of its parameters.
The nth moment of WGE can be obtained by

$$E(X^n) = \frac{\Gamma(n+k)[1-(1+\alpha)^{-(n+k)}]}{\Gamma(k)\lambda^n[1-(1+\alpha)^{-k}]},$$

and the MGF of (3) is given by

$$M_X(t) = \frac{\lambda^k\{(\lambda-t)^{-k} - [\lambda(\alpha+1)-t]^{-k}\}}{1-(1+\alpha)^{-k}}, \quad t < \lambda.$$

3. Weighted Generalized Exponential-Exponential Model

Take f to be the PDF of the generalized exponential random variable proposed by Gupta and Kundu [7], denoted by $GE(\beta, \lambda)$, as follows

$$f(u) = \beta\lambda e^{-\lambda u}(1-e^{-\lambda u})^{\beta-1}, \quad u > 0, \lambda > 0, \beta > 0. \tag{4}$$

Then (2) yields to the PDF of weighted generalized exponential based on exponential distribution, denoted by $WGEE(\alpha, \beta, \lambda)$

$$f_X(x) = \frac{\beta\lambda e^{-\lambda x}(1-e^{-\lambda x})^{\beta-1}(1-e^{-\lambda\alpha x})}{1-\beta B(\alpha+1, \beta)}, \quad x > 0,$$

where $B(a, b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt$ is the beta function.

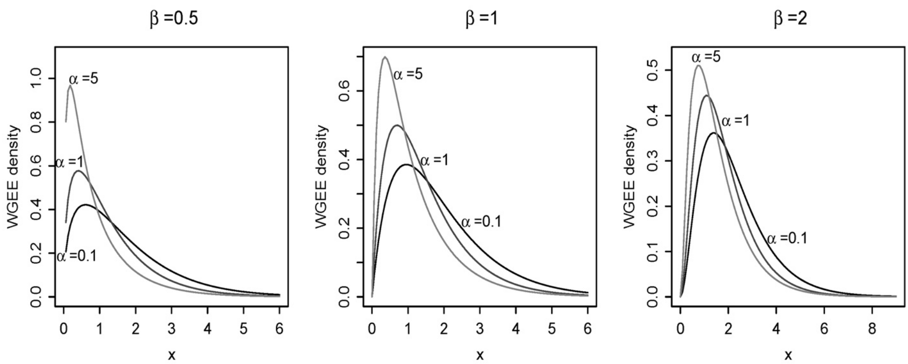

Figure 3. Plots of the WGEE density function for fixed scale parameter $\lambda = 1$ and some selected shape parameters.

Remark 3.1. *When $\beta = 1$, then $WGEE(\alpha, \beta = 1, \lambda) = WE(\alpha, \lambda)$. The CDF of WGEE distribution is given by*

$$F(x) = \frac{1}{1 - \beta B(\alpha + 1, \beta)} \{(1 - e^{-\lambda x})^{\beta} - \beta b_{e^{-\lambda x}}(\alpha + 1, \beta)\}, \quad x > 0,$$

where $b_x(a, b) = \int_x^1 t^{a-1}(1-t)^{b-1} dt$ is the incomplete beta function. The survival reliability function and HRF for WGEE distribution are given by

$$S(x) = \frac{1 - \beta B(\alpha + 1, \beta) - (1 - e^{-\lambda x})^{\beta} + \beta b_{e^{-\lambda x}}(\alpha + 1, \beta)}{1 - \beta B(\alpha + 1, \beta)}, \quad x > 0,$$

$$h(x) = \frac{\beta \lambda e^{-\lambda x}(1 - e^{-\lambda x})^{\beta - 1}(1 - e^{-\lambda \alpha x})}{1 - \beta B(\alpha + 1, \beta) - (1 - e^{-\lambda x})^{\beta} + \beta b_{e^{-\lambda x}}(\alpha + 1, \beta)}, \quad x > 0.$$

The graphs of $WGEE(\alpha, \beta, \lambda)$ density in Figure 3 show that the function is an unimodal function for various values of the shape parameters. In Figure 4, we plotted the HRF of the WGEE distribution for selected values of the shape parameters and fixed scale parameter $\lambda = 1$. It can be unimodal, decreasing or increasing depending on the values of its parameters.

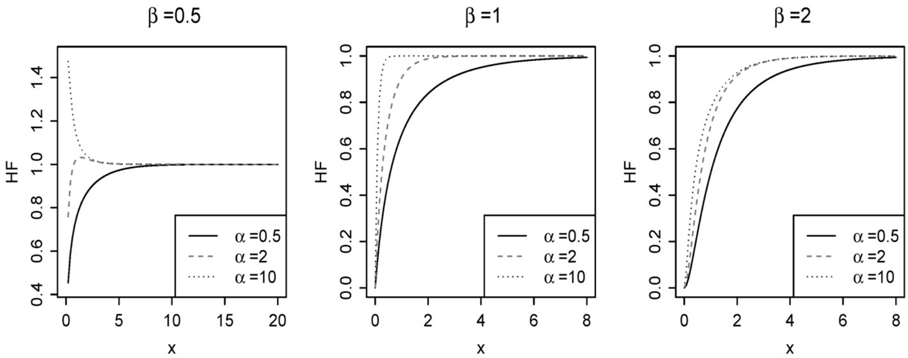

Figure 4. Plots of the WGEE hazard rate function for fixed scale parameter $\lambda = 1$ and some selected shape parameters.

Using the series representation

$$(1 - e^{-\lambda x})^{\beta - 1} = \sum_{i=0}^{\infty} (-1)^i C(\beta - 1, i) e^{-i\lambda x},$$

where $C(\beta - 1, i) = \frac{(\beta-1)...(\beta-i)}{i!}$, the nth moment of the WGEE can be obtained by

$$E(X^n) = \frac{\beta\Gamma(n+1)}{\lambda^n[1 - \beta B(\alpha+1, \beta)]} \sum_{i=0}^{\infty} (-1)^i C(\beta-1, i) \{\frac{1}{(1+i)^{n+1}} - \frac{1}{(1+\alpha+i)^{n+1}}\}.$$

The MGF of WGEE is given by

$$M_X(t) = \frac{\beta\{B(1 - t/\lambda, \beta) - B(\alpha + 1 - t/\lambda, \beta)\}}{1 - \beta B(\alpha+1, \beta)}, \ \ t < \lambda.$$

4. Data Analysis

In this section, we analysis two real data sets to demonstrate the performance of the proposed distributions in practice. For each data set, we compare the results of the fited proposed models with six other models

- WE distribution with PDF (1)
- Gamma distribution with PDF (3)
- GE distribution with PDF (4)
- Gamma exponentiated exponential (GEE) distribution introduced by Ristić and Balakrishnan [13] with PDF

$$f(x) = \frac{\lambda\beta^k}{\Gamma(k)} e^{-\lambda x}(1-e^{-\lambda x})^{\beta-1}(-\log(1-e^{-\lambda x}))^{k-1}, \quad x, \beta, k, \lambda > 0.$$

- Generalized gamma distribution introduced by Stacy [14] with PDF

$$f(x) = \frac{1}{\Gamma(k)} \beta\lambda^{k\beta} x^{k\beta-1} e^{-(\lambda x)^\beta}, \quad x, \beta, k, \lambda > 0.$$

- Beta exponential (BE) distribution introduced by Nadarajah and Kotz [11] with PDF

$$f(x) = \frac{\lambda}{B(a,b)} e^{-\lambda b x}(1 - e^{-\lambda x})^{a-1}, \quad x, a, b, \lambda > 0.$$

Table 1: The MLEs of parameters, AIC and BIC for the kidney data.

The model	MLEs of the parameters	AIC	BIC
WE(α, λ)	39.7157, 0.0087	673.5795	677.7004
gamma(k, λ)	0.7902, 0.0067	671.8474	675.9683
GE(β, λ)	0.7904, 0.0072	671.9841	676.105
GEE(β, k, λ)	1.0103, 1.7459, 0.0038	672.7146	678.896
generalized gamma(β, k, λ)	0.3157, 6.0174 , 4.0233	669.7248	675.9061
BE(a, b, λ)	0.7902, 2.0150, 0.0035	673.8774	680.0588
WGE(α, k, λ)	19.0394, 0.4376, 0.0052	668.2001	674.3814
WGEE(α, β, λ)	19.7047, 0.3866, 0.0059	667.5430	673.7243

To see which one of these models is more appropriate to fit the data set, we calculate the MLEs of parameters, Akaike information criterion (AIC) and Bayesian Information Criterion (BIC). The MLEs of the unknown parameters cannot be obtained explicitly. They have to be obtained by solving some numerical methods, like Newton-Raphson or Gauss-Newton methods or their variants. In this paper we use the optim function from the statistical software R [15] to estimate the unknown parameters.

4.1 Kidney data

The following data set were analysed by McGilchris and Aisbett [8]. The data consists of 58 recurrence times to infection, at the point of insertion of the catheter, for kidney patients using portable dialysis equipment. 8, 16, 23, 22, 28, 447, 318, 30, 12, 24, 245, 7, 9, 511, 30, 53, 196, 15, 154, 7, 333, 141, 96, 38, 536, 17, 185, 177, 292, 114, 15, 152, 562, 402, 13, 66, 39, 12, 40, 201, 132, 156, 34, 30, 2, 25, 130, 26, 27, 58, 43, 152, 30, 190, 119, 8, 78, 63.

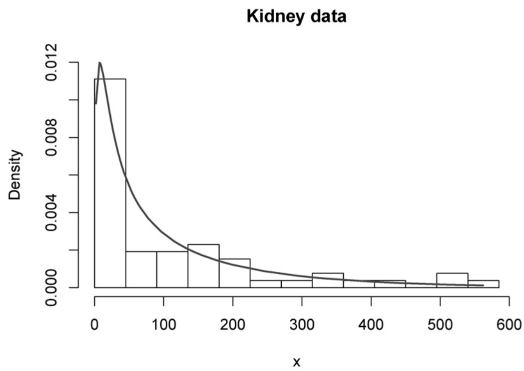

Figure 5. The WGEE distribution seems to fit the kidney data well.

Table 1 shows the results of fitted models. These results indicate that both WGE and WGEE distributions have the lowest AIC and BIC values among those of the fitted models, but the the WGEE distribution had the best fit. Figure 5 displays the histogram of kidney data and the fitted WGEE model. It suggests that WGEE distribution fit the data well.

4.2 Air conditioning failure data

The data set consists of successive failure intervals of air conditioning system of each member of a fleet of 13 Boeing 720 jet planes. The pooled data of 213 observations were first analyzed by Proschan [12]. The results are given in Table 2. As we can see the lowest values of AIC and BIC obtained for WGEE and WGE distributions. Based on the both AIC and BIC the WGEE is the best distribution among all those used here to fit the data set. In order to assess if the model is appropriate, the histogram of the data and the plot of fitted WGEE model are displayed in Figure 6. This figure shows the WGEE distribution is very suitable to this data.

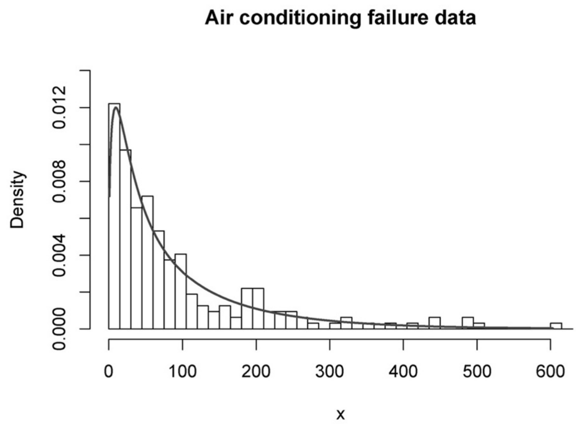

Figure 6. The WGEE distribution seems to fit the Air conditioning failure data well.

Table 2: The MLEs of parameters, AIC and BIC for the Air conditioning failure data.

The model	MLEs of the parameters	AIC	BIC
WE(α, λ)	66.451, 0.011	2359.806	2366.528
gamma(k, λ)	0.9216, 0.010	2360.582	2367.304
GE(β, λ)	0.930, 0.010	2360.808	2367.53
GEE(β, k, λ)	1.224, 2.094, 0.004	2359.281	2369.365
generalized gamma(β, k, λ)	3.571, 0.457, 0.238	2355.020	2365.104
BE(a, b, λ)	0.931, 2.393, 0.004	2362.639	2372.723
WGE(α, k, λ)	12.879, 0.420, 0.007	2353.685	2363.769
WGEE(α, β, λ)	9.997, 0.393, 0.008	2353.662	2363.746

5. Conclusion

We propose two new classes of weighted distributions generated by exponential distribution. The WGE and WGEE models contain WE model as their sub models. The failure rate function of proposed distributions can have the following three forms depending on its shape parameters: unimodal, increasing and decreasing. The moment properties of these distributions are studied. The flexibility of the proposed distributions and increased range of skewness were able to fit and capture features in two real data sets much better than WE and other popular distributions.

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Abbas Mahdavi
Department of Statistics
Instructor of Statistics
Vali-e-Asr University of Rafsanjan
Rafsanjan, Iran
E-mail: a.mahdavi@vru.ac.ir