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Original Research Paper

Local Fractional Yang-Laplace Variational Method for Solving KdV Equation on Cantor Set

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Abstract. Fractional calculus is a branch of classical mathematics, which deals with the generalization of fractional order derivative and integral operator. Recently, a great deal of research has been carried out on the use of fractional calculus to study the phenomena associated with fractal structures and processes. Fractals have a fractional dimension and occur naturally in non-linear and imbalanced phenomena in various forms and contexts. In recent years, various types of derivatives and fractional and fractal calculus have been proposed by many scientists and have been extensively utilized. Measurements are localized in physical processes, and local fractional calculus is a useful tool for solving some type of physical and engineering problems. In this article, we applied

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the local fractional Yang-Laplace variational for solving the local fractional linear and nonlinear KdV equation on a Cantor set within local fractional derivative. we emphasize on the LFYLVM method which is a combination form of local fractional variational iteration method and Yang-Laplace transform. The non-differentiable exact and approximate solutions are obtained for kind of local fractional linear and nonlinear KdV equations. Most of the solutions obtained from this method are obtained in series form that converge rapidly in physical problems. Illustrative examples are included to demonstrate the high accuracy and convergence of this algorithm It is shown that the used method is an efficient and easy method to implement for linear and nonlinear problems arising in science and engineering.

AMS Subject Classification: 35R11 Fractional Partial Differential Equation

Keywords and Phrases: Local fractional KdV equation, Yang-Laplace transform, Local fractional variational iteration method, Cantor sets.

1 Introduction

Fractional calculus is a field of applied mathematics that deals with derivatives and integrals of arbitrary orders and has important applications in physics and engineering problems [18, 26, 8].

In dealing with some domains, it cannot be defined by smooth functions; both the classical and fractional approaches based on Riemann-Liouville (or Caputo) derivatives are unacceptable [31, 16]. In such cases, the local fractional calculus is an efficient tool for modeling and solving these problems. Local fractional calculus is a generalization of differentiation and integration of the functions defined on fractal sets. The theory of various versions of local fractional calculus considered to describe the non-differentiable problems of local fractional PDEs in physics and fundamental science due to the surface and structure of materials, which are so-called fractal [22]. A fractal phenomenon characterized by striking irregularities, and described by a continuously non-differentiable function. After Mandelbrot [22] described the fractals, the fractional and local fractional calculus used to real world problems based on them. For example, Burgers' equation (BE) [33], Parabolic Fokker-Planck equation (PFPE) [4], Oscillator equation (OE) [35], Diffusion equation (DE) [32, 17], and others [27, 16].

As long as the Cantor set is in some sense one of the best examples of a fractal, it is usually used for local fractional calculus.

The mathematical model of shallow water waves, acquired by Boussinesq [6], was rediscovered by Korteweg and de Vries [19]. It is commonly known as Korteweg-de Vries equation (KdV): [19, 7]

$$\frac{\partial}{\partial t}\Phi\left(x,t\right)+\frac{\partial^{3}}{\partial x^{3}}\Phi\left(x,t\right)-6\Phi\left(x,t\right)\frac{\partial}{\partial x}\Phi\left(x,t\right)=0. \tag{1}$$

Recently, the fractional KdV equations have been studied by several authors [24, 1]. Zhang [38] presented a formulation of the time fractional generalized Korteweg-de Vries equation using the Euler -Lagrange variational technique in the Riemann-Liouville derivative sense and found an approximate solitary wave solution.It is imperative to note that the above mentioned works are based on the global fractional calculus of differentiable functions. Yang et al. [34] have derived the local fractional Korteweg-de Vries equation related to fractal waves on shallow water surfaces from the local fractional calculus point of view. The local fractional series expansion method is used to solve a kind of linear local fractional KdV by Zhang et al. [39].

The main model of local fractional KdV equation is the following mathematical model of shallow water waves

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}}-Ru\frac{\partial^{\alpha}u}{\partial x^{\alpha}}+S\frac{\partial^{3\alpha}u}{\partial x^{3\alpha}}+\frac{\partial^{\alpha}u}{\partial x^{\alpha}}=0, \tag{2}$$

where $u\left(x,t\right)$ is a non-differentiable function and R and S are scalers. On the case in equation (2) for $R=1,\ S=1$, we have a particular issue of the local fractional Korteweg-de Vries equation.

In the present work, we investigate the application of local fractional variational iteration transform method (LFVYTM) to solve the local fractional Korteweg-de Vries equation related to the fractal waves on shallow water surfaces from the local fractional calculus point of view [34]. The main advantage of this method is its capability to combine two powerful methods, namely the local fractional variational iteration method and the Yang-Laplace transform for obtaining rapid convergent series for fractional partial differential equations.

The structure of the paper is as follows. In Section 2, we review the concept of local fractional calculus and the theory of the Yang-Laplace transform. The local fractional variational iteration method and its convergence are explained in section 3. Section 4 focuses on analysis of the method which is used. Several illustrative examples are explained in Section5. Finally, Section 6 assigned to the conclusion.

2 Preliminaries [31, 28]

In this part, we remind the fundamental theory of Local fractional operators (LFO).

Definition 2.1. [31, 28] A function $f(x)$ is said to be local fractional continuous at $x = x_0$ if for each positive $\varepsilon > 0$, there exists for $\delta > 0$ such that

$$|f(x) - f(x_0)| < \varepsilon^{\alpha}, \tag{3}$$

whenever $|x - x_0| < \delta$ and $0 < \alpha \leq 1$. It is written as

$$\lim_{x \longrightarrow x_0} f(x) = f(x_0). \tag{4}$$

Definition 2.2. [31, 28] A non-differentiable function $f : \mathbb{R} \to \mathbb{R}$, $x \to f(x)$ is called to be local fractional continuous of order $\alpha, 0 < \alpha \leq 1$, when we have

$$f(x) - f(x_0) = o\left((x - x_0)^{\alpha}\right). \tag{5}$$

Definition 2.3. [31, 28] A function $f(x)$ belongs to the space $C_\alpha[a, b]$ if and only if it can be written as (5) for any $x_0 \in C_\alpha[a, b]$, $0 < \alpha \leq 1$, and we now write $f(x) \in C_\alpha[a, b]$.

Definition 2.4. [31, 28] Suppose that $f(x) \in C_\alpha(a, b)$ and $0 < \alpha \leq 1$. For $\varepsilon > 0$ and
$0 < |x - x_0| < \delta$, the limit

$$D_x^{\alpha} f(x_0) = \left.\frac{d^{\alpha} f(x)}{dx^{\alpha}}\right|_{x=x_0} = \lim_{x \to x_0} \frac{\triangle^{\alpha}\left(f(x) - f(x_0)\right)}{(x - x_0)^{\alpha}} \ , \tag{6}$$

exists and is finite, where $\triangle^{\alpha}(f(x) - f(x_0)) \cong \Gamma(\alpha + 1)(f(x) - f(x_0))$. In this case, $D_x^{\alpha} f(x_0)$ denotes for the local fractional derivative of $f(x)$ of order α at $x = x_0$.

Definition 2.5. [31, 28] Suppose $f(x) \in C_\alpha[a,b]$. Then, the local fractional integral $f(x)$ of order α $(0<\alpha\le 1)$ is defined as follows:

$$ {}_aI_b^{(\alpha)} f(x) = \frac{1}{\Gamma(1+\alpha)} \int_a^b f(t)(dt)^\alpha = \frac{1}{\Gamma(\alpha+1)} \lim_{\triangle t_k \to 0} \sum_{k=0}^{N-1} f(t_k)(\triangle t_k)^\alpha , \tag{7} $$

where $\triangle t_k = t_{k+1} - t_k$ with $t_0 = a < t_1 < \cdots < t_{N-1<} \, t_N = b$.

Definition 2.6. [31, 28] The Mittag-Lefer function defined on the fractal set is given by

$$ E_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{x^{\alpha k}}{\Gamma(1+k\alpha)}, $$

x$\in \mathbb{R}, 0<\alpha\le 1.$(8)

For more details on the relationship between LFOs and fractals, see (for example [4, 17, 27, 29, 15, 20]).

The basic operators of several non-differentiable functions [31, 28] defined on Cantor sets have been listed in Table 1.

Table 1: List of the operators of non-differentiable functions

$\frac{d^\alpha f(x)}{dx^\alpha}$	${}_0I_x^{(\alpha)} f(t)$	Notations
$\frac{d^\alpha}{dx^\alpha} E_\alpha(x^\alpha) = E_\alpha(x^\alpha)$	${}_0I_x^{(\alpha)} E_\alpha(t^\alpha)$ $= E_\alpha(x^\alpha) - 1$	$E_\alpha(x^\alpha)$ $= \sum_{k=0}^\infty \frac{x^{k\alpha}}{\Gamma(1+k\alpha)}$
$\frac{d^\alpha}{dx^\alpha}\left[\frac{x^{(n+1)\alpha}}{\Gamma(1+(n+1)\alpha)}\right] = \frac{x^{n\alpha}}{\Gamma(1+n\alpha)}$	${}_0I_x^{(\alpha)} \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}$ $= \frac{x^{(n+1)\alpha}}{\Gamma(1+(n+1)\alpha)}$	x^α is a Cantor function
$\frac{d^\alpha sin_\alpha(x^\alpha)}{dx^\alpha} = cos_\alpha(x^\alpha)$	${}_0I_x^{(\alpha)} cos_\alpha(t^\alpha)$ $= sin_\alpha(x^\alpha)$	$sin_\alpha(x^\alpha)$ $= \sum_{k=0}^\infty \frac{(-1)^k x^{(2k+1)\alpha}}{\Gamma(1+(2k+1)\alpha)}$
$\frac{d^\alpha cos_\alpha(x^\alpha)}{dx^\alpha} = -sin_\alpha(x^\alpha)$	${}_0I_x^{(\alpha)} sin_\alpha(t^\alpha)$ $= 1 - cos_\alpha(x^\alpha)$	$cos_\alpha(x^\alpha)$ $= \sum_{k=0}^\infty \frac{(-1)^\alpha x^{2k\alpha}}{\Gamma(1+2k\alpha)}$

Definition 2.7. [31, 28] A generalized normed linear space on X of fractional dimension α, is a mapping $\|\|_\alpha : X \to \mathbb{R}^\alpha$, if it satisfies the following properties:

1. $\|x^\alpha\|_\alpha \ge 0$, $\|x^\alpha\|_\alpha = 0$ if and only if $x^\alpha = 0^\alpha$.

2. $\|k^\alpha x^\alpha\|_\alpha = |k^\alpha| \, \|x^\alpha\|_\alpha$.

3. $\|x^\alpha + y^\alpha\|_\alpha \leq \|x^\alpha\|_\alpha + \|y^\alpha\|_\alpha, for\ x^\alpha, y^\alpha \in\ X\ and\ k\ \in\ \mathbb{R}$.

Definition 2.8 (Generalized Banach space)**.** [40]

Let X be a generalized normed linear space. Because of the completeness of X, the Cauchy sequence $\{x_n^\alpha\}_{n=1}^\infty$ is convergent, i.e. ,for every $\varepsilon > 0$, there exists a positive integer N such that [28, 40]

$$\|x_n^\alpha - x_m^\alpha\|_\alpha < \varepsilon^\alpha, \tag{9}$$

whenever $m, n \geq N$. This is equivalent to the requirement that

$$\lim_{m,n \longrightarrow \infty} \|x_n^\alpha - x_m^\alpha\|_\alpha = 0. \tag{10}$$

Theorem 2.9 (Generalized Contraction Mapping Theorem in Generalized Banach Space)**.** *(for proof, see [40])*

Suppose that $T : X \longrightarrow X$ is a map on a generalized Banach space X such that for some

$m \geq 1,\ T^m$ is a contraction, i.e

$$\|T^m (y^\alpha) - T^m (x^\alpha)\|_\alpha \ \leq \beta^\alpha \|x^\alpha - y^\alpha\|_\alpha \ , \tag{11}$$

for all $x^\alpha,\ y^\alpha \in X, \beta \in (0, 1)$, then T has a unique fixed point.

Definition 2.10. Let $\frac{1}{\Gamma(\alpha+1)} \int_0^\infty |f(x)| \, (dx)^\alpha < k < \infty$. The Yang-Laplace transform of $f(x)$ is given by

$$\begin{aligned} L_\alpha \{f(x)\} &= f_s^{L.\alpha}(s) \\ &= \frac{1}{\Gamma(\alpha+1)} \int_0^\infty E_\alpha (-s^\alpha x^\alpha) \, f(x) \, (dx)^\alpha, \ 0 < \alpha \leq 1. \end{aligned} \tag{12}$$

Definition 2.11. The inverse formula of the Yang-Laplace transforms of $f(x)$ is given by

$$\begin{aligned} L_\alpha^{-1} \left\{f_s^{L.\alpha}(s)\right\} &= f(x) \\ &= \frac{1}{(2\pi)^\alpha} \int_{\beta - i\omega}^{\beta + i\omega} E_\alpha (s^\alpha x^\alpha) \, f_s^{L.\alpha}(s) \, (ds)^\alpha \ 0 < \alpha \leq 1 \end{aligned} \tag{13}$$

where $s^\alpha = \beta^\alpha + i^\alpha \omega^\alpha$, fractal imaginary unit i^α and $Re(s) = \beta > 0$.

Some Basic Properties of Local Fractional Laplace Transform [31, 28]

Let $L_\alpha \{f(x)\} = f_s^{L,\alpha}(s)$ and $L_\alpha \{g(x)\} = g_s^{L,\alpha}$, then we have the following formula

$$L_\alpha \{af(x) + bg(x)\} = af_s^{L,\alpha}(s) + bg_s^{L,\alpha} \tag{14}$$

$$L_\alpha \{E_\alpha (c^\alpha x^\alpha) f(x)\} = f_s^{L,\alpha}(s - c) \tag{15}$$

$$L_\alpha \left\{f^{(\alpha)}(x)\right\} = s^\alpha f_s^{L,\alpha}(s) - f(0) \tag{16}$$

$$L_\alpha \{E_\alpha (a^\alpha x^\alpha)\} = \frac{1}{s^\alpha - a^\alpha} \tag{17}$$

$$L_\alpha \left\{x^{k\alpha}\right\} = \frac{\Gamma(1 + k\alpha)}{s^{(k+1)\alpha}} \tag{18}$$

For more explanations and examples see ([31]).

3 Local Fractional Variational Iteration Method [31, 28]

In this section, the variational iteration method of the local fractional operator briefly introduce. In 1998, the variational iteration method has been adopted to solve fractional differential equations for the first time [12]. The variational iteration method of the local fractional operator was employed to solve the local fractional partial differential equations [27, 29, 13, 30, 24, 11].

Now, we consider a local fractional variational principle [31, 28] as follows

$$I(y) = {}_aI_b^{(\alpha)} f(x, y(x), y^\alpha(x)), \tag{19}$$

where $y^\alpha(x)$ is local fractional differential operator on $a \leq x \leq b$.

The stationary condition of Eq. (19) reads [31, 28]

$$\frac{\partial f}{\partial y} - \frac{d^\alpha}{dx^\alpha}\left(\frac{\partial f}{\partial y^{(\alpha)}}\right) = 0 \tag{20}$$

Eq. (20) is useful for the identification of the Lagrange multiplier in the local fractional variational iteration method.

We suppose a public Nonlinear local fractional partial differential equation:

$$L_{(m\alpha)}(x,t)+R_\alpha u(x,t)+N_\alpha u(x,t)=f(x,t),\ t>0,\ x\in\mathbb{R},\ 0<\alpha\leq 1, \tag{21}$$

Where $L_{(m\alpha)}=\frac{\partial^{(m\alpha)}}{\partial t^{(m\alpha)}}$, $m\in N$ and R_α is a linear local fractional operator, N_α indicates the general Nonlinear local fractional operator, and $f(x,t)$ is the source term.

Let us suppose the Local fractional variational iteration algorithm given in [31, 18, 37]

$$\begin{aligned} u_{n+1}(x,t) = u_n + \int_0^t \Big\{ \frac{\lambda^\alpha}{\Gamma(\alpha+1)} (L_{(m\alpha)}u_n(x,\tau) \\ +R_\alpha u_n(x,\tau)+N_\alpha u_n(x,\tau)-f(x,\tau))\Big\}(d\tau)^\alpha. \end{aligned} \tag{22}$$

We can write the local fractional correction functional as

$$\begin{aligned} u_{n+1}(x,t)=u_n(x,t)+\int_0^t \{\frac{\lambda^\alpha}{\Gamma(\alpha+1)}(L_{(m\alpha)}u_n(x,\tau)+R_\alpha\tilde{u}_n(x,\tau) \\ +N_\alpha\tilde{u}_n(x,\tau)-f(x,\tau))\}(d\tau)^\alpha, \end{aligned} \tag{23}$$

where $\tilde{u}_n$ is a restricted local fractional variational and λ^α is a fractal Lagrange multiplier. The determination of λ^α requires stationary conditions of the functional, that is $\delta^\alpha\tilde{u}_n=0$ [31, 28, 37].

Extremizing the variation of the correction functional (23) leads to the Lagrangian multiplier λ^α as follows:

$$\lambda^\alpha=(-1)^m\frac{(\tau-t)^{(m-1)\alpha}}{\Gamma(1+(m-1)\alpha)} \tag{24}$$

Substitute, (24) in (23), we get

$$\begin{aligned} u_{n+1}(x,t)=u_n(x,t)+{}_0I_t^{(\alpha)}\Big\{\frac{(-1)^m(\tau-t)^{(m-1)\alpha}}{\Gamma(1+(m-1)\alpha)}\big(L_{(m\alpha)}u_n(x,\tau) \\ +R_\alpha\tilde{u}_n(x,\tau)+N_\alpha\tilde{u}_n(x,\tau)-f(x,\tau))\} \end{aligned} \tag{25}$$

The initial iteration $u_0(x,t)$ can be used as the initial value $u(x,0)$. In (25), we let $\to \infty$, to get

$$u(x,t) = \lim_{n\to\infty} u_n(x,t). \tag{26}$$

Convergence of The local fractional variational method [37]

Yang and Zhang [37] have proved Convergence of the local fractional variational method based on the theory of the generalized Banach space.

Assume that X is a generalized Banach space, $T : X \to X$ is a Nonlinear mapping, and suppose that

$$\|T(u) - T(\overline{u})\|_\alpha \;\le\; \beta^\alpha \|u - \overline{u}\|_\alpha. \tag{27}$$

For some constant $0 < \beta < 1$. Then T has a unique fixed point.

Furthermore, we suppose that the sequence is

$$u_{n+1} = T(u_n). \tag{28}$$

According to the above subject, we consider the Nonlinear mapping
$u_{n+1}(x,t) = T(u_n(x,t)) = u_n(x,t) + {}_0I_t^{(\alpha)}\left\{\frac{(-1)^m(\tau-t)^{(m-1)\alpha}}{\Gamma(1+(m-1)\alpha)}\left(L_{(m\alpha)}u_n(x,\tau)\right.\right.$
$\left.\left.+R_\alpha\tilde{u}_n(x,\tau) + N_\alpha\tilde{u}_n(x,\tau) - f(x,\tau)\right)\right\}.$
To discuss the Nonlinear mapping, we let
$u_0 \in X,\ T(u_0) = u_1,\ u_2 = T(u_1) = T^2(u_0),$
and, in general, $u_{n+1} = T(u_n) = T^{n+1}(u_0).$
We show that the sequence $\{u_n\}_{n=1}^{\infty}$ is a Cauchy sequence.
Let $n < m$, then
$\|T(u_m) - T(u_n)\|_\alpha = \|T^m(u_0) - T^n(u_0)\|_\alpha$
$\le \beta^{n\alpha}\|u_{m-n} - u_0\|_\alpha \le \beta^{n\alpha}\left(\|u_1 - u_0\|_\alpha + \|u_2 - u_1\|_\alpha + \cdots + \|u_{m-n} - u_{m-n-1}\|_\alpha\right)$
$\le \beta^{n\alpha}\|u_1 - u_0\|_\alpha\left(1^\alpha + \beta^\alpha + \cdots + \beta^{(m-n)\alpha}\right)$
$\le \frac{\beta^{n\alpha}}{1^\alpha - \beta^\alpha}\|u_1 - u_0\|_\alpha.$
Hence, we have $u_{n+1} = T(u_n)$, with an arbitrary choice of u_0 converges to the fixed point of T and
$\|u_{m+1} - u_{n+1}\|_\alpha = \|T(u_m) - T(u_n)\|_\alpha \le \frac{\beta^{n\alpha}}{1^\alpha - \beta^\alpha}\|u_1 - u_0\|_\alpha.$
Here we have $0 < \beta < 1$ and such that
$\lim_{m,n\to\infty}\|u_{m+1} - u_{n+1}\|_\alpha \le \lim_{m,n\to\infty}\frac{\beta^{n\alpha}}{1^\alpha - \beta^\alpha}\|u_1 - u_0\|_\alpha = 0.$
Hence, we have

$\mathrm{u}(x,t) = \lim_{n\to\infty} u_n(x,t)$.
The method creates the solution in the form of rapidly convergent series that lead to the exact solution in linear local fractional differential equations and an efficient numerical solution with high accuracy for the Nonlinear equations.

4 Analysis of Method

We consider a general Nonlinear local fractional partial differential equation:

$$L_\alpha u(x,t) + R_\alpha u(x,t) + N_\alpha u(x,t) = f(x,t)\,,\ t>0,\ x\in\mathbb{R},\ 0<\alpha\le 1, \tag{29}$$

Where $L_\alpha = \frac{\partial^\alpha}{\partial t^\alpha}$, R_α is a linear local fractional operator; N_α represents the general Nonlinear local fractional operator and $f(x,t)$ is the source term.

Applying the Yang-Laplace transform (denoted in this paper by $\mathcal{L}_\alpha$) on both sides of (29), we get

$$\mathcal{L}_\alpha\{L_\alpha u(x,t)\} + \mathcal{L}_\alpha\{R_\alpha u(x,t)\} + \mathcal{L}_\alpha\{N_\alpha u(x,t)\} = \mathcal{L}_\alpha\{f(x,t)\}\,. \tag{30}$$

Using the property of the Yang-Laplace transform, we have

$$s^\alpha \mathcal{L}_\alpha\{u(x,t)\} - u(x,0) = \mathcal{L}_\alpha\{f(x,t)\} - \mathcal{L}_\alpha\{R_\alpha u(x,t)\} - \mathcal{L}_\alpha\{N_\alpha u(x,t)\}\,, \tag{31}$$

or

$$\begin{aligned}\mathcal{L}_\alpha\{u(x,t)\} =& \frac{1}{s^\alpha}u(x,0) + \frac{1}{s^\alpha}(\mathcal{L}_\alpha\{f(x,t)\} \\ &- \mathcal{L}_\alpha\{R_\alpha u(x,t)\} - \mathcal{L}_\alpha\{N_\alpha u(x,t)\}).\end{aligned} \tag{32}$$

Operating with the Yang-Laplace inverse on both sides of (31) yield,

$$u(x,t) = u(x,0) + \mathcal{L}_\alpha^{-1}\left(\frac{1}{s^\alpha}\mathcal{L}_\alpha\{f(x,t) - R_\alpha u(x,t) - N_\alpha u(x,t)\}\right). \tag{33}$$

Taking derivative $\frac{\partial^\alpha}{\partial t^\alpha}$ from sides of (33), we have

$$u_t^\alpha(x,t) - \frac{\partial^\alpha}{\partial t^\alpha}\mathcal{L}_\alpha^{-1}\left(\frac{1}{s^\alpha}\mathcal{L}_\alpha\{f(x,t) - R_\alpha u(x,t) - N_\alpha u(x,t)\}\right) = 0. \tag{34}$$

By the correction function of the variational method

$$\begin{aligned} u_{n+1}(x,t) =& u_n(x,t) - \frac{1}{\Gamma(\alpha+1)} \int_0^t ((u_n)_\xi^\alpha (x,\xi) \\ & - \frac{\partial^\alpha}{\partial \xi^\alpha} \mathcal{L}_\alpha^{-1} (\frac{1}{s^\alpha} \mathcal{L}_\alpha \{ f(x,\xi) - R_\alpha u_n(x,\xi) \\ & - N_\alpha u_n(x,\xi) \}))(d\xi)^\alpha. \end{aligned} \tag{35}$$

Finally, the solution $u(x,t)$ is given by

$$u(x,t) = \lim_{n\to\infty} u_n(x,t). \tag{36}$$

5 Illustrative Examples

In this part two examples for the KdV equation on Cantor sets are offered to demonstrate the convenience and the performance of the above method.

Example 5.1. We consider the following linear KdV equation on Cantor sets with the local fractional operator

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \frac{\partial^{3\alpha} u}{\partial x^{3\alpha}} + \frac{\partial^\alpha u}{\partial x^\alpha} = 0, \qquad u(x,0) = E_\alpha(x^\alpha). \tag{37}$$

Now we use the algorithm of Yang-Laplace transform on equation (37), we have

$$s^\alpha \mathcal{L}_\alpha \{u(x,t)\} - u(x,0) = -\mathcal{L}_\alpha \left\{ \frac{\partial^{3\alpha} u(x,t)}{\partial x^{3\alpha}} + \frac{\partial^\alpha u(x,t)}{\partial x^\alpha} \right\}. \tag{38}$$

Using the given initial condition on Eq. (38), we obtain

$$\mathcal{L}_\alpha \{u(x,t)\} = \frac{1}{s^\alpha} E_\alpha(x^\alpha) - \frac{1}{s^\alpha} \mathcal{L}_\alpha \left\{ \frac{\partial^{3\alpha} u(x,t)}{\partial x^{3\alpha}} + \frac{\partial^\alpha u(x,t)}{\partial x^\alpha} \right\}. \tag{39}$$

Next apply the inverse Laplace transform to Eq. (39), we get

$$u(x,t) = E_\alpha(x,t) - \mathcal{L}_\alpha^{-1} \left\{ \frac{1}{s^\alpha} \mathcal{L}_\alpha \left\{ \frac{\partial^{3\alpha} u(x,t)}{\partial x^\alpha} + \frac{\partial^\alpha u(x,t)}{\partial x^\alpha} \right\} \right\} \tag{40}$$

Taking derivative $\frac{\partial^\alpha}{\partial t^\alpha}$ from both sides of (40), we find

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}+\frac{\partial^\alpha}{\partial t^\alpha}\left[\mathcal{L}_\alpha^{-1}\left\{\frac{1}{s^\alpha}\mathcal{L}_\alpha\left\{\frac{\partial^{3\alpha}\ u(x,t)}{\partial x^{3\alpha}}+\frac{\partial^\alpha u(x,t)}{\partial x^\alpha}\right\}\right\}\right]=0. \quad (41)$$

The correction function using (41) in to (25), yields,

$$\begin{aligned}u_{n+1}\left(x,t\right)=&u_n\left(x,t\right)-\frac{1}{\Gamma\left(\alpha+1\right)}\int_0^t[(u_n)_\xi^\alpha\left(x,\xi\right)\\&+\frac{\partial^\alpha}{\partial\xi^\alpha}\{\mathcal{L}_\alpha^{-1}(\frac{1}{s^\alpha}\mathcal{L}_\alpha\{\frac{\partial^{3\alpha}u_n\left(x,\xi\right)}{\partial x^{3\alpha}}+\\&\frac{\partial^\alpha u_n\left(x,\xi\right)}{\partial x^\alpha}\})\}](d\xi)^\alpha.\end{aligned} \quad (42)$$

We can use the initial condition in (42), $u_0\left(x,t\right)=u\left(x,0\right)=E_\alpha\left(x^\alpha\right)$. Now by using this selection into the correction function gives the following successive approximations

$$u_0\left(x,t\right)=E_\alpha\left(x^\alpha\right) \quad (43)$$

$$\begin{aligned}u_1\left(x,t\right)=&u_0\left(x,t\right)-\frac{1}{\Gamma\left(\alpha+1\right)}\int_0^t[(u_0)_\xi^\alpha\left(x,\xi\right)\\&+\frac{\partial^\alpha}{\partial\xi^\alpha}\left\{\mathcal{L}_\alpha^{-1}\left(\frac{1}{s^\alpha}\mathcal{L}_\alpha\left\{\frac{\partial^{3\alpha}u_0\left(x,\xi\right)}{\partial x^{3\alpha}}+\frac{\partial^\alpha u_0\left(x,\xi\right)}{\partial x^\alpha}\right\}\right)\right\}](d\xi)^\alpha\end{aligned}$$

Therefore, we select

$$\begin{aligned}u_1\left(x,t\right)&=E_\alpha\left(x^\alpha\right)-2\frac{E_\alpha\left(x^\alpha\right)}{\Gamma\left(\alpha+1\right)}\int_0^t\frac{\partial^\alpha}{\partial\xi^\alpha}\left(\mathcal{L}_\alpha^{-1}\left(\frac{1}{s^{2\alpha}}\right)\right)(d\xi)^\alpha\\&=E_\alpha\left(x^\alpha\right)\left(1-\frac{2t^\alpha}{\Gamma\left(\alpha+1\right)}\right)\end{aligned} \quad (44)$$

$$
\begin{aligned}
u_2(x,t) &= u_1(x,t) - \frac{1}{\Gamma(\alpha+1)}\int_0^t [(u_1)_\xi^\alpha(x,\xi) \\
&\quad + \frac{\partial^\alpha}{\partial \xi^\alpha}\{\mathcal{L}_\alpha^{-1}(\frac{1}{s^\alpha}\mathcal{L}_\alpha\{\frac{\partial^{3\alpha}u_1(x,\xi)}{\partial x^{3\alpha}} + \frac{\partial^\alpha u_1(x,\xi)}{\partial x^\alpha}\})\}](d\xi)^\alpha \\
&= E_\alpha(x^\alpha)[(1-\frac{2t^\alpha}{\Gamma(\alpha+1)}) + \frac{2}{\Gamma(\alpha+1)}\int_0^t (1- \\
&\quad \frac{\partial^\alpha}{\partial \xi^\alpha}\{\mathcal{L}_\alpha^{-1}(\frac{1}{s^\alpha}\mathcal{L}_\alpha(1-\frac{2\xi^\alpha}{\Gamma(\alpha+1)}))\})(d\xi)^\alpha] \\
&= E_\alpha(x^\alpha)[1-\frac{2t^\alpha}{\Gamma(\alpha+1)} + \frac{2}{\Gamma(\alpha+1)}\int_0^t (1- \\
&\quad \frac{\partial^\alpha}{\partial \xi^\alpha}[\mathcal{L}_\alpha^{-1}(\frac{1}{s^{2\alpha}}-\frac{2}{s^{3\alpha}})])(d\xi)^\alpha] \\
&= E_\alpha(x^\alpha)[1-\frac{2t^\alpha}{\Gamma(\alpha+1)} + \frac{2}{\Gamma(\alpha+1)}\int_0^t (1- \\
&\quad \frac{\partial^\alpha}{\partial \xi^\alpha}[\frac{\xi^\alpha}{\Gamma(\alpha+1)}-\frac{2\xi^{2\alpha}}{\Gamma(2\alpha+1)}])(d\xi)^\alpha].
\end{aligned}
$$

$$
u_2(x,t) = E_\alpha(x^\alpha)\left[1-\frac{2t^\alpha}{\Gamma(\alpha+1)} + \frac{4t^{2\alpha}}{\Gamma(1+2\alpha)}\right]. \tag{45}
$$

By continuing this process, we find the non-differentiable solution of Eq.(37) in the following

$$
u_n(x,t) = E_\alpha(x^\alpha)\sum_{k=0}^{n}(-2)^k\frac{t^{k\alpha}}{\Gamma(1+k\alpha)}. \tag{46}
$$

Finally, the answer is

$$
\begin{aligned}
u(x,t) &= \lim_{n\to\infty} u_n(x,t) = E_\alpha(x^\alpha)\sum_{k=0}^{\infty}(-2)^k\frac{t^{k\alpha}}{\Gamma(1+k\alpha)} \\
&= E_\alpha(x^\alpha)\,E_\alpha(-2t^\alpha) = E_\alpha((x-2t)^\alpha)),
\end{aligned} \tag{47}
$$

which is the exact solution of Eq.(37).

Example 5.2. Let us consider the following Nonlinear Kdv equation on Cantor sets with the local fractional operator

$$
\frac{\partial^\alpha u}{\partial t^\alpha} - u\frac{\partial^\alpha u}{\partial x^\alpha} + \frac{\partial^{3\alpha}u}{\partial x^{3\alpha}} + \frac{\partial^\alpha u}{\partial x^\alpha} = 0, \tag{48}
$$

and subject to the initial condition

$$u\left(x,0\right)= \frac{x^{\alpha}}{\Gamma\left(\alpha+1\right)}. \tag{49}$$

Applying the algorithm of Yang-Laplace transform on equation (48), we have

$$s^{\alpha}\mathcal{L}\left\{u(x,t)\right\}-u\left(x,0\right)=\mathcal{L}_{\alpha}\left\{u\frac{\partial^{\alpha}u\left(x,t\right)}{\partial x^{\alpha}}-\frac{\partial^{3\alpha}u\left(x,t\right)}{\partial x^{3\alpha}}-\frac{\partial^{\alpha}u\left(x,t\right)}{\partial x^{\alpha}}\right\}. \tag{50}$$

Using the given initial condition on Eq. (50), we get

$$\begin{aligned}\mathcal{L}_{\alpha}\left\{u\left(x,t\right)\right\}&=\frac{1}{s^{\alpha}}\frac{x^{\alpha}}{\Gamma\left(\alpha+1\right)}\\ &+\frac{1}{s^{\alpha}}\mathcal{L}_{\alpha}\left\{u\left(x,t\right)\frac{\partial^{\alpha}u\left(x,t\right)}{\partial x^{\alpha}}-\frac{\partial^{3\alpha}u\left(x,t\right)}{\partial x^{3\alpha}}-\frac{\partial^{\alpha}u\left(x,t\right)}{\partial x^{\alpha}}\right\}.\end{aligned} \tag{51}$$

Next, apply the inverse Laplace transform to Eq. (51), yields

$$\begin{aligned}u\left(x,t\right)&=\frac{x^{\alpha}}{\Gamma\left(\alpha+1\right)}\\ &+\mathcal{L}_{\alpha}^{-1}\left\{\frac{1}{s^{\alpha}}\mathcal{L}_{\alpha}\left[u\left(x,t\right)\frac{\partial^{\alpha}u\left(x,t\right)}{\partial x^{\alpha}}-\frac{\partial^{3\alpha}u\left(x,t\right)}{\partial x^{3\alpha}}-\frac{\partial^{\alpha}u\left(x,t\right)}{\partial x^{\alpha}}\right]\right\}.\end{aligned} \tag{52}$$

Taking derivative $\frac{\partial^{\alpha}}{\partial t^{\alpha}}$ from both sides of (52), we find

$$\begin{aligned}&\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}-\frac{\partial^{\alpha}}{\partial t^{\alpha}}[\mathcal{L}_{\alpha}^{-1}\{\frac{1}{s^{\alpha}}\mathcal{L}_{\alpha}[u(x,t)\frac{\partial^{\alpha}u(x,t)}{\partial x^{\alpha}}\\ &-\frac{\partial^{3\alpha}u(x,t)}{\partial x^{3\alpha}}-\frac{\partial^{\alpha}u(x,t)}{\partial x^{\alpha}}]\}]=0.\end{aligned} \tag{53}$$

The correction function using (53) into (25), yields,

$$\begin{aligned}u_{n+1}\left(x,t\right)=u_{n}\left(x,t\right)-\frac{1}{\Gamma\left(\alpha+1\right)}\int_{0}^{t}&\{\left(u_{n}\right)_{\xi}^{\alpha}\left(x,\xi\right)\\ &-\frac{\partial^{\alpha}}{\partial\xi^{\alpha}}[\mathcal{L}_{\alpha}^{-1}(\frac{1}{s^{\alpha}}\mathcal{L}_{\alpha}[u\left(x,\xi\right)\frac{\partial^{\alpha}u\left(x,\xi\right)}{\partial x^{\alpha}}\\ &-\frac{\partial^{3\alpha}u\left(x,\xi\right)}{\partial x^{3\alpha}}-\frac{\partial^{\alpha}u\left(x,\xi\right)}{\partial x^{\alpha}}])]\}(d\xi)^{\alpha}.\end{aligned} \tag{54}$$

We can use the initial condition to select $u_0(x,t) = u(x,0) = \frac{x^\alpha}{\Gamma(\alpha+1)}$. Now, using this selection to the correction function gives the following successive approximations

$$u_0(x,t) = \frac{x^\alpha}{\Gamma(\alpha+1)}, \tag{55}$$

$$\begin{aligned} u_1(x,t) &= u_0(x,t) - \frac{1}{\Gamma(\alpha+1)}\int_0^t \{(u_0)_\xi^\alpha(x,\xi) - \frac{\partial^\alpha}{\partial\xi^\alpha}[\mathcal{L}_\alpha^{-1}(\frac{1}{s^\alpha}\mathcal{L}_\alpha \\ &\qquad [u_0(x,\xi)\frac{\partial^\alpha u_0(x,\xi)}{\partial x^\alpha} - \frac{\partial^{3\alpha}u_0(x,\xi)}{\partial x^{3\alpha}} - \frac{\partial^\alpha u_0(x,\xi)}{\partial x^\alpha}])]\}(d\xi)^\alpha \\ &= \frac{x^\alpha}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha+1)}\int_0^t \frac{\partial^\alpha}{\partial\xi^\alpha}[\mathcal{L}_\alpha^{-1}(\frac{1}{s^\alpha}\mathcal{L}_\alpha[\frac{x^\alpha}{\Gamma(\alpha+1)} - 1])](d\xi)^\alpha \\ u_1(x,t) &= \frac{x^\alpha}{\Gamma(\alpha+1)} + (\frac{x^\alpha}{\Gamma(\alpha+1)} - 1)\frac{t^\alpha}{\Gamma(\alpha+1)}. \end{aligned} \tag{56}$$

$$\begin{aligned} u_2(x,t) &= u_1(x,t) - \frac{1}{\Gamma(\alpha+1)}\int_0^t \{(u_1)_\xi^\alpha(x,\xi) - \frac{\partial^\alpha}{\partial\xi^\alpha}[\mathcal{L}_\alpha^{-1}(\frac{1}{s^\alpha}\mathcal{L}_\alpha \\ &\qquad [u_1(x,\xi)\frac{\partial^\alpha u_1(x,\xi)}{\partial x^\alpha} - \frac{\partial^{3\alpha}u_1(x,\xi)}{\partial x^{3\alpha}} - \frac{\partial^\alpha u_1(x,\xi)}{\partial x^\alpha}])]\}(d\xi)^\alpha \\ &= \frac{x^\alpha}{\Gamma(1+\alpha)} + (\frac{x^\alpha}{\Gamma(1+\alpha)} - 1)\frac{t^\alpha}{\Gamma(1+\alpha)} - \frac{1}{\Gamma(1+\alpha)} \\ &\qquad \int_0^t [(\frac{x^\alpha}{\Gamma(1+\alpha)} - 1) - \frac{\partial^\alpha}{\partial\xi^\alpha}\{\mathcal{L}_\alpha^{-1}(\frac{1}{s^\alpha}\mathcal{L}_\alpha[(\frac{x^\alpha}{\Gamma(1+\alpha)} + \\ &\qquad (\frac{x^\alpha}{\Gamma(1+\alpha)} - 1)\frac{t^\alpha}{\Gamma(1+\alpha)} - 1)(1 + \frac{t^\alpha}{\Gamma(1+\alpha)})])\}](d\xi)^\alpha \\ u_2(x,t) &= \frac{x^\alpha}{\Gamma(1+\alpha)} + (\frac{x^\alpha}{\Gamma(1+\alpha)} - 1)(\frac{t^\alpha}{\Gamma(1+\alpha)} + \frac{2t^{2\alpha}}{\Gamma(1+2\alpha)} \\ &\qquad + \frac{t^{3\alpha}}{\Gamma(1+3\alpha)}\frac{\Gamma(1+2\alpha)}{\Gamma^2(1+\alpha)}). \end{aligned} \tag{57}$$

Similarly, we can obtain the solution $u_3(x,t)$ as follows

$$
\begin{aligned}
u_3(x,t) = {} & \frac{x^\alpha}{\Gamma(1+\alpha)} + \left(\frac{x^\alpha}{\Gamma(1+\alpha)} - 1\right) \\
& (\frac{t^\alpha}{\Gamma(1+\alpha)} + \frac{2t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{t^{3\alpha}}{\Gamma(1+3\alpha)}\left[4 + \frac{\Gamma(1+2\alpha)}{\Gamma^2(1+\alpha)}\right] \\
& + \left(\frac{2t^{4\alpha}}{\Gamma(1+4\alpha)\Gamma(1+\alpha)}\right)\left(\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)} + \frac{2\Gamma(1+3\alpha)}{\Gamma(1+2\alpha)}\right) \\
& + \left(\frac{2\Gamma(1+4\alpha)t^{5\alpha}}{\Gamma(1+5\alpha)\Gamma(1+\alpha)}\right)(\frac{2}{\Gamma(1+2\alpha)} \\
& + \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)\Gamma^3(1+\alpha)}) + \frac{4t^{6\alpha}}{\Gamma(1+6\alpha)}\frac{\Gamma(1+5\alpha)}{\Gamma(1+3\alpha)\Gamma^2(1+\alpha)} \\
& + \frac{\Gamma(1+6\alpha)t^{7\alpha}}{\Gamma(1+7\alpha)}\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)\Gamma(1+\alpha)}).
\end{aligned} \tag{58}
$$

By continuing this process, we can find the other terms of this sequence. By the convergence of the local fractional variational method, as $n \to \infty$, the sequence approaches the exact solution of the equation (48) on the Cantor set.

6 Conclusions

Local fractional calculus is created on fractal, and the local fractional variational iteration transform method is derived from local fractional calculus. This method is efficacious for the applied Sciences to solve differential and integral equations involving the local fractional operators.

In this paper, it is applied the local fractional differential operators in an equation (2). In the procedure, it is considered the linear and Nonlinear local fractional KdV equation. Based on the local fractional variational iteration transform method, the solutions of the KdV equations were offered. The iteration functions, which are local fractional continuous, are obtained easily within the fractal Lagrange multipliers, which can be optimally specified by the local fractional variational theory [31, 28].

The results show that the presented method is a strong mathematical tool for finding other numerical and exact solutions to many linear and

Nonlinear local fractional differential equations with initial and boundary conditions.

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