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Original Research Paper

Bayesian Estimation of Parameters under Two-Parameter Extended Marshall-Olkin Family of Distributions based on Different Loss Functions

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Abstract. Estimation of different parameters from two parameters extended Marshall-Olkin distribution was performed based on Lindly approximation and MCMC method under loss functions including square error, Linex, modified Linex, square error in logarithm and entropy. For producing samples from complete conditional distribution parameters, the Metropolis-Hastings algorithm was used. And we have used various informative prior distributions and uninformative prior. Based on the observations of the present study, it was observed that the Lindley approximation method has provided more appropriate estimates from the view of MSE reduction by the MCMC method, and as a result, compatible estimations have been produced. The optimality of the proposed model is described by using a practical example..

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1 Introduction

Marshall and Olkin [5] have introduced a new distribution by adding a new parameter to a distribution family. They have started from $\bar{F}(x)$ (survival function) and they introduced a new family of survival functions:

$$\bar{G}(x;a) = \frac{a\bar{F}(x)}{F(x) + a\bar{F}(x)}, \tag{1}$$

in equation (1), $-\infty < x < \infty$ and $0 < a < \infty$. The interesting aspect of this distribution is that the random variable of N having geometric distribution with parameter a $(0 < a < 1)$ (in other words $P(N = n) = a(1-a)^{n-1},\ n = 1, 2, 3, \cdots$), then $U_N = \min(X_1, \cdots, X_N)$ has survival function (1). Also, if $a > 1$ and $P(N = n) = a^{-1}(1 - a^{-1})^{n-1},\ n = 1, 2, \cdots$ then $V_N = \max(X_1, \cdots, X_N)$ has the survival equation (1). Some statistics experts have tried to add a new parameter to Marshall-Olkin distribution. Jayakumar and Thomas [3] have introduced and assessed the bellow survival function:

$$\bar{G}(x;a,b) = \left[\frac{a\bar{F}(x)}{F(x) + a\bar{F}(x)}\right]^b. \tag{2}$$

Nadarajeh et al. [6] have introduced a new family of survival function:

$$\bar{G}(x;a,b) = \frac{a^b}{1-a^b}\left[\left(F(x) + a\bar{F}(x)\right)^{-b} - 1\right],$$

in which $x \in R$, $a > 0$ and $b > 0$. Note that if $a \to 1$ then $\bar{G}(x;a,b) \to \bar{F}(x)$ and if $b \to 1$ then $\bar{G}(x;a,b)$ approaches to survival function of Marshall-Olkin.

The distribution function, the probability density function and the hazard unction of this distribution are as follows:

$$G(x;a,b) = \frac{[1 - \bar{F}(x)\bar{a}]^b - a^b}{[1 - \bar{F}(x)\bar{a}]^b(1 - a^b)} = (1 - a^b)^{-1} - a^b[1 - \bar{F}(x)\bar{a}]^{-b} \tag{3}$$

$$g(x;a,b) = \frac{\bar{a}ba^b f(x)}{(1-a^b)[1-\bar{F}(x)\bar{a}]^{b+1}} = \bar{a}ba^b f(x)(1-a^b)^{-1} \times[1-\bar{F}(x)\bar{a}]^{-(b+1)} \tag{4}$$

$$\begin{aligned} r_G(x;a,b) &= \frac{\bar{a}b\bar{F}(x)r_F(x)}{\Big[1-\bar{F}(x)\bar{a}\Big]\Big[1-\Big(F(x)+a\bar{F}(x)\Big)^b\Big]} \\ &= \bar{a}b\bar{F}(x)r_F(x)\Big[1-\bar{F}(x)\bar{a}\Big]^{-1}\Big[1-\Big(1-\bar{F}(x)\bar{a}\Big)^b\Big]^{-1} \end{aligned} \tag{5}$$

in which $\bar{a} = 1-a$, $\bar{F}(x) = 1-F(x)$ and $r_F(x)$ is the hazard function of X.
For analyzing the shape of $\frac{r_G(x;a,b)}{r_F(x)}$, we have that when $0 < a < 1$ decreases and $1 \leq \frac{r_G(x;a,b)}{r_F(x)} \leq \frac{b\bar{a}}{(a(1-a^b))}$, and when $a > 1$ the function increases and $\frac{b\bar{a}}{(a(1-a^b))} \leq \frac{r_G(x;a,b)}{r_F(x)} \leq 1$.
It is worth mentioning that

$$\lim_{x\to-\infty} r_G(x;a,b) = \frac{b\bar{a}}{a(1-a^b)} \lim_{x\to-\infty} r_F(x)$$

and

$$\lim_{x\to+\infty} r_G(x;a,b) = \lim_{x\to+\infty} r_F(x).$$

This article consists of three sections. In the second part, we will deal with the Two-Parameter Marshall-Olkin Extended Weibull family, and in the third part, we will deal with the Bayesian statistics of this family.

2 Two-Parameter Marshall-Olkin Extended Weibull Family (TPMOEW)

The extended Weibull (EW) distribution class was introduced and discussed by Gorvich et al. [2] They gained a prominent position in lifetime models. Cumulative distribution function (cdf) of these models is as follow:

$$F(x;a,\boldsymbol{\xi}) = 1-\exp[-\alpha H(x;\boldsymbol{\xi})], \qquad x \in D \subseteq R_+,\ a > 0, \tag{6}$$

That $H(x;\boldsymbol{\xi})$ is a non-negative differentiable and ascending function that depends on the parametric vector $\boldsymbol{\xi}$. The probability density function F is denoted by the following equation.

$$f(x;a,\boldsymbol{\xi}) = \alpha h(x;\boldsymbol{\xi})\exp[-\alpha H(x;\boldsymbol{\xi})], \tag{7}$$

where $h(x;\boldsymbol{\xi})$ is the derivative of $H(x;\boldsymbol{\xi})$ equation. The survival equation of this distribution is defined based on bellow equation:

$$\bar{F}(x;\alpha,\boldsymbol{\xi}) = \exp[-\alpha H(x;\boldsymbol{\xi})], \tag{8}$$

Special and important cases of $H(x;\boldsymbol{\xi})$ are brought in (6) equation bellow:
(A) $H(x;\boldsymbol{\xi}) = x$ gives the exponential function.
(B) $H(x;\lambda,\gamma) = x^{\gamma}\exp(\lambda x)$ prepares extended Weibull function.
(C) $H(x;\beta) = \beta^{-1}[\exp(\beta x) - 1]$ provides Gompertz distribution.
In this paper, we obtain a new family of distributions by combining the two-parameter Marshall-Olkin class and the exponential Weibull class. The family of two-parameter, Extended Marshall-Olkin Weibull (TPMOEW) distributions includes special models.
The TPMOEW distribution function is described by the following equation:

$$G(x;a,b,\alpha,\boldsymbol{\xi}) = \frac{\Big[1-\bar{a}\exp[-\alpha H(x;\boldsymbol{\xi})]\Big]^{b} - a^{b}}{(1-a^{b})\Big[1-\bar{a}\exp[-\alpha H(x;\boldsymbol{\xi})]\Big]^{b}}, \quad x \in D \subseteq R_{+}, \quad a,b,\alpha > 0.$$

The survival function, density function and hazard function are determined by Equations (9), (10) and (11), respectively.

$$\bar{G}(x;a,b,\alpha,\boldsymbol{\xi}) = \frac{a^{b} - a^{b}\Big[1-\bar{a}\exp[-\alpha H(x;\boldsymbol{\xi})]\Big]^{b}}{(1-a^{b})\Big[1-\bar{a}\exp[-\alpha H(x;\boldsymbol{\xi})]\Big]^{b}} \tag{9}$$

$$g(x;a,b,\alpha,\boldsymbol{\xi}) = \frac{a^{b}b\bar{a}\alpha h(x;\boldsymbol{\xi})\exp[-\alpha H(x;\boldsymbol{\xi})]}{(1-a^{b})\Big[1-\bar{a}\exp[-\alpha H(x;\boldsymbol{\xi})]\Big]^{b+1}} \tag{10}$$

$$
\begin{aligned}
r_G(x;a,b,\alpha,\boldsymbol{\xi}) &= \frac{b\bar{a}\alpha h(x;\boldsymbol{\xi})\exp[-\alpha H(x;\boldsymbol{\xi})]}{\Big[1-\bar{a}\exp[-\alpha H(x;\boldsymbol{\xi})]\Big]\Big\{1-\Big[1-\bar{a}\exp[-\alpha H(x;\boldsymbol{\xi})]\Big]^{b}\Big\}} \\
&= \frac{b\bar{a}\exp[-\alpha H(x;\boldsymbol{\xi})]r_F(x)}{\Big[1-\bar{a}\exp[-\alpha H(x;\boldsymbol{\xi})]\Big]\Big\{1-\Big[1-\bar{a}\exp[-\alpha H(x;\boldsymbol{\xi})]\Big]^{b}\Big\}}
\end{aligned} \tag{11}
$$

Three Special Models
We look at three special TPMOEW models. Case 1: Two-Parameter Marshall-Olkin Exponential Distribution (TPMOEX), Case 2: Two-Parameter Marshall-Olkin Modified Weibull (TPMOMW), Case 3: Two-Parameter Marshall-Olkin Gompertz (TPMOGO).

2.1 Two-parameter Marshall-Olkin Exponential distribution (TPMOEX)

In this distribution $H(x;\xi)=x$, $h(x;\xi)=1$, $\bar{a}=1-a$ and $\bar{G}=1-G$ also:

$$G(x;a,b,\lambda)=\frac{\Big[1-\bar{a}\exp(-\lambda x)\Big]^{b}-a^{b}}{(1-a^{b})\Big[1-\bar{a}\exp(-\lambda x)\Big]^{b}},$$

$$\bar{G}(x;a,b,\lambda)=\frac{a^{b}-a^{b}\Big[1-\bar{a}\exp(-\lambda x)\Big]^{b}}{(1-a^{b})\Big[1-\bar{a}\exp(-\lambda x)\Big]^{b}},$$

$$g(x;a,b,\lambda)=\frac{a^{b}b\bar{a}\lambda\exp(-\lambda x)}{(1-a^{b})\Big[1-\bar{a}\exp(-\lambda x)\Big]^{b+1}},$$

$$r_G(x;a,b,\lambda)=\frac{b\bar{a}\lambda\exp(-\lambda x)}{\Big[1-\bar{a}\exp(-\lambda x)\Big]\Big[1-(1-\bar{a}\exp(-\lambda x))^{-b}\Big]},$$

2.2 Two-parameter Marshall-Olkin modified Weibull (TPMOMW)

For $H(x;\lambda,\gamma) = x^{\gamma}\exp(\lambda x)$ and $h(x;\lambda,\gamma) = x^{\gamma-1}\exp(\lambda x)(\gamma+\lambda x)$, we will have TPMOMW distribution. Probability density function is based on bellow:

$$\begin{aligned} g(x;a,b,\alpha,\lambda,\gamma) &= \frac{a^b b\bar{a}\alpha x^{\gamma-1}\exp(\lambda x)(\gamma+\lambda x)\exp[-\alpha x^{\gamma}\exp(\lambda x)]}{(1-a^b)\Big[1-\bar{a}\exp[-\alpha x^{\gamma}\exp(\lambda x)]\Big]^b} \\ &= \frac{a^b b\bar{a}\alpha x^{\gamma-1}(\gamma+\lambda x)\exp[\lambda x-\alpha x^{\gamma}\exp(\lambda x)]}{(1-a^b)\Big[1-\bar{a}\exp[-\alpha x^{\gamma}\exp(\lambda x)]\Big]^b}, \end{aligned}$$

in which $\lambda,\gamma \geq 0$ is distribution function and hazard rate function is based on bellow, respectively:

$$G(x;a,b,\alpha,\lambda,\gamma) = \frac{1}{1-a^b} - \frac{a^b}{1-a^b}\Big[1-\bar{a}\exp[-\alpha x^{\gamma}\exp(\lambda x)]\Big]^{-1}$$

$$\begin{aligned} r(x;a,b,\alpha,\lambda,\gamma) &= \frac{b\bar{a}\alpha x^{\gamma-1}\exp(\lambda x)(\gamma+\lambda x)}{\Big[1-\bar{a}\exp[-\alpha x^{\gamma}\exp(\lambda x)]\Big]} \\ &\qquad \times \frac{\exp[-\alpha x^{\gamma}\exp(\lambda x)]}{\Big\{1-\Big[1-\bar{a}\exp[-\alpha x^{\gamma}\exp(\lambda x)]\Big]^b\Big\}} \end{aligned}$$

2.3 Two-parameter Marshall-Olkin Gompertz (TPMOGO)

For $H(x;\beta) = \beta^{-1}[\exp(\beta x)-1]$ and $h(x;\beta) = \exp(\beta x)$ TPMOGO distribution is obtained. The density function of this distribution is as follows:

$$g(x;a,b,\alpha,\beta) = \frac{a^b b\bar{a}\exp(\beta x)\exp\Big[-\alpha\beta^{-1}[\exp(\beta x)-1]\Big]}{(1-a^b)\Big[1-\bar{a}\exp\Big[-\alpha\beta^{-1}[\exp(\beta x)-1]\Big]\Big]}, \quad \beta\in R, \quad x\in R_+$$

The distribution function and the hazard rate function of the TPMOGO model are as follows:

$$G(x;a,b,\alpha,\beta)=\frac{1}{1-a^b}-\frac{a^b}{1-a^b}\Bigg[1-\bar{a}\exp\Bigg[-\alpha\beta^{-1}[\exp(\beta x)-1]\Bigg]\Bigg]^{-b},$$

$$\begin{aligned} r(x;a,b,\alpha,\beta) &= \frac{b\alpha\bar{a}\exp(\beta x)}{\Bigg[1-\bar{a}\exp\Bigg[-\alpha\beta^{-1}[\exp(\beta x)-1]\Bigg]\Bigg]} \\ &\quad\times\frac{\exp\Bigg[-\alpha\beta^{-1}[\exp(\beta x)-1]\Bigg]}{\Bigg\{1-\Bigg[1-\bar{a}\exp\Bigg[-\alpha\beta^{-1}[\exp(\beta x)-1]\Bigg]\Bigg]\Bigg\}}, \end{aligned}$$

3 Bayesian Statistics

Since it plays an important role in estimating the Bayesian loss function, the Bayesian estimation of Two-Parameter Marshall-Olkin Exponential Distribution (TPMOEX), Two-Parameter Marshall-Olkin Modified Weibull (TPMOMW) and Two-Parameter Marshall-Olkin Gompertz (TPMOGO) is presented under the functions of loss error square, entropy, Linex, error square in logarithm and modified Linex. In the following, we express the Bayesian estimator of θ under the different loss functions in Table 1.
Suppose we are looking for Bayesian estimators for the Gorvich distribution. A formal choice for the previous distribution of α, λ and γ could be three independent gamma distributions, namely gamma(δ,β), gamma(θ,ζ) and gamma(η,ε), respectively.

Suppose we are looking for Bayesian estimators for the Gorvich distribution. A formal choice for the previous distribution of α, λ and γ could be three independent gamma distributions, namely gamma(δ,β), gamma(θ,ζ) and gamma(η,ε), respectively.

Table 1: The Bayesian estimators of θ under different loss functions

Loss function	Formula	Estimator
Square error	$L_s = k(\hat{\theta} - \theta)^2$	$\hat{\theta}_s = E(\theta \mid x)$ $= \int \theta \pi(\theta \mid x) d\theta$
Entropy	$L_E = \frac{\hat{\theta}}{\theta} - ln\frac{\hat{\theta}}{\theta} - 1$	$\hat{\theta}_E = \frac{1}{E(\frac{1}{\theta} \vert x)}$
Linex	$L_L = e^{(\hat{\theta} - \theta)} - (\hat{\theta} - \theta) - 1$	$\hat{\theta}_L = -lnE(e^{-\theta} \mid x)$
Modified Linex	$L_{mL} = e^{(\frac{\hat{\theta}}{\theta} - 1)} - (\frac{\hat{\theta}}{\theta} - 1) - 1$	$E\left(\frac{1}{\theta} e^{\frac{\hat{\theta}_{mL}}{\theta}} \mid x\right)$ $= \exp E(\frac{1}{\theta} \mid x)$
Square error in Logarithm	$L_{Sl} = (ln\hat{\theta} - ln\theta)^2$	$\hat{\theta}_{Sl} = e^{E(ln\theta \vert x)}$

Therefore, the prior densities α, λ and γ are as follows.

$$\begin{aligned} f_{\delta,\beta}(\alpha) &\propto \alpha^{\delta-1} e^{-\alpha\beta} \\ f_{\theta,\zeta}(\lambda) &\propto \lambda^{\theta-1} e^{-\lambda\zeta} \\ f_{\eta,\varepsilon}(\gamma) &\propto \gamma^{\eta-1} e^{-\gamma\varepsilon} \end{aligned}$$

Therefore, the prior density function of the parameters α, λ and γ are as follows.

$$\begin{aligned} f(\alpha, \lambda, \gamma \mid x) &= k\alpha^{\delta-1}\lambda^{\theta-1}\gamma^{\eta-1} e^{-\alpha\beta-\lambda\zeta-\gamma\varepsilon} \alpha^n \prod_{i=1}^{n} x_i^{(\gamma-1)} \prod_{i=1}^{n} (\gamma + \lambda x_i) \\ &\quad \times \prod_{i=1}^{n} \exp(\lambda x_i - \alpha x_i^{\gamma} e^{\lambda x_i}) \end{aligned}$$

that

$$\begin{aligned} k^{-1} &= \int_{\alpha}\int_{\lambda}\int_{\gamma} \alpha^{\delta-1}\lambda^{\theta-1}\gamma^{\eta-1} e^{-\alpha\beta-\lambda\zeta-\gamma\varepsilon} \alpha^n \prod_{i=1}^{n} x_i^{(\gamma-1)} \prod_{i=1}^{n} (\gamma + \lambda x_i) \\ &\quad \times \prod_{i=1}^{n} \exp(\lambda x_i - \alpha x_i^{\gamma} e^{\lambda x_i}) d\alpha \, d\lambda \, d\gamma \end{aligned}$$

Table 2: The estimation of MCMC based on different loss functions

Loss function	Estimation
Square error	$\hat{\theta}_s = \frac{1}{N-M}\sum_{i=M+1}^{N}\theta^{(i)}$
Entropy	$\hat{\theta}_E = \frac{N-M}{\sum_{i=M+1}^{N}\frac{1}{\theta^{(i)}}}$
Linex	$\hat{\theta}_L = -ln\left\{\frac{\sum_{i=M+1}^{N}e^{-\theta^{(i)}}}{N-M}\right\}$
Square error in logarithm	$\hat{\theta}_{SL} = e^{\frac{1}{N-M}\sum_{i=M+1}^{N}ln\theta^{(i)}}$
Modified linex	$\hat{\theta}_{MLINEX} = \left(\frac{1}{N-M}\sum_{i=M+1}^{N}(\theta^{(i)})^{-\frac{1}{c}}\right)^{-\frac{1}{c}}$

The posterior density function has an integral in the denominator that is not closed form and therefore the parameters of Bayesian statistics has not explicit form. There are various methods for solving such integrals. We use the Lindley method to approximate those integrals.

3.1 Bayesian estimators for TPMOEX

For the TPMOEX distribution, the Bayesian estimation of the parameters under the loss functions of square error, entropy, Linex, square error in logarithm and modified Linex is presented in Table 2, where in general form $\hat{\theta}$ is the estimator of the parameter θ.

Mont Carlo Markov Chain (MCMC) for TPMOEX Chain Method

As observed in Bayesian statistics, calculating the posterior distribution is not always simple and in many cases the denominator of the Bayesian fraction is not analytically possible due to computational complexity. Even if the posterior distribution can be computed analytically, characteristics of the posterior distribution mean, variance, etc. can not be calculated. In general, the calculation of posterior expectations have encountered with various troubles due to the inability in calculating the posterior distribution or inability to calculate the corresponding integral.

$$E(g(\boldsymbol{\theta} \mid \boldsymbol{y})) = \int g(\boldsymbol{\theta})\pi(\boldsymbol{\theta} \mid \boldsymbol{y})d\boldsymbol{\theta}$$

Therefore, in the previous section, the Lindley approximation method was examined based on different loss functions. This section introduces Mont Carlo Markov Chain methods for sampling posterior distributions and for approximating such integrals. There are many algorithms for producing Markov chains with a specific mean distribution. Among these algorithms, Gibbs and Metropolis-Hastings sampling algorithms are more acceptable in the statistical literature.

Gibbs sampling algorithm

The Gibbs algorithm uses conditional posterior distributions of various parameters provided that other parameters that are called Posterior distribution full conditional, are used to sample the posterior distribution. This algorithm for TPMOEX model is done in three steps as follows:

1. Consider the initial amount $\boldsymbol{\theta}^{(0)} = (\boldsymbol{a}^{(0)}, \boldsymbol{b}^{(0)}, \lambda^{(0)})$.

2. In the $(t+1)$ stage based on the obtained amounts from (t) stage, a, b and λ are updated based as bellow:

$$\begin{aligned} \boldsymbol{a}^{(t+1)} &\sim P(a \mid \boldsymbol{b}^{(t)}, \lambda^{(t)}, \boldsymbol{x}) \\ b^{(t+1)} &\sim P(\boldsymbol{b} \mid \boldsymbol{a}^{(t+1)}, \lambda^{(t)}, \boldsymbol{x}) \\ \lambda^{(t+1)} &\sim P(\boldsymbol{\lambda} \mid \boldsymbol{a}^{(t+1)}, \boldsymbol{b}^{(t+1)}, \boldsymbol{x}) \end{aligned}$$

3. Repeat the second stage until the Marcov chain reaches to its Stationary distribution. Posterior distribution full conditional of the parameters from the model are obtained as follow:

$$P(a \mid \boldsymbol{b}, \boldsymbol{\lambda}) \propto a^{\alpha-1} \exp(-\alpha\beta) \frac{a^{nb}\bar{a}^{n}}{(1-a^{b})^{n}} \prod_{i=1}^{n} (1-\bar{a}\exp(-\lambda x_i))^{-(b+1)} \quad (12)$$

$$\begin{aligned} P(b \mid \boldsymbol{a}, \boldsymbol{\lambda}) &\propto b^{\theta-1} \exp(-b\gamma) \frac{a^{nb}b^{n}}{(1-a^{b})^{n}} \\ &\quad \times \prod_{i=1}^{n} (1-\bar{a}\exp(-\lambda x_i))^{-(b+1)} \quad (13) \end{aligned}$$

$$\begin{aligned} P(\lambda \mid \boldsymbol{a}, \boldsymbol{b}) &\propto \lambda^{n+\eta-1} \exp\left(-\lambda(\sum_{i=1}^{n} x_i + \varepsilon)\right) \frac{a^{nb}b^{n}}{(1-a^{b})^{n}} \\ &\quad \times \prod_{i=1}^{n} (1-\bar{a}\exp(-\lambda x_i))^{-(b+1)} \quad (14) \end{aligned}$$

When the conditional posterior distributions, such as distributions (12) to (14), do not have a known shape, the metropolis-Hastings algorithm can be used to the sample of the posterior distribution. In this study, because Posterior distribution full conditional of the parameters have not a known shape, the Metropolis-Hastings algorithm is used for Bayesian inferences.

Metropolis-Hastings algorithm

This algorithm has a structure of rejection and acceptance and evaluates the observations proposed by a proposed distribution as possible examples of a posterior distribution based on a probabilistic rule and accepts them with a certain probability. Therefore, to use this algorithm, a proposed distribution is required, the choice of which affects the accuracy and speed of convergence of the algorithm. One of the desirable features of this algorithm is that it is sufficient to know the posterior distribution proportionally. The steps of this algorithm for TPMOEX model are as follows:

1. Consider the initial amount $\boldsymbol{\theta}^{(0)} = (\boldsymbol{a}^{(0)}, \boldsymbol{b}^{(0)}, \lambda^{(0)})$.

2. In the $(t+1)$ stage based on the (t) stage, update a, b and λ as follow:

a. From the proposed distribution $q(a^*, b^*, \lambda^* \mid a^{(t)}, b^{(t)}, \lambda^{(t)})$, produce (a^*, b^*, λ^*) amount.

b. Calculate the Hastings ratio based on bellow:

$$H = \frac{\pounds(a^*, b^*, \lambda^* \mid \boldsymbol{x}) P(a^*, b^*, \lambda^*) q(a^{(t)}, b^{(t)}, \lambda^{(t)} \mid a^*, b^*, \lambda^*)}{\pounds(a^{(t)}, b^{(t)}, \lambda^{(t)} \mid \boldsymbol{x}) P(a^{(t)}, b^{(t)}, \lambda^{(t)}) q(a^*, b^*, \lambda^* \mid a^{(t)}, b^{(t)}, \lambda^{(t)})} \quad (15)$$

Calculate the amount of $\tilde{u} = \min(H, 1)$, then produce a sample from homogeneous distribution namely U, such that:

$$\begin{cases} (a^{(t+1)}, b^{(t+1)}, \lambda^{(t+1)}) = (a^*, b^*, \lambda^*) & \text{if } \tilde{u} \leq U \ , \\ \\ (a^{(t+1)}, b^{(t+1)}, \lambda^{(t+1)}) = (a^{(t)}, b^{(t)}, \lambda^{(t)}) & \text{if } \tilde{u} \geq U \end{cases}$$

3. Repeat the second stage until the Marcov chain reaches to its stationary distribution.

Pay attention that in equation (15), $l(\cdot \mid \boldsymbol{x})$ is considered the probability function, $P(\cdot)$ is considered the posterior's distribution and $q(a^{(t)}, b^{(t)}, \lambda^{(t)} \mid a^*, b^*, \lambda^*)$ is the proposed distribution that here is considered independent semi-normal triple variable.

Based on this, if this chain is repeated N times, and M number of them is considered as burnt variables, we will have:

$$E(g(\boldsymbol{\theta} \mid \boldsymbol{y})) = \int g(\boldsymbol{\theta})\pi(\boldsymbol{\theta} \mid \boldsymbol{y})d\boldsymbol{\theta} = \frac{1}{N-M}\sum_{i=M+1}^{N} g(\boldsymbol{\theta}_i)$$

As a result, the parameter estimates based on different loss functions are summarized in Table 3 and 4.

Simulation studies

In this section, a simulation study is performed to apply Bayesian mcmc and Lindley methods based on different loss functions. Since in Bayesian approach, the selection of posterior distribution super parameters is very important, then in order to examine the different values of the super parameters on the final results, informative and ignorant backgrounds are used. In the case of previous informative backgrounds, the hyper parameters are adjusted so that the average of the previous distributions is equal to a guess value of the model parameters. For further investigation, two sets of hyper parameters can be considered, in the first case the variance of posteriors is small and in the second case the variance of the posteriors is large. But in the unconscious state, it was assumed that no information about the super parameters was available and their value was considered zero. In this simulation study, the model parameters are considered as $a = b = \lambda = 2$. In the case of informational backgrounds with small variance, the super parameters are assumed to be $\alpha = \theta = \eta = 4$, $\beta = \gamma = \varepsilon = 2$, and as a result for this case the mean and variance of the previous distributions are 2 and 1, respectively. However, in the case of informative distributions with large variance, the super parameters are assumed as $\alpha = \theta = \eta = 0.4$, $\beta = \gamma = \varepsilon = 0.2$, in this case, the average of the posteriors is obtained as before 2, but their variance is larger than the previous case and equal to 10.

There are different methods to show the convergence of Monte Carlo chains to their meaning distribution, including the effect diagram, which is shown in Figure 1 for a simulated sample type with size of $n = 50$ and for each unconscious state, this figure shows the created convergence as well.

Table 3: The Bayesian estimators of the parameters under different loss functions

Loss function	Parameters	TPMOEX
Square error	a	$\hat{a}+\hat{\rho}_a\hat{\sigma}_{aa}+\frac{1}{2}\Big[\hat{\sigma}_{aa}\Big(\hat{L}_{aaa}\hat{\sigma}_{aa}+\hat{L}_{bba}\hat{\sigma}_{bb}+\hat{L}_{\lambda\lambda a}\hat{\sigma}_{\lambda\lambda}\Big)\Big]$
	b	$\hat{b}+\hat{\rho}_b\hat{\sigma}_{bb}+\frac{1}{2}\Big[\hat{\sigma}_{bb}\Big(\hat{L}_{aab}\hat{\sigma}_{aa}+\hat{L}_{bbb}\hat{\sigma}_{bb}+\hat{L}_{\lambda\lambda b}\hat{\sigma}_{\lambda\lambda}\Big)\Big]$
	λ	$\hat{\lambda}+\hat{\rho}_\lambda\hat{\sigma}_{\lambda\lambda}+\frac{1}{2}\Big[\hat{\sigma}_{\lambda\lambda}\Big(\hat{L}_{aa\lambda}\hat{\sigma}_{aa}+\hat{L}_{bb\lambda}\hat{\sigma}_{bb}+\hat{L}_{\lambda\lambda\lambda}\hat{\sigma}_{\lambda\lambda}\Big)\Big]$
Entropy	a	$\hat{a}\Big[1+(\frac{1}{\hat{a}^2}-\frac{1}{\hat{a}}\hat{\rho}_a)\hat{\sigma}_{aa}-\frac{1}{2\hat{a}}\hat{\sigma}_{aa}\Big(\hat{L}_{aaa}\hat{\sigma}_{aa}+\hat{L}_{bba}\hat{\sigma}_{bb}+\hat{L}_{\lambda\lambda a}\hat{\sigma}_{\lambda\lambda}\Big)\Big]^{-1}$
	b	$\hat{b}\Big[1+(\frac{1}{\hat{b}^2}-\frac{1}{\hat{b}}\hat{\rho}_b)\hat{\sigma}_{bb}-\frac{1}{2\hat{b}}\hat{\sigma}_{bb}\Big(\hat{L}_{aab}\hat{\sigma}_{aa}+\hat{L}_{bbb}\hat{\sigma}_{bb}+\hat{L}_{\lambda\lambda b}\hat{\sigma}_{\lambda\lambda}\Big)\Big]^{-1}$
	λ	$\hat{\lambda}\Big[1+(\frac{1}{\hat{\lambda}^2}-\frac{1}{\hat{\lambda}}\hat{\rho}_\lambda)\hat{\sigma}_{\lambda\lambda}-\frac{1}{2\hat{\lambda}}\hat{\sigma}_{\lambda\lambda}\Big(\hat{L}_{aa\lambda}\hat{\sigma}_{aa}+\hat{L}_{bb\lambda}\hat{\sigma}_{bb}+\hat{L}_{\lambda\lambda\lambda}\hat{\sigma}_{\lambda\lambda}\Big)\Big]^{-1}$
Linex	a	$\hat{a}-Ln\Big\{1+\Big[(\frac{1}{2}-\hat{\rho}_a)\hat{\sigma}_{aa}-\frac{1}{2}\hat{\sigma}_{aa}\Big(\hat{L}_{aaa}\hat{\sigma}_{aa}+\hat{L}_{bba}\hat{\sigma}_{bb}+\hat{L}_{\lambda\lambda a}\hat{\sigma}_{\lambda\lambda}\Big)\Big]\Big\}$
	b	$\hat{b}-Ln\Big\{1+\Big[(\frac{1}{2}-\hat{\rho}_b)\hat{\sigma}_{bb}-\frac{1}{2}\hat{\sigma}_{bb}\Big(\hat{L}_{aab}\hat{\sigma}_{aa}+\hat{L}_{bbb}\hat{\sigma}_{bb}+\hat{L}_{\lambda\lambda b}\hat{\sigma}_{\lambda\lambda}\Big)\Big]\Big\}$
	λ	$\hat{\lambda}-Ln\Big\{1+\Big[(\frac{1}{2}-\hat{\rho}_\lambda)\hat{\sigma}_{\lambda\lambda}-\frac{1}{2}\hat{\sigma}_{\lambda\lambda}\Big(\hat{L}_{aa\lambda}\hat{\sigma}_{aa}+\hat{L}_{bb\lambda}\hat{\sigma}_{bb}+\hat{L}_{\lambda\lambda\lambda}\hat{\sigma}_{\lambda\lambda}\Big)\Big]\Big\}$
Square log error	a	$\hat{a}\exp\Big[\frac{\hat{\sigma}_{aa}}{2\hat{a}}\Big(-\frac{1}{\hat{a}}+2\hat{\rho}_a+(\hat{L}_{aaa}\hat{\sigma}_{aa}+\hat{L}_{bba}\hat{\sigma}_{bb}+\hat{L}_{\lambda\lambda a}\hat{\sigma}_{\lambda\lambda})\Big)\Big]$
	b	$\hat{b}\exp\Big[\frac{\hat{\sigma}_{bb}}{2\hat{b}}\Big(-\frac{1}{\hat{b}}+2\hat{\rho}_b+(\hat{L}_{aab}\hat{\sigma}_{aa}+\hat{L}_{bbb}\hat{\sigma}_{bb}+\hat{L}_{\lambda\lambda b}\hat{\sigma}_{\lambda\lambda})\Big)\Big]$
	λ	$\hat{\lambda}\exp\Big[\frac{\hat{\sigma}_{\lambda\lambda}}{2\hat{\lambda}}\Big(-\frac{1}{\hat{\lambda}}+2\hat{\rho}_\lambda+(\hat{L}_{aa\lambda}\hat{\sigma}_{aa}+\hat{L}_{bb\lambda}\hat{\sigma}_{bb}+\hat{L}_{\lambda\lambda\lambda}\hat{\sigma}_{\lambda\lambda})\Big)\Big]$

Table 4: Continue Table 3

Loss function	Parameters	TPMOEX
	a	$\hat{a}\Big[1+\Big(\frac{c(c+1)}{2\hat{a}^2}-\frac{c}{\hat{a}}\hat{\rho}_a\Big)\hat{\sigma}_{aa}-\frac{c}{2\hat{a}}\hat{\sigma}_{aa}\Big(\hat{L}_{aaa}\hat{\sigma}_{aa}+\hat{L}_{bba}\hat{\sigma}_{bb}+\hat{L}_{\lambda\lambda a}\hat{\sigma}_{\lambda\lambda}\Big)\Big]^{-\frac{1}{c}}$
Modified Linex	b	$\hat{b}\Big[1+\Big(\frac{c(c+1)}{2\hat{b}^2}-\frac{c}{\hat{b}}\hat{\rho}_b\Big)\hat{\sigma}_{bb}-\frac{c}{2\hat{b}}\hat{\sigma}_{bb}\Big(\hat{L}_{aab}\hat{\sigma}_{aa}+\hat{L}_{bbb}\hat{\sigma}_{bb}+\hat{L}_{\lambda\lambda b}\hat{\sigma}_{\lambda\lambda}\Big)\Big]^{-\frac{1}{c}}$
	λ	$\hat{\lambda}\Big[1+\Big(\frac{c(c+1)}{2\hat{\lambda}^2}-\frac{c}{\hat{\lambda}}\hat{\rho}_\lambda\Big)\hat{\sigma}_{\lambda\lambda}-\frac{c}{2\hat{\lambda}}\hat{\sigma}_{\lambda\lambda}\Big(\hat{L}_{aa\lambda}\hat{\sigma}_{aa}+\hat{L}_{bb\lambda}\hat{\sigma}_{bb}+\hat{L}_{\lambda\lambda\lambda}\hat{\sigma}_{\lambda\lambda}\Big)\Big]^{-\frac{1}{c}}$

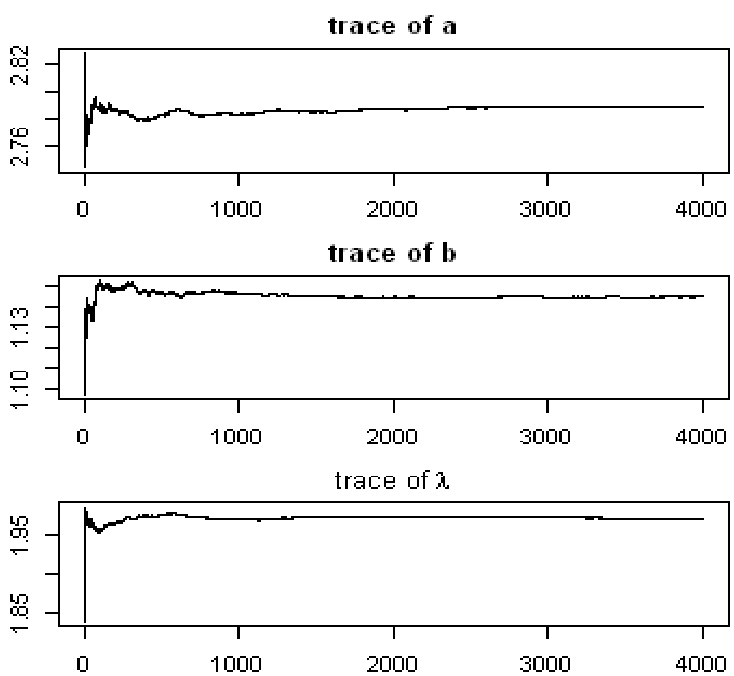

Figure 1: Diagram of the effect of TPMOEX distribution parameters for unconsciousness with size of $n = 50$.

Table 5: Average error for the prior case informative with low variance

		MCMC			LINDLEY		
Sample size	Loss function	a	b	λ	a	b	λ
20	Square error	3.72	0.19	0.60	4.82	0.39	0.97
	Entropy	3.36	0.18	0.57	4.71	0.35	0.87
	Linex	3.13	0.18	0.55	4.65	0.31	0.84
	Square error in logarithm	3.39	0.18	0.58	4.58	0.30	0.82
30	Square error	2.69	0.17	0.47	3.09	0.23	0.60
	Entropy	2.63	0.16	0.45	3.37	0.30	0.74
	Linex	2.51	0.16	0.51	3.33	0.28	0.67
	Square error in logarithm	2.66	0.17	0.56	3.21	0.28	0.77
50	Square error	2.31	0.15	0.41	2.87	0.14	0.53
	Entropy	1.69	0.14	0.39	2.13	0.16	0.69
	Linex	1.62	0.14	0.48	2.31	0.20	0.53
	Square error in logarithm	1.71	0.14	0.50	2.30	0.13	0.61

In the following, 1000 simulated samples with sizes of $n = 20, 30, 50$ will be generated from the TPMOEX distribution and Bayesian estimates are obtained based on both methods and different loss functions, and finally their average error squares are reported in the relevant tables. Note that to implement the MCMC method for obtaining Bayesian estimates of the parameters and to achieve convergence of the parameters as well as convergence of the generated Markov chain to its stationary distributions, this chain is repeated 4000 times and the initial 1000 repeats are disregarded to calculate the estimates.

According to Tables 5 to 7, it is clear that not much difference in different loss functions in Bayesian estimates in both methods can be found. Also, it is clear that both Lindley and Monte Carlo Markov chains have almost the same function in estimating the parameters, but as it is clear, MCMC method estimations have a lower MSE than Lindley estimation with the same loss function, however this difference in b and λ is negligible. In estimating the (a) parameter, this difference can be used as a criterion for comparing the performance of the two methods and it can be concluded that MCMC estimates are more optimal than Lindley approximation. It should also be noted that in all cases, with increasing sample volume the MSE values decrease, and this indicates that the estimates obtained in this study are consistent. It is worth mentioning that the lower the prior information of the hyper parameters, their MSE increases, and this subject can be clearly seen in the comparison of Tables 5, 6 and 7.

Table 6: Average error for the prior case informative with high variance

		MCMC			LINDLEY		
Sample size	Loss function	a	b	λ	a	b	λ
20	Square error	6.95	0.57	0.93	7.84	0.76	1.28
	Entropy	5.89	0.59	0.92	7.06	0.75	1.12
	Linex	5.69	0.57	0.81	7.64	0.76	1.10
	Square error in logarithm	5.92	0.67	0.93	7.44	0.78	1.14
30	Square error	5.67	0.45	0.78	6.57	0.51	0.90
	Entropy	5.62	0.46	0.78	6.56	0.56	0.88
	Linex	5.49	0.45	0.77	6.16	0.69	0.93
	Square error in logarithm	5.64	0.45	0.78	6.23	0.58	0.99
50	Square error	4.02	0.36	0.63	5.33	0.48	0.87
	Entropy	3.87	0.37	0.61	4.33	0.39	0.74
	Linex	3.79	0.29	0.50	4.16	0.25	0.87
	Square error in logarithm	3.89	0.30	0.52	5.86	0.39	0.90

Table 7: Average error for the prior cases in unconsciousness

		MCMC			LINDLEY		
Sample size	Loss function	a	b	λ	a	b	λ
20	Square error	10.87	1.589	1.58	11.68	1.89	2.68
	Entropy	9.23	1.52	1.72	11.82	1.46	2.21
	Linex	8.18	1.63	1.73	10.32	1.98	2.15
	Square error in logarithm	8.82	1.74	1.25	9.65	1.85	1.32
30	Square error	10.03	1.12	1.41	10.57	1.45	2.23
	Entropy	8.84	0.98	1.20	9.56	1.12	2.01
	Linex	7.87	1.25	1.02	8.12	1.23	1.93
	Square error in logarithm	8.14	1.41	1.11	9.24	1.58	2.02
50	Square error	9.23	0.96	1.12	9.58	1.14	1.98
	Entropy	7.74	0.95	1.11	8.96	1.02	1.85
	Linex	6.84	1.08	0.97	7.96	1.05	1.23
	Square error in logarithm	7.92	1.13	1.08	8.71	1.29	1.58

Table 8: The results of the good fit criteria

Loss function	MCMC			LINDLEY		
	-loglik	AIC	BIC	-loglik	AIC	BIC
Square error	41.65	89.30	91.61	46.49	98.98	101.30
Entropy	41.21	88.43	90.74	47.32	100.65	102.97
Linex	41.06	88.12	90.44	47.81	101.62	103.94
Square error in logarithm	42.22	90.45	92.77	47.54	101.09	103.41

Practical example
In this section a series of real data sets to implement the proposed method are used. This data actually shows the failure time of software release with an average length of 1000 hours.

0.519	0.968	1.43	1.893
2.49	3.058	3.625	4.442
5.218	5.823	6.539	7.083
7.485	7.846	8.205	8.564

Table 8 shows the values with good criteria for logLik, AIC and BIC fit from the fit of the TPMOEX distribution with respect to different loss functions and both Lindley MCMC methods that considering them, it is conducted that Markov Mont Karlov chain method considering all loss functions have better fit and among these, Bayesian estimation under the Linux loss function has the best fit criteria than others, so it is conducted that it has created a better fit.

As a result, the amounts of parameter estimations are reported in Table 9:

3.2 Bayesian statistics for TPMOMW

The estimation of parameters for both models will be done based on Lindley and MCMC methods based on different loss functions. The Metropolis-Hastings algorithm will be used to generate a sample from the complete conditional distributions of the parameters. Consider that if the Markov chain is repeated N times, then the first M samples are considered as the burnt samples. In this case, the estimation of MCMC

Table 9: Parameter estimations

Loss function	MCMC			LINDLEY		
	a	b	λ	a	b	λ
Square error	8.48	2.23	0.49	8.24	2.99	0.48
Entropy	8.47	2.10	0.48	8.38	3.04	0.47
Linex	8.43	2.12	0.49	8.61	3.08	0.47
Square error in logarithm	8.47	2.17	0.46	8.63	3.02	0.47

parameters is obtained based on different loss functions in Tables 10 to 12.

For the reason that in Bayesian approach, selection of super parameters in posterior distribution is very important, hence for analyzing various amounts of super parameters on the final results, consciousness and unconsciousness posteriors are used. In the case of previous backgrounds, the super parameters are adjusted so that the average of the previous distributions is equal to a conjuctured value of the model parameters. For further investigations, two sets of super parameters can be considered, in the first case the variance of the antecedents is small and in the second case the variance of the antecedents is large. But in the unconscious state, it was assumed that no information about the super parameters was available and their value was considered zero. In this research, for simulation, the actual value of all parameters is considered equal to 2, and based on this, the super parameters of the previous gamma distributions can be defined as follow:
A. Awareness with low variance: In this case, the parameters of distribution shape and gamma distribution scale are considered 4 and 2, respectively. In this case, the previous mathematical expectation of the parameters will be equal to their actual value (i.e, 2) and the variance of the previous distribution of the parameters will be equal to one.

B. Awareness with high variance: In this case, the shape parameters and the scale of the gamma distribution parameters are considered equal to 0.4 and 0.2, respectively. It is clear that in this case the average of the parameters as in the previous case is equal to 2 but their variance is equal to 10.

Table 10: The Bayesian estimators of the parameters under different loss functions

Loss functions	Parameters	TPMOMW
	a	$\hat{a}+\hat{\rho}_a\hat{\sigma}_{aa}+\frac{1}{2}\Big[\hat{\sigma}_{aa}\Big(\hat{L}_{aaa}\hat{\sigma}_{aa}+\hat{L}_{bba}\hat{\sigma}_{bb}+\hat{L}_{\alpha\alpha a}\hat{\sigma}_{\alpha\alpha}+\hat{L}_{\lambda\lambda a}\hat{\sigma}_{\lambda\lambda}+\hat{L}_{\gamma\gamma a}\hat{\sigma}_{\gamma\gamma}\Big)\Big]$
	b	$\hat{b}+\hat{\rho}_b\hat{\sigma}_{bb}+\frac{1}{2}\Big[\hat{\sigma}_{bb}\Big(\hat{L}_{aab}\hat{\sigma}_{aa}+\hat{L}_{bbb}\hat{\sigma}_{bb}+\hat{L}_{\alpha\alpha b}\hat{\sigma}_{\alpha\alpha}+\hat{L}_{\lambda\lambda b}\hat{\sigma}_{\lambda\lambda}+\hat{L}_{\gamma\gamma b}\hat{\sigma}_{\gamma\gamma}\Big)\Big]$
Square error	α	$\hat{\alpha}+\hat{\rho}_\alpha\hat{\sigma}_{\alpha\alpha}+\frac{1}{2}\Big[\hat{\sigma}_{\alpha\alpha}\Big(\hat{L}_{aa\alpha}\hat{\sigma}_{aa}+\hat{L}_{bb\alpha}\hat{\sigma}_{bb}+\hat{L}_{\alpha\alpha\alpha}\hat{\sigma}_{\alpha\alpha}+\hat{L}_{\lambda\lambda\alpha}\hat{\sigma}_{\lambda\lambda}+\hat{L}_{\gamma\gamma\alpha}\hat{\sigma}_{\gamma\gamma}\Big)\Big]$
	λ	$\hat{\lambda}+\hat{\rho}_\lambda\hat{\sigma}_{\lambda\lambda}+\frac{1}{2}\Big[\hat{\sigma}_{\lambda\lambda}\Big(\hat{L}_{aa\lambda}\hat{\sigma}_{aa}+\hat{L}_{bb\lambda}\hat{\sigma}_{bb}+\hat{L}_{\alpha\alpha\lambda}\hat{\sigma}_{\alpha\alpha}+\hat{L}_{\lambda\lambda\lambda}\hat{\sigma}_{\lambda\lambda}+\hat{L}_{\gamma\gamma\lambda}\hat{\sigma}_{\gamma\gamma}\Big)\Big]$
	γ	$\hat{\gamma}+\hat{\rho}_\gamma\hat{\sigma}_{\gamma\gamma}+\frac{1}{2}\Big[\hat{\sigma}_{\gamma\gamma}\Big(\hat{L}_{aa\gamma}\hat{\sigma}_{aa}+\hat{L}_{bb\gamma}\hat{\sigma}_{bb}+\hat{L}_{\alpha\alpha\gamma}\hat{\sigma}_{\alpha\alpha}+\hat{L}_{\lambda\lambda\gamma}\hat{\sigma}_{\lambda\lambda}+\hat{L}_{\gamma\gamma\gamma}\hat{\sigma}_{\gamma\gamma}\Big)\Big]$
	a	$\hat{a}\Big[1+\Big(\frac{1}{\hat{a}^2}-\frac{1}{\hat{a}}\hat{\rho}_a\Big)\hat{\sigma}_{aa}-\frac{1}{2\hat{a}}\hat{\sigma}_{aa}\Big(\hat{L}_{aaa}\hat{\sigma}_{aa}+\hat{L}_{bba}\hat{\sigma}_{bb}+\hat{L}_{\alpha\alpha a}\hat{\sigma}_{\alpha\alpha}+\hat{L}_{\lambda\lambda a}\hat{\sigma}_{\lambda\lambda}+\hat{L}_{\gamma\gamma a}\hat{\sigma}_{\gamma\gamma}\Big)\Big]^{-1}$
	b	$\hat{b}\Big[1+\Big(\frac{1}{\hat{b}^2}-\frac{1}{\hat{b}}\hat{\rho}_b\Big)\hat{\sigma}_{bb}-\frac{1}{2\hat{b}}\hat{\sigma}_{bb}\Big(\hat{L}_{aab}\hat{\sigma}_{aa}+\hat{L}_{bbb}\hat{\sigma}_{bb}+\hat{L}_{\alpha\alpha b}\hat{\sigma}_{\alpha\alpha}+\hat{L}_{\lambda\lambda b}\hat{\sigma}_{\lambda\lambda}+\hat{L}_{\gamma\gamma b}\hat{\sigma}_{\gamma\gamma}\Big)\Big]^{-1}$
Entropy	α	$\hat{\alpha}\Big[1+\Big(\frac{1}{\hat{\alpha}^2}-\frac{1}{\hat{\alpha}}\hat{\rho}_\alpha\Big)\hat{\sigma}_{\alpha\alpha}-\frac{1}{2\hat{\alpha}}\hat{\sigma}_{\alpha\alpha}\Big(\hat{L}_{aa\alpha}\hat{\sigma}_{aa}+\hat{L}_{bb\alpha}\hat{\sigma}_{bb}+\hat{L}_{\alpha\alpha\alpha}\hat{\sigma}_{\alpha\alpha}+\hat{L}_{\lambda\lambda\alpha}\hat{\sigma}_{\lambda\lambda}+\hat{L}_{\gamma\gamma\alpha}\hat{\sigma}_{\gamma\gamma}\Big)\Big]^{-1}$
	λ	$\hat{\lambda}\Big[1+\Big(\frac{1}{\hat{\lambda}^2}-\frac{1}{\hat{\lambda}}\hat{\rho}_\lambda\Big)\hat{\sigma}_{\lambda\lambda}-\frac{1}{2\hat{\lambda}}\hat{\sigma}_{\lambda\lambda}\Big(\hat{L}_{aa\lambda}\hat{\sigma}_{aa}+\hat{L}_{bb\lambda}\hat{\sigma}_{bb}+\hat{L}_{\alpha\alpha\lambda}\hat{\sigma}_{\alpha\alpha}+\hat{L}_{\lambda\lambda\lambda}\hat{\sigma}_{\lambda\lambda}+\hat{L}_{\gamma\gamma\lambda}\hat{\sigma}_{\gamma\gamma}\Big)\Big]^{-1}$
	γ	$\hat{\gamma}\Big[1+\Big(\frac{1}{\hat{\gamma}^2}-\frac{1}{\hat{\gamma}}\hat{\rho}_\gamma\Big)\hat{\sigma}_{\gamma\gamma}-\frac{1}{2\hat{\gamma}}\hat{\sigma}_{\gamma\gamma}\Big(\hat{L}_{aa\gamma}\hat{\sigma}_{aa}+\hat{L}_{bb\gamma}\hat{\sigma}_{bb}+\hat{L}_{\alpha\alpha\gamma}\hat{\sigma}_{\alpha\alpha}+\hat{L}_{\lambda\lambda\gamma}\hat{\sigma}_{\lambda\lambda}+\hat{L}_{\gamma\gamma\gamma}\hat{\sigma}_{\gamma\gamma}\Big)\Big]^{-1}$

Table 11: Continue Table 10

Loss functions	Parameters	TPMOMW
	a	$\hat{a}\exp\Big[\frac{\hat{\sigma}_{aa}}{2\hat{a}}\Big(-\frac{1}{\hat{a}}+2\hat{\rho}_a+\Big(\hat{L}_{aaa}\hat{\sigma}_{aa}+\hat{L}_{bba}\hat{\sigma}_{bb}+\hat{L}_{\alpha\alpha a}\hat{\sigma}_{\alpha\alpha}+\hat{L}_{\lambda\lambda a}\hat{\sigma}_{\lambda\lambda}+\hat{L}_{\gamma\gamma a}\hat{\sigma}_{\gamma\gamma}\Big)\Big)\Big]$
	b	$\hat{b}\exp\Big[\frac{\hat{\sigma}_{bb}}{2\hat{b}}\Big(-\frac{1}{\hat{b}}+2\hat{\rho}_b+\Big(\hat{L}_{aab}\hat{\sigma}_{aa}+\hat{L}_{bbb}\hat{\sigma}_{bb}+\hat{L}_{\alpha\alpha b}\hat{\sigma}_{\alpha\alpha}+\hat{L}_{\lambda\lambda b}\hat{\sigma}_{\lambda\lambda}+\hat{L}_{\gamma\gamma b}\hat{\sigma}_{\gamma\gamma}\Big)\Big)\Big]$
Square error in logarithm	α	$\hat{\alpha}\exp\Big[\frac{\hat{\sigma}_{\alpha\alpha}}{2\hat{\alpha}}\Big(-\frac{1}{\hat{\alpha}}+2\hat{\rho}_\alpha+\Big(\hat{L}_{aa\alpha}\hat{\sigma}_{aa}+\hat{L}_{bb\alpha}\hat{\sigma}_{bb}+\hat{L}_{\alpha\alpha\alpha}\hat{\sigma}_{\alpha\alpha}+\hat{L}_{\lambda\lambda\alpha}\hat{\sigma}_{\lambda\lambda}+\hat{L}_{\gamma\gamma\alpha}\hat{\sigma}_{\gamma\gamma}\Big)\Big)\Big]$
	λ	$\hat{\lambda}\exp\Big[\frac{\hat{\sigma}_{\lambda\lambda}}{2\hat{\lambda}}\Big(-\frac{1}{\hat{\lambda}}+2\hat{\rho}_\lambda+\Big(\hat{L}_{aa\lambda}\hat{\sigma}_{aa}+\hat{L}_{bb\lambda}\hat{\sigma}_{bb}+\hat{L}_{\alpha\alpha\lambda}\hat{\sigma}_{\alpha\alpha}+\hat{L}_{\lambda\lambda\lambda}\hat{\sigma}_{\lambda\lambda}+\hat{L}_{\gamma\gamma\lambda}\hat{\sigma}_{\gamma\gamma}\Big)\Big)\Big]$
	γ	$\hat{\gamma}\exp\Big[\frac{\hat{\sigma}_{\gamma\gamma}}{2\hat{\gamma}}\Big(-\frac{1}{\hat{\gamma}}+2\hat{\rho}_\gamma+\Big(\hat{L}_{aa\gamma}\hat{\sigma}_{aa}+\hat{L}_{bb\gamma}\hat{\sigma}_{bb}+\hat{L}_{\alpha\alpha\gamma}\hat{\sigma}_{\alpha\alpha}+\hat{L}_{\lambda\lambda\gamma}\hat{\sigma}_{\lambda\lambda}+\hat{L}_{\gamma\gamma\gamma}\hat{\sigma}_{\gamma\gamma}\Big)\Big)\Big]$
	a	$\hat{a}\Big[1+\Big(\frac{c(c+1)}{2\hat{a}^2}-\frac{c}{\hat{a}}\hat{\rho}_a\Big)\hat{\sigma}_{aa}-\frac{c}{2\hat{a}}\hat{\sigma}_{aa}\Big(\hat{L}_{aaa}\hat{\sigma}_{aa}+\hat{L}_{bba}\hat{\sigma}_{bb}+\hat{L}_{\alpha\alpha a}\hat{\sigma}_{\alpha\alpha}+\hat{L}_{\lambda\lambda a}\hat{\sigma}_{\lambda\lambda}+\hat{L}_{\gamma\gamma a}\hat{\sigma}_{\gamma\gamma}\Big)\Big]^{-\frac{1}{c}}$
	b	$\hat{b}\Big[1+\Big(\frac{c(c+1)}{2\hat{b}^2}-\frac{c}{\hat{b}}\hat{\rho}_b\Big)\hat{\sigma}_{bb}-\frac{c}{2\hat{b}}\hat{\sigma}_{bb}\Big(\hat{L}_{aab}\hat{\sigma}_{aa}+\hat{L}_{bbb}\hat{\sigma}_{bb}+\hat{L}_{\alpha\alpha b}\hat{\sigma}_{\alpha\alpha}+\hat{L}_{\lambda\lambda b}\hat{\sigma}_{\lambda\lambda}+\hat{L}_{\gamma\gamma b}\hat{\sigma}_{\gamma\gamma}\Big)\Big]^{-\frac{1}{c}}$
Modified Linex	α	$\hat{\alpha}\Big[1+\Big(\frac{c(c+1)}{2\hat{\alpha}^2}-\frac{c}{\hat{\alpha}}\hat{\rho}_\alpha\Big)\hat{\sigma}_{\alpha\alpha}-\frac{c}{2\hat{\alpha}}\hat{\sigma}_{\alpha\alpha}\Big(\hat{L}_{aa\alpha}\hat{\sigma}_{aa}+\hat{L}_{bb\alpha}\hat{\sigma}_{bb}+\hat{L}_{\alpha\alpha\alpha}\hat{\sigma}_{\alpha\alpha}+\hat{L}_{\lambda\lambda\alpha}\hat{\sigma}_{\lambda\lambda}+\hat{L}_{\gamma\gamma\alpha}\hat{\sigma}_{\gamma\gamma}\Big)\Big]^{-\frac{1}{c}}$
	λ	$\hat{\lambda}\Big[1+\Big(\frac{c(c+1)}{2\hat{\lambda}^2}-\frac{c}{\hat{\lambda}}\hat{\rho}_\lambda\Big)\hat{\sigma}_{\lambda\lambda}-\frac{c}{2\hat{\lambda}}\hat{\sigma}_{\lambda\lambda}\Big(\hat{L}_{aa\lambda}\hat{\sigma}_{aa}+\hat{L}_{bb\lambda}\hat{\sigma}_{bb}+\hat{L}_{\alpha\alpha\lambda}\hat{\sigma}_{\alpha\alpha}+\hat{L}_{\lambda\lambda\lambda}\hat{\sigma}_{\lambda\lambda}+\hat{L}_{\gamma\gamma\lambda}\hat{\sigma}_{\gamma\gamma}\Big)\Big]^{-\frac{1}{c}}$
	γ	$\hat{\gamma}\Big[1+\Big(\frac{c(c+1)}{2\hat{\gamma}^2}-\frac{c}{\hat{\gamma}}\hat{\rho}_\gamma\Big)\hat{\sigma}_{\gamma\gamma}-\frac{c}{2\hat{\gamma}}\hat{\sigma}_{\gamma\gamma}\Big(\hat{L}_{aa\gamma}\hat{\sigma}_{aa}+\hat{L}_{bb\gamma}\hat{\sigma}_{bb}+\hat{L}_{\alpha\alpha\gamma}\hat{\sigma}_{\alpha\alpha}+\hat{L}_{\lambda\lambda\gamma}\hat{\sigma}_{\lambda\lambda}+\hat{L}_{\gamma\gamma\gamma}\hat{\sigma}_{\gamma\gamma}\Big)\Big]^{-\frac{1}{c}}$

Table 12: Continue Table 10

Loss functions	Parameters	TPMOMW
	a	$\hat{a} - ln\Big[1 + (\frac{1}{2} - \hat{\rho}_a)\hat{\sigma}_{aa} - \frac{1}{2}\hat{\sigma}_{aa}\Big(\hat{L}_{aaa}\hat{\sigma}_{aa} + \hat{L}_{bba}\hat{\sigma}_{bb} + \hat{L}_{\alpha\alpha a}\hat{\sigma}_{\alpha\alpha} + \hat{L}_{\lambda\lambda a}\hat{\sigma}_{\lambda\lambda} + \hat{L}_{\gamma\gamma a}\hat{\sigma}_{\gamma\gamma}\Big)\Big]$
	b	$\hat{b} - ln\Big[1 + (\frac{1}{2} - \hat{\rho}_b)\hat{\sigma}_{bb} - \frac{1}{2}\hat{\sigma}_{bb}\Big(\hat{L}_{aab}\hat{\sigma}_{aa} + \hat{L}_{bbb}\hat{\sigma}_{bb} + \hat{L}_{\alpha\alpha b}\hat{\sigma}_{\alpha\alpha} + \hat{L}_{\lambda\lambda b}\hat{\sigma}_{\lambda\lambda} + \hat{L}_{\gamma\gamma b}\hat{\sigma}_{\gamma\gamma}\Big)\Big]$
Linex	α	$\hat{\alpha} - ln\Big[1 + (\frac{1}{2} - \hat{\rho}_\alpha)\hat{\sigma}_{\alpha\alpha} - \frac{1}{2}\hat{\sigma}_{\alpha\alpha}\Big(\hat{L}_{aa\alpha}\hat{\sigma}_{aa} + \hat{L}_{bb\alpha}\hat{\sigma}_{bb} + \hat{L}_{\alpha\alpha\alpha}\hat{\sigma}_{\alpha\alpha} + \hat{L}_{\lambda\lambda\alpha}\hat{\sigma}_{\lambda\lambda} + \hat{L}_{\gamma\gamma\alpha}\hat{\sigma}_{\gamma\gamma}\Big)\Big]$
	λ	$\hat{\lambda} - ln\Big[1 + (\frac{1}{2} - \hat{\rho}_\lambda)\hat{\sigma}_{\lambda\lambda} - \frac{1}{2}\hat{\sigma}_{\lambda\lambda}\Big(\hat{L}_{aa\lambda}\hat{\sigma}_{aa} + \hat{L}_{bb\lambda}\hat{\sigma}_{bb} + \hat{L}_{\alpha\alpha\lambda}\hat{\sigma}_{\alpha\alpha} + \hat{L}_{\lambda\lambda\lambda}\hat{\sigma}_{\lambda\lambda} + \hat{L}_{\gamma\gamma\lambda}\hat{\sigma}_{\gamma\gamma}\Big)\Big]$
	γ	$\hat{\gamma} - ln\Big[1 + (\frac{1}{2} - \hat{\rho}_\gamma)\hat{\sigma}_{\gamma\gamma} - \frac{1}{2}\hat{\sigma}_{\gamma\gamma}\Big(\hat{L}_{aa\gamma}\hat{\sigma}_{aa} + \hat{L}_{bb\gamma}\hat{\sigma}_{bb} + \hat{L}_{\alpha\alpha\gamma}\hat{\sigma}_{\alpha\alpha} + \hat{L}_{\lambda\lambda\gamma}\hat{\sigma}_{\lambda\lambda} + \hat{L}_{\gamma\gamma\gamma}\hat{\sigma}_{\gamma\gamma}\Big)\Big]$

Table 13: The parameter estimates based on different loss functions

Loss function	Parameter estimates
Square error	$\hat{a}_s = \frac{1}{N-M}\sum_{i=M+1}^{N} a^{(i)}$ $\hat{b}_s = \frac{1}{N-M}\sum_{i=M+1}^{N} \lambda^{(i)}$ $\hat{\lambda}_s = \frac{1}{N-M}\sum_{i=M+1}^{N} \lambda^{(i)}$
Entropy	$\hat{a}_E = \frac{N-M}{\sum_{i=M+1}^{N}\frac{1}{a^{(i)}}}$ $\hat{b}_E = \frac{N-M}{\sum_{i=M+1}^{N}\frac{1}{b^{(i)}}}$ $\hat{\lambda}_E = \frac{N-M}{\sum_{i=M+1}^{N}\frac{1}{\lambda^{(i)}}}$
Linex	$\hat{a}_L = -Ln\left\{\frac{\sum_{i=M+1}^{N} e^{-a^{(i)}}}{N-M}\right\}$ $\hat{b}_L = -Ln\left\{\frac{\sum_{i=M+1}^{N} e^{-b^{(i)}}}{N-M}\right\}$ $\hat{\lambda}_L = -Ln\left\{\frac{\sum_{i=M+1}^{N} e^{-\lambda^{(i)}}}{N-M}\right\}$
Square error in logarithm	$\hat{a}_{SL} = e^{\frac{1}{N-M}\sum_{i=M+1}^{N} Lna^{(i)}}$ $\hat{b}_{SL} = e^{\frac{1}{N-M}\sum_{i=M+1}^{N} Lnb^{(i)}}$ $\hat{\lambda}_{SL} = e^{\frac{1}{N-M}\sum_{i=M+1}^{N} Ln\lambda^{(i)}}$

C. Unconsciousness: In this value, all super parameters are considered zero. This mode is used when the researcher has no information about the actual value of the previous distribution super parameters. And, therefore for showing convergence of the generated chains to its stationary distribution, the plot of parameter effect will be used. Figure 2 shows the effect of the TPMOMW parameters for a simulated sample with $n = 50$ size and for the unconciousness mode, where the created convergence by the Monte Carlo Markov chain is seen as well.

In the following, 1000 simulated samples with sizes of $n = 30, 50, 70$ will be produced from TPMOMW and Bayesian estimates and it is obtained based on both methods and different loss functions, and finally the mean squares of their error are reported in the relevant tables. Note that in order to implement the MCMC method to obtain Bayesian estimates of the parameters and to achieve convergence of the parameters as well as convergence of the generated Markov chain to its mean distributions, this chain is repeated 4000 times and the initial 1000 iterations are omitted to calculate the estimates.
Considering Tables 14 to 16, it is clear that there is not much difference in various loss functions in Bayesian estimates in both methods. It is also defined that the Lindley method has produced more appropriate estimates because it has a smaller MSE than the MCMC method, and it can be seen that by increasing the sample size, the quadratic values of the squares error become smalle, which indicates the compatibility of the estimators. It is also observed that with decreasing prior information, the MSE estimators' values increase.

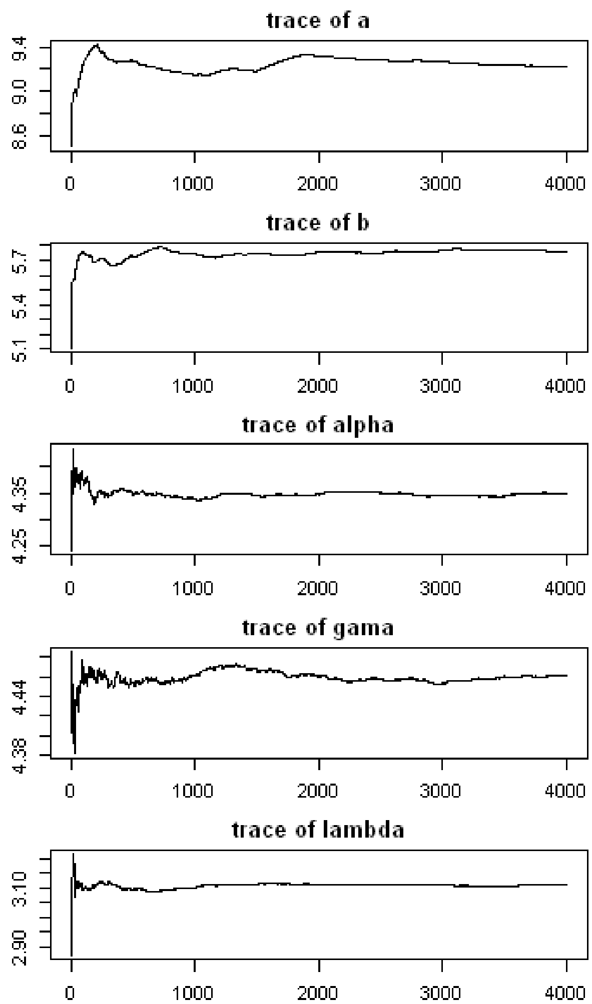

Figure 2: Diagram of the effect of TPMOMW distribution parameters for unconciousness with size $n = 50$.

Table 14: The amount of total square error in the previous cases of consciousness with low variance (TPMOMW)

		Square error	Entropy	Linex	Square error in logarithm	Modified linex
$n = 30$						
LINDLEY	a	8.2184	7.3135	8.2761	7.5778	7.4097
	b	8.5979	8.3028	8.3875	8.3671	7.5179
	α	7.0113	6.2331	6.3274	6.4246	6.9259
	γ	9.8631	9.8971	9.1321	10.3501	9.7418
	λ	5.8436	4.8618	5.0181	5.4814	5.7123
MCMC	a	14.8828	15.8604	16.0664	14.5365	15.4199
	b	12.0279	11.3904	11.4569	11.6387	12.8003
	α	12.0498	11.6041	12.1761	11.7003	11.5330
	γ	11.3189	9.9851	11.1938	11.2490	10.5979
	λ	8.7283	8.0027	8.1950	8.2865	8.6764
$n = 50$						
		Square error	Entropy	Linex	Square error in logarithm	Modified linex
LINDLEY	a	5.0680	6.1482	5.7079	4.5039	5.2269
	b	5.6194	5.6321	6.2656	4.6564	6.4466
	α	4.1204	4.0825	4.6564	4.6868	4.2785
	γ	6.1905	7.8764	7.3787	6.5224	6.7595
	λ	1.6368	2.2915	2.9264	2.3722	3.8301
MCMC	a	12.0750	13.6014	13.5839	12.1522	13.6306
	b	10.0379	9.1825	9.1279	10.2037	10.1124
	α	9.1458	8.2067	9.7831	7.5984	9.7724
	γ	8.4768	7.8344	8.3746	7.5886	8.2571
	λ	6.4948	5.1409	5.4786	6.3033	5.8315
$n = 70$						
		Square error	Entropy	Linex	Square error in logarithm	Modified linex
LINDLEY	a	3.4265	3.1411	4.4089	2.8112	3.3820
	b	4.2670	3.9709	3.8853	3.4916	4.4514
	α	1.9585	2.2564	3.3654	2.9032	2.0262
	γ	5.7972	6.1664	5.2831	5.1886	5.1244
	λ	0.6319	0.6734	1.1897	0.6382	2.0504
MCMC	a	10.3469	11.9402	11.7098	10.1217	11.4566
	b	7.8353	7.3869	7.7387	9.0312	7.9452
	α	8.0291	7.3405	7.4899	7.4759	7.8920
	γ	7.0665	6.2230	7.0547	6.6189	7.2703
	λ	5.5671	4.0208	3.9761	4.0385	5.2993

Table 15: Total values of square error in the case of high-variance consciousness case (TPMOMW)

		Square error	Entropy	Linex	Square error in logarithm	Modified linex
		$n=30$				
LINDLEY	a	10.5462	9.9581	10.4080	9.9990	10.3731
	b	11.2691	10.8650	11.1901	10.8801	10.3416
	α	9.5678	8.7241	9.2349	8.7066	9.1856
	γ	11.9213	12.1923	11.7355	12.5431	11.8550
	λ	7.9913	7.6303	7.2656	7.5252	7.9398
MCMC	a	17.3437	18.3015	18.1825	17.2390	18.3107
	b	14.6019	14.2093	14.1603	14.5034	15.0004
	α	14.1915	13.8062	14.5870	14.3145	14.0956
	γ	13.7307	12.9726	13.8894	13.7954	13.4282
	λ	11.3786	10.9441	10.8305	11.1559	10.7117
		$n=50$				
		Square error	Entropy	Linex	Square error in logarithm	Modified linex
LINDLEY	a	7.6903	8.3593	8.2233	7.4030	8.1871
	b	7.7419	7.7921	8.2719	7.6546	8.6864
	α	7.0116	6.8519	6.8944	7.3346	6.5061
	γ	8.7122	9.9246	9.5523	8.7462	9.4475
	λ	4.6128	4.4340	5.5532	5.3652	6.0003
MCMC	a	14.4989	15.7353	16.1969	14.5878	15.8437
	b	12.3819	11.6488	12.1223	12.6688	12.2586
	α	11.6218	10.7564	12.4803	10.5327	12.0337
	γ	10.6661	10.4015	11.2674	10.4415	10.6460
	λ	9.2144	8.0690	7.6920	8.4756	8.6077
		$n=70$				
		Square error	Entropy	Linex	Square error in logarithm	Modified linex
LINDLEY	a	5.9605	6.1401	6.8552	5.7735	5.8885
	b	6.3809	6.1411	6.8366	6.3539	7.1692
	α	4.2511	5.2489	5.8686	5.3123	4.3528
	γ	8.1545	8.5507	7.7888	7.3689	7.9817
	λ	3.1710	3.0139	3.6596	3.6212	4.4135
MCMC	a	12.4563	13.9733	14.5366	12.3945	13.5952
	b	10.3722	9.5846	10.4824	11.3793	10.6800
	α	10.6391	9.8666	10.3398	9.7635	10.3919
	γ	9.4135	8.9513	10.0450	9.0261	9.7471
	λ	8.1745	6.3927	6.8051	6.6646	7.5478

Table 16: Total error values in unconscious previous case (TPMOMW)

		Square error	Entropy	Linex	Square error in logarithm	Modified linex
				$n = 30$		
LINDLEY	a	12.7748	12.7558	12.7385	12.7654	12.7558
	b	13.2768	13.2368	13.2040	13.2569	13.2368
	α	11.6383	11.6009	11.5778	11.6196	11.6009
	γ	14.6463	14.6086	14.5853	14.6274	14.6086
	λ	10.2267	10.1956	10.1788	10.2111	10.1956
MCMC	a	19.4113	20.8696	20.6423	19.3178	20.8696
	b	17.3456	17.0027	17.1513	17.3698	17.0027
	α	16.3236	16.2548	16.7895	16.3249	16.3256
	γ	15.9025	15.9060	15.9043	15.9043	15.9060
	λ	13.9864	13.5221	12.9685	13.5211	13.5723
				$n = 50$		
		Square error	Entropy	Linex	Square error in logarithm	Modified linex
LINDLEY	a	10.1365	10.6977	10.7038	10.2362	10.4445
	b	10.7016	10.5660	10.7584	10.5343	10.8595
	α	9.0119	9.2466	9.5602	9.3821	9.0400
	γ	11.6636	12.4175	12.0266	11.6467	11.7413
	λ	7.5255	7.2469	7.7725	7.8043	8.0035
MCMC	a	17.0042	18.1156	18.6282	16.6011	18.3759
	b	14.8463	14.1712	14.7426	15.2163	14.7539
	α	14.0875	13.7026	14.5586	13.4191	14.3156
	γ	13.3729	13.2529	13.7688	12.9498	13.2316
	λ	11.9252	10.6303	10.1970	11.2982	11.1585
				$n = 70$		
		Square error	Entropy	Linex	Square error in logarithm	Modified linex
LINDLEY	a	8.8681	8.7613	9.5471	8.5475	8.7574
	b	9.2415	8.9832	9.3274	8.6123	9.8407
	α	7.1383	8.0344	8.3299	7.4730	7.2189
	γ	10.5596	10.6604	10.4766	10.1171	10.5062
	λ	6.1267	5.9774	6.3517	6.6008	6.4147
MCMC	a	15.2826	16.8429	17.2891	15.1303	16.5823
	b	13.3040	12.3917	13.0748	14.0324	12.9003
	α	12.9994	12.0412	12.6152	11.8609	12.7879
	γ	12.0986	11.4566	12.4962	11.6787	12.1152
	λ	10.7255	8.9300	8.8431	9.4993	10.1236

Table 17: The Bayesian estimators for TPMOGO under different loss functions

Loss function	Parameters	TPMOGO
	a	$\hat{a}+\hat{\rho}_a\hat{\sigma}_{aa}+\frac{1}{2}\Big[\hat{\sigma}_{aa}\Big(\hat{L}_{aaa}\hat{\sigma}_{aa}+\hat{L}_{bba}\hat{\sigma}_{bb}+\hat{L}_{\alpha\alpha a}\hat{\sigma}_{\alpha\alpha}+\hat{L}_{\beta\beta a}\hat{\sigma}_{\beta\beta}\Big)\Big]$
	b	$\hat{b}+\hat{\rho}_b\hat{\sigma}_{bb}+\frac{1}{2}\Big[\hat{\sigma}_{bb}\Big(\hat{L}_{aab}\hat{\sigma}_{aa}+\hat{L}_{bbb}\hat{\sigma}_{bb}+\hat{L}_{\alpha\alpha b}\hat{\sigma}_{\alpha\alpha}+\hat{L}_{\beta\beta b}\hat{\sigma}_{\beta\beta}\Big)\Big]$
Square error	α	$\hat{\alpha}+\hat{\rho}_\alpha\hat{\sigma}_{\alpha\alpha}+\frac{1}{2}\Big[\hat{\sigma}_{\alpha\alpha}\Big(\hat{L}_{aa\alpha}\hat{\sigma}_{aa}+\hat{L}_{bb\alpha}\hat{\sigma}_{bb}+\hat{L}_{\alpha\alpha\alpha}\hat{\sigma}_{\alpha\alpha}+\hat{L}_{\beta\beta\alpha}\hat{\sigma}_{\beta\beta}\Big)\Big]$
	β	$\hat{\beta}+\hat{\rho}_\beta\hat{\sigma}_{\beta\beta}+\frac{1}{2}\Big[\hat{\sigma}_{\beta\beta}\Big(\hat{L}_{aa\beta}\hat{\sigma}_{aa}+\hat{L}_{bb\beta}\hat{\sigma}_{bb}+\hat{L}_{\alpha\alpha\beta}\hat{\sigma}_{\alpha\alpha}+\hat{L}_{\beta\beta\beta}\hat{\sigma}_{\beta\beta}\Big)\Big]$
	a	$\hat{a}\Big[1+\Big(\frac{1}{\hat{a}^2}-\frac{1}{\hat{a}}\hat{\rho}_a\Big)\hat{\sigma}_{aa}-\frac{1}{2\hat{a}}\hat{\sigma}_{aa}\Big(\hat{L}_{aaa}\hat{\sigma}_{aa}+\hat{L}_{bba}\hat{\sigma}_{bb}+\hat{L}_{\alpha\alpha a}\hat{\sigma}_{\alpha\alpha}+\hat{L}_{\beta\beta a}\hat{\sigma}_{\beta\beta}\Big)\Big]^{-1}$
	b	$\hat{b}\Big[1+\Big(\frac{1}{\hat{b}^2}-\frac{1}{\hat{b}}\hat{\rho}_b\Big)\hat{\sigma}_{bb}-\frac{1}{2\hat{b}}\hat{\sigma}_{bb}\Big(\hat{L}_{aab}\hat{\sigma}_{aa}+\hat{L}_{bbb}\hat{\sigma}_{bb}+\hat{L}_{\alpha\alpha b}\hat{\sigma}_{\alpha\alpha}+\hat{L}_{\beta\beta b}\hat{\sigma}_{\beta\beta}\Big)\Big]^{-1}$
Entropy	α	$\hat{\alpha}\Big[1+\Big(\frac{1}{\hat{\alpha}^2}-\frac{1}{\hat{\alpha}}\hat{\rho}_\alpha\Big)\hat{\sigma}_{\alpha\alpha}-\frac{1}{2\hat{\alpha}}\hat{\sigma}_{\alpha\alpha}\Big(\hat{L}_{aa\alpha}\hat{\sigma}_{aa}+\hat{L}_{bb\alpha}\hat{\sigma}_{bb}+\hat{L}_{\alpha\alpha\alpha}\hat{\sigma}_{\alpha\alpha}+\hat{L}_{\beta\beta\alpha}\hat{\sigma}_{\beta\beta}\Big)\Big]^{-1}$
	β	$\hat{\beta}\Big[1+\Big(\frac{1}{\hat{\beta}^2}-\frac{1}{\hat{\beta}}\hat{\rho}_\beta\Big)\hat{\sigma}_{\beta\beta}-\frac{1}{2\hat{\beta}}\hat{\sigma}_{\beta\beta}\Big(\hat{L}_{aa\beta}\hat{\sigma}_{aa}+\hat{L}_{bb\beta}\hat{\sigma}_{bb}+\hat{L}_{\alpha\alpha\beta}\hat{\sigma}_{\alpha\alpha}+\hat{L}_{\beta\beta\beta}\hat{\sigma}_{\beta\beta}\Big)\Big]^{-1}$
	a	$\hat{a}-ln\Big[1+(\frac{1}{2}-\hat{\rho}_a)\hat{\sigma}_{aa}-\frac{1}{2}\hat{\sigma}_{aa}\Big(\hat{L}_{aaa}\hat{\sigma}_{aa}+\hat{L}_{bba}\hat{\sigma}_{bb}+\hat{L}_{\alpha\alpha a}\hat{\sigma}_{\alpha\alpha}+\hat{L}_{\beta\beta a}\hat{\sigma}_{\beta\beta}\Big)\Big]$
	b	$\hat{b}-ln\Big[1+(\frac{1}{2}-\hat{\rho}_b)\hat{\sigma}_{bb}-\frac{1}{2}\hat{\sigma}_{bb}\Big(\hat{L}_{aab}\hat{\sigma}_{aa}+\hat{L}_{bbb}\hat{\sigma}_{bb}+\hat{L}_{\alpha\alpha b}\hat{\sigma}_{\alpha\alpha}+\hat{L}_{\beta\beta b}\hat{\sigma}_{\beta\beta}\Big)\Big]$
Linex	α	$\hat{\alpha}-ln\Big[1+(\frac{1}{2}-\hat{\rho}_\alpha)\hat{\sigma}_{\alpha\alpha}-\frac{1}{2}\hat{\sigma}_{\alpha\alpha}\Big(\hat{L}_{aa\alpha}\hat{\sigma}_{aa}+\hat{L}_{bb\alpha}\hat{\sigma}_{bb}+\hat{L}_{\alpha\alpha\alpha}\hat{\sigma}_{\alpha\alpha}+\hat{L}_{\beta\beta\alpha}\hat{\sigma}_{\beta\beta}\Big)\Big]$
	β	$\hat{\beta}-ln\Big[1+(\frac{1}{2}-\hat{\rho}_\beta)\hat{\sigma}_{\beta\beta}-\frac{1}{2}\hat{\sigma}_{\beta\beta}\Big(\hat{L}_{aa\beta}\hat{\sigma}_{aa}+\hat{L}_{bb\beta}\hat{\sigma}_{bb}+\hat{L}_{\alpha\alpha\beta}\hat{\sigma}_{\alpha\alpha}+\hat{L}_{\beta\beta\beta}\hat{\sigma}_{\beta\beta}\Big)\Big]$

3.3 Bayesian statistics for TPMOGO

Figure 3 shows the effect of the TPMOGO model parameters for a simulated sample with $n = 50$, which shows the convergence created by the model Monte Carlo Markov chains.

As before, 1000 simulated samples with sizes of $n = 30, 50, 70$ will be generated from the TPMOGO distribution and Bayesian estimates are obtained based on both methods and based on different loss functions, and finally their mean squares of errors are reported in the relevant tables. Note that in order to implement the MCMC method to obtain Bayesian estimates of the parameters and to achieve convergence of the parameters as well as convergence of the generated Markov chain to its mean distributions, this chain is repeated 4000 times and the initial 1000 iterations are omitted to calculate the estimates.
According to Tables 19 to 21, it is clear that not much difference in various loss functions can be found in Bayesian estimates in both methods. It is also concluded that the Lindley method has generated more appropriate estimates because it has smaller MSE than the MCMC method, and it is also observed that as the sample size increases, the quadratic values of the squares error become smaller, which indicates the compatibility of the estimators.

Table 18: Continue Table 17

Loss function	Parameters	TPMOGO
	a	$\hat{a}\exp\Big[\frac{\hat{\sigma}_{aa}}{2\hat{a}}\Big(-\frac{1}{\hat{a}}+2\hat{\rho}_a+\Big(\hat{L}_{aaa}\hat{\sigma}_{aa}+\hat{L}_{bba}\hat{\sigma}_{bb}+\hat{L}_{\alpha\alpha a}\hat{\sigma}_{\alpha\alpha}+\hat{L}_{\beta\beta a}\hat{\sigma}_{\beta\beta}\Big)\Big)\Big]$
	b	$\hat{b}\exp\Big[\frac{\hat{\sigma}_{bb}}{2\hat{b}}\Big(-\frac{1}{\hat{b}}+2\hat{\rho}_b+\Big(\hat{L}_{aab}\hat{\sigma}_{aa}+\hat{L}_{bbb}\hat{\sigma}_{bb}+\hat{L}_{\alpha\alpha b}\hat{\sigma}_{\alpha\alpha}+\hat{L}_{\beta\beta b}\hat{\sigma}_{\beta\beta}\Big)\Big)\Big]$
Square error in logarithm	α	$\hat{\alpha}\exp\Big[\frac{\hat{\sigma}_{\alpha\alpha}}{2\hat{\alpha}}\Big(-\frac{1}{\hat{\alpha}}+2\hat{\rho}_\alpha+\Big(\hat{L}_{aa\alpha}\hat{\sigma}_{aa}+\hat{L}_{bb\alpha}\hat{\sigma}_{bb}+\hat{L}_{\alpha\alpha\alpha}\hat{\sigma}_{\alpha\alpha}+\hat{L}_{\beta\beta\alpha}\hat{\sigma}_{\beta\beta}\Big)\Big)\Big]$
	β	$\hat{\beta}\exp\Big[\frac{\hat{\sigma}_{\beta\beta}}{2\hat{\beta}}\Big(-\frac{1}{\hat{\beta}}+2\hat{\rho}_\beta+\Big(\hat{L}_{aa\beta}\hat{\sigma}_{aa}+\hat{L}_{bb\beta}\hat{\sigma}_{bb}+\hat{L}_{\alpha\alpha\beta}\hat{\sigma}_{\alpha\alpha}+\hat{L}_{\beta\beta\beta}\hat{\sigma}_{\beta\beta}\Big)\Big)\Big]$
	a	$\hat{a}\Big[1+\Big(\frac{c(c+1)}{2\hat{a}^2}-\frac{c}{\hat{a}}\hat{\rho}_a\Big)\hat{\sigma}_{aa}-\frac{c}{2\hat{a}}\hat{\sigma}_{aa}\Big(\hat{L}_{aaa}\hat{\sigma}_{aa}+\hat{L}_{bba}\hat{\sigma}_{bb}+\hat{L}_{\alpha\alpha a}\hat{\sigma}_{\alpha\alpha}+\hat{L}_{\beta\beta a}\hat{\sigma}_{\beta\beta}\Big)\Big]^{-\frac{1}{c}}$
	b	$\hat{b}\Big[1+\Big(\frac{c(c+1)}{2\hat{b}^2}-\frac{c}{\hat{b}}\hat{\rho}_b\Big)\hat{\sigma}_{bb}-\frac{c}{2\hat{b}}\hat{\sigma}_{bb}\Big(\hat{L}_{aab}\hat{\sigma}_{aa}+\hat{L}_{bbb}\hat{\sigma}_{bb}+\hat{L}_{\alpha\alpha b}\hat{\sigma}_{\alpha\alpha}+\hat{L}_{\beta\beta b}\hat{\sigma}_{\beta\beta}\Big)\Big]^{-\frac{1}{c}}$
Modified Linex	α	$\hat{\alpha}\Big[1+\Big(\frac{c(c+1)}{2\hat{\alpha}^2}-\frac{c}{\hat{\alpha}}\hat{\rho}_\alpha\Big)\hat{\sigma}_{\alpha\alpha}-\frac{c}{2\hat{\alpha}}\hat{\sigma}_{\alpha\alpha}\Big(\hat{L}_{aa\alpha}\hat{\sigma}_{aa}+\hat{L}_{bb\alpha}\hat{\sigma}_{bb}+\hat{L}_{\alpha\alpha\alpha}\hat{\sigma}_{\alpha\alpha}+\hat{L}_{\beta\beta\alpha}\hat{\sigma}_{\beta\beta}\Big)\Big]^{-\frac{1}{c}}$
	β	$\hat{\beta}\Big[1+\Big(\frac{c(c+1)}{2\hat{\beta}^2}-\frac{c}{\hat{\beta}}\hat{\rho}_\beta\Big)\hat{\sigma}_{\beta\beta}-\frac{c}{2\hat{\beta}}\hat{\sigma}_{\beta\beta}\Big(\hat{L}_{aa\beta}\hat{\sigma}_{aa}+\hat{L}_{bb\beta}\hat{\sigma}_{bb}+\hat{L}_{\alpha\alpha\beta}\hat{\sigma}_{\alpha\alpha}+\hat{L}_{\beta\beta\beta}\hat{\sigma}_{\beta\beta}\Big)\Big]^{-\frac{1}{c}}$

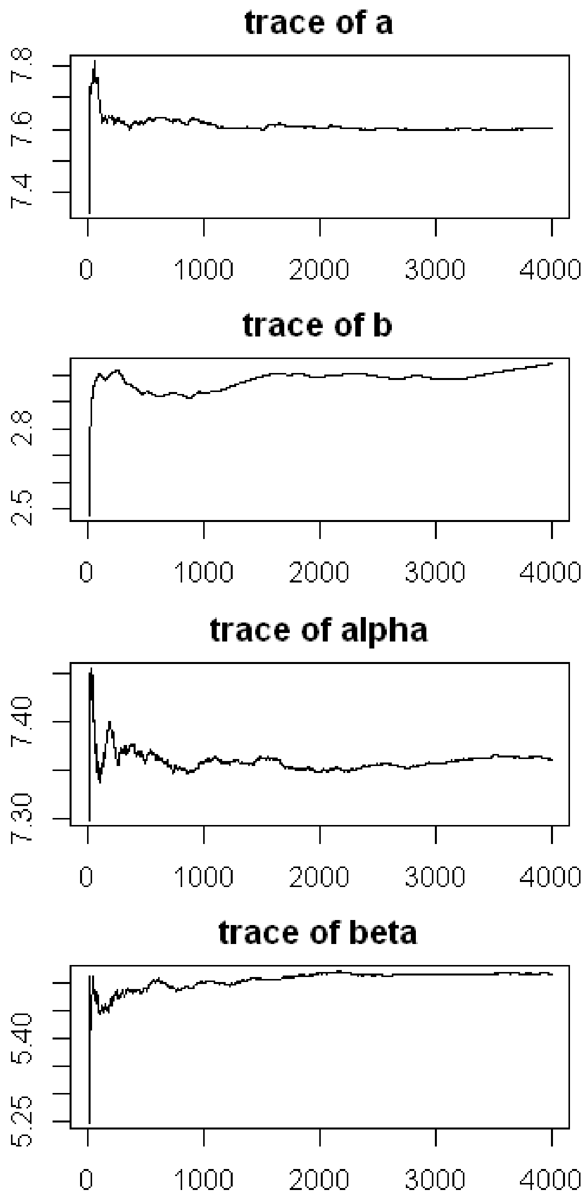

Figure 3: Diagram of the effect of TPMOGO distribution parameters for unconsciousness mode with size of $n = 50$.

Table 19: the amount of total error squares in the case of low-variance consciousness case (TPMOGO)

		Square error	Entropy	Linex	Square error in logarithm	Modified linex
$n=30$						
LINDLEY	a	10.589	10.229	9.720	10.276	9.355
	b	10.551	9.550	9.538	9.473	9.781
	β	10.339	9.509	9.215	10.270	10.710
	α	10.011	10.180	9.869	9.692	9.692
MCMC	a	12.770	13.766	12.655	13.474	12.561
	b	13.958	12.738	12.365	12.763	13.838
	β	12.871	13.470	14.031	12.954	13.724
	α	14.158	12.580	13.164	13.023	14.121
$n=50$						
		Square error	Entropy	Linex	Square error in logarithm	Modified linex
LINDLEY	a	7.741	6.902	6.413	6.472	8.352
	b	6.737	6.523	7.794	6.426	8.492
	β	6.695	7.239	7.717	7.285	6.743
	α	7.524	7.111	7.248	6.908	8.142
MCMC	a	9.992	11.576	9.374	12.253	10.921
	b	11.545	8.441	8.891	8.062	10.587
	β	9.969	10.298	10.442	9.883	10.355
	α	11.998	8.623	10.520	9.678	9.967
$n=70$						
		Square error	Entropy	Linex	Square error in logarithm	Modified linex
LINDLEY	a	5.178	4.434	3.470	2.902	3.928
	b	4.291	3.127	3.384	3.758	5.005
	β	2.630	4.191	5.231	4.681	4.150
	α	4.285	3.983	5.335	4.287	4.648
MCMC	a	7.225	9.535	5.455	7.331	5.954
	b	8.630	5.379	5.492	6.319	6.631
	β	5.993	6.229	6.992	7.867	6.761
	α	9.387	6.370	7.428	7.573	8.194

Table 20: Total values of error squares in the case of high-variance consciousness case (TPMOGO)

		Square error	Entropy	Linex	Square error in logarithm	Modified linex
$n = 30$						
LINDLEY	a	12.801	12.618	12.672	12.830	12.177
	b	12.566	12.364	12.442	12.202	12.415
	β	12.898	12.291	12.198	12.540	12.910
	α	12.856	12.848	12.615	12.235	12.238
MCMC	a	14.813	16.759	15.121	16.358	15.109
	b	16.270	14.821	15.204	15.639	15.942
	β	14.982	15.834	16.785	15.753	15.935
	α	16.622	14.749	15.210	15.837	16.542
$n = 50$						
		Square error	Entropy	Linex	Square error in logarithm	Modified linex
LINDLEY	a	10.625	9.506	8.785	9.192	10.717
	b	9.443	8.879	10.250	9.042	10.994
	β	8.812	9.971	10.490	9.678	9.732
	α	10.124	9.232	10.135	9.278	10.334
MCMC	a	12.553	14.122	11.768	14.306	12.931
	b	13.838	11.294	11.244	10.869	13.221
	β	12.076	12.879	13.389	12.564	12.499
	α	14.178	11.293	12.523	12.146	12.409
$n = 70$						
		Square error	Entropy	Linex	Square error in logarithm	Modified linex
LINDLEY	a	7.815	7.232	5.866	5.868	6.613
	b	6.385	5.771	6.314	5.925	7.283
	β	5.460	6.359	7.946	7.187	6.976
	α	6.625	6.946	7.922	6.469	7.371
MCMC	a	9.283	11.917	8.349	9.874	8.944
	b	11.444	7.955	7.969	9.128	9.612
	β	8.595	8.711	9.751	10.400	9.036
	α	11.874	8.603	9.981	9.587	10.229

Table 21: Total values of square error in the previous state of unconscious (TPMOGO)

		Square error	Entropy	Linex	Square error in logarithm	Modified linex
				$n = 30$		
		Square error	Entropy	Linex	Square error in logarithm	Modified linex
LINDLEY	a	14.109	14.109	14.109	14.109	14.109
	b	14.109	14.109	14.109	14.109	14.109
	β	14.109	14.109	14.109	14.109	14.109
	α	14.109	14.109	14.109	14.109	14.109
MCMC	a	16.186	18.034	16.135	17.957	16.526
	b	17.718	16.201	16.260	16.662	17.219
	β	16.524	17.380	17.819	17.190	17.114
	α	17.963	16.520	16.421	17.065	17.780
				$n = 50$		
		Square error	Entropy	Linex	Square error in logarithm	Modified linex
LINDLEY	a	12.085	10.916	10.281	10.280	11.970
	b	11.046	10.175	11.894	10.386	12.001
	β	10.404	11.157	11.711	11.629	11.444
	α	11.589	11.007	12.066	10.925	11.824
MCMC	a	14.101	16.022	12.909	15.626	13.939
	b	15.499	13.263	12.657	12.815	15.125
	β	13.238	14.278	14.889	14.409	13.787
	α	15.346	12.946	14.374	13.429	14.218
				$n = 70$		
		Square error	Entropy	Linex	Square error in logarithm	Modified linex
LINDLEY	a	9.620	8.257	7.537	7.700	8.467
	b	7.462	7.520	8.190	7.216	8.614
	β	6.547	7.429	9.376	9.085	8.280
	α	7.779	8.776	9.154	7.938	9.065
MCMC	a	11.244	13.206	10.266	11.810	10.871
	b	12.932	9.434	9.635	10.753	11.229
	β	10.562	10.534	11.048	11.898	10.081
	α	12.961	9.783	11.583	11.169	11.624

Table 22: Goodness of fit statistics for TPMOMW model

	LIND			MCMC		
Loss function	LOG	AIC	BIC	LOG	AIC	BIC
Square error	73.80	155.61	160.15	220.23	448.47	445.91
Entropy	59.37	124.87	130.58	201.42	410.84	408.29
Linex	68.26	144.52	149.06	196.68	401.376	398.82
Square error in logarithm	59.01	126.03	130.57	209.12	426.24	423.69
Modified Linex	57.39	122.78	127.32	210.56	429.12	426.56

Practical example

This section uses a series of real data sets to implement the proposed methods. These data actually include 23 secondary failure times of reactor pumps that have been used in studies such as Beibington et al. [1].

0.0620	0.0700	0.1010	0.1500
0.1990	0.2730	0.3470	0.3580
0.4020	0.4910	0.6050	0.6140
0.7460	0.9540	1.0600	1.3590
1.9210	2.1600	3.4650	4.0820
4.9920	5.3200	6.5600	

The Tables 22-24 show the values of good criteria logLik, AIC and BIC fits resulting from the fit of the two proposed models with respect to different loss functions, according to which it is concluded that Lindley estimates have a better fit with respect to all loss functions, and hence Bayesian Lindley estimates is the best model for the TPMOMW model under the modified Lynx loss function because it has better fit criteria than the others.

Table 23: Goodness of fit statistics for TPMOGO model

	LIND			MCMC		
Loss function	LOG	AIC	BIC	LOG	AIC	BIC
Square error	1010.66	2031.33	2037.01	1420.77	2851.54	2848.35
Entropy	984.73	1979.46	1985.14	1246.41	2502.82	2499.63
Linex	984.73	1979.46	1985.1	1330.77	2671.54	2668.35
Square error in logarithm	999.67	2009.34	2015.02	1430.77	2871.54	2868.35
Modified Linex	964.34	1938.68	1944.35	1176.78	2363.56	2360.36

Table 24: Lindley estimation of parameters of TPMOMW model based on modified Linex loss function

a	b	γ	α	λ
2.1278	0.5450	3.3166	0.7631	0.5308

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Appendix: Lindley Estimation In [4], Lindley showed that the ratio of two integrals can be estimated and the omitted part is $o(n^{-2})$ and the main part that is used for estimation is $o(n^{-1})$.

$$\frac{\int w(\boldsymbol{\theta})e^{L(\boldsymbol{\theta})}d\boldsymbol{\theta}}{\int \nu(\boldsymbol{\theta})e^{L(\boldsymbol{\theta})}d\boldsymbol{\theta}}$$

In the above integral ratio $\boldsymbol{\theta} = (\theta_1, \theta_2, \cdots, \theta_m)$ is the vector parameter and $L(\boldsymbol{\theta})$ is the logarithm of the likelihood function

$$L(\boldsymbol{\theta}) = \sum \log p(x_i \mid \boldsymbol{\theta})$$

where $x_1, x_2, \cdots, x_n$, observation from $p(\cdot \mid \boldsymbol{\theta})$. In the application we should place $\nu(\boldsymbol{\theta}) = \pi(\boldsymbol{\theta})$, that $\pi(\boldsymbol{\theta})$ is the prior density function of the $\boldsymbol{\theta}$ parametric vector. Therefore, the dominator is the integral ratio of the normalizer constant in the Bayesian theory and $w(\boldsymbol{\theta}) = u(\boldsymbol{\theta})\pi(\boldsymbol{\theta})$ such that the integral ratio becomes $E(u(\boldsymbol{\theta}) \mid x_1, x_2, \cdots, x_n)$.
In Bayesian estimation, by placing $\rho(\theta) = \log \pi(\theta)$, the integral ratio changes to

$$\frac{\int u(\boldsymbol{\theta})e^{L(\boldsymbol{\theta})+\rho(\boldsymbol{\theta})}d\boldsymbol{\theta}}{\int e^{L(\boldsymbol{\theta})+\rho(\boldsymbol{\theta})}d\boldsymbol{\theta}}$$

That the new integral ratio is estimated as follow by Lindly:

$$\hat{u} + \frac{1}{2}\sum(\hat{u}_{ij} + 2\hat{u}_i\hat{\rho}_i)\hat{\sigma}_{ij} + \frac{1}{2}\sum \hat{L}_{ijk}\hat{u}_L\hat{\sigma}_{ij}\hat{\sigma}_{kL}$$

The first term is $o(1)$ and the other terms are $o(n^{-1})$ in the above phrase.

$$\begin{aligned}
\hat{u} &= u(\hat{\theta}_1, \hat{\theta}_2, \cdots, \hat{\theta}_m) \\
\hat{g}_i &= g_i(\hat{\theta}_1, \hat{\theta}_2, \cdots, \hat{\theta}_m) = \frac{\partial g}{\partial \theta_i} \mid_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}} \\
\hat{g}_{ij} &= g_{ij}(\hat{\theta}_1, \hat{\theta}_2, \cdots, \hat{\theta}_m) = \frac{\partial^2 g}{\partial \theta_i \partial \theta_j} \mid_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}} \\
\hat{g}_{ijk} &= g_{ijk}(\hat{\theta}_1, \hat{\theta}_2, \cdots, \hat{\theta}_m) = \frac{\partial^3 g}{\partial \theta_i \partial \theta_j \partial \theta_k} \mid_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}}
\end{aligned}$$

And $\hat{\boldsymbol{\theta}} = (\hat{\theta}_1, \hat{\theta}_2, \cdots, \hat{\theta}_m)$ is the maximum probability estimation of $\boldsymbol{\theta} = (\theta_1, \theta_2, \cdots, \theta_m)$ and $\hat{\sigma}_{ij}$ is obtained from the matrix relation $(\hat{\sigma}_{ij}) =$

$(-\hat{L}_{ij})^{-1}$. Therefore:

$$\begin{aligned} I(x) &= E(u(\theta_1, \theta_2, \cdots, \theta_m) \mid \boldsymbol{X} = \boldsymbol{x}) \\ &= \frac{\int u(\theta_1, \theta_2, \cdots, \theta_m) e^{L(\theta_1, \theta_2, \cdots, \theta_m) + \rho(\theta_1, \theta_2, \cdots, \theta_m)} d\theta_1 \cdots d\theta_3}{\int e^{L(\theta_1, \theta_2, \cdots, \theta_m) + \rho(\theta_1, \theta_2, \cdots, \theta_m)} d\theta_1 \cdots d\theta_3} \end{aligned}$$

But in this equation $\theta_1, \theta_2, \cdots, \theta_m$ are independent, so

$$\hat{\sigma}_{\theta_1\theta_2} = \cdots = \hat{\sigma}_{\theta_{m-1}\theta_m} = 0$$

And by considering page 227 of Lindly article, this estimation is decreased to the following equation:

$$\hat{u} + \frac{1}{2} \sum (\hat{u}_{ii} + 2\hat{u}_i \hat{\rho}_i) \hat{\sigma}_{ii} + \frac{1}{2} \sum \hat{L}_{iik} \hat{u}_k \hat{\sigma}_{ii} \hat{\sigma}_{kk}$$

According to Lindley's approximation, this mathematical expectation for the number of different random variables (m) is as follows

m=2

$$\begin{aligned} I(\boldsymbol{x}) = u(\hat{\theta}_1, \hat{\theta}_2) &+ \frac{1}{2}\Big[(\hat{u}_{\theta_1\theta_1} + 2\hat{u}_{\theta_1}\hat{\rho}_{\theta_1})\hat{\sigma}_{\theta_1\theta_1} + (\hat{u}_{\theta_2\theta_2} + 2\hat{u}_{\theta_2}\hat{\rho}_{\theta_2})\hat{\sigma}_{\theta_2\theta_2}\Big] \\ &+ \frac{1}{2}\Big[\hat{u}_{\theta_1}\Big(\hat{L}_{\theta_1\theta_1\theta_1}\hat{\sigma}_{\theta_1\theta_1}\hat{\sigma}_{\theta_1\theta_1} + \hat{L}_{\theta_2\theta_2\theta_1}\hat{\sigma}_{\theta_2\theta_2}\hat{\sigma}_{\theta_1\theta_1}\Big) \\ &+ \hat{u}_{\theta_2}\Big(\hat{L}_{\theta_1\theta_1\theta_2}\hat{\sigma}_{\theta_1\theta_1}\hat{\sigma}_{\theta_2\theta_2} + \hat{L}_{\theta_2\theta_2\theta_2}\hat{\sigma}_{\theta_2\theta_2}\hat{\sigma}_{\theta_2\theta_2}\Big)\Big] \end{aligned}$$

m=3

$$\begin{aligned} I(\boldsymbol{x}) &= u(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3) \\ &+ \frac{1}{2}\Big[(\hat{u}_{\theta_1\theta_1} + 2\hat{u}_{\theta_1}\hat{\rho}_{\theta_1})\hat{\sigma}_{\theta_1\theta_1} + (\hat{u}_{\theta_2\theta_2} + 2\hat{u}_{\theta_2}\hat{\rho}_{\theta_2})\hat{\sigma}_{\theta_2\theta_2} + (\hat{u}_{\theta_3\theta_3} + 2\hat{u}_{\theta_3}\hat{\rho}_{\theta_3})\hat{\sigma}_{\theta_3\theta_3}\Big] \\ &+ \frac{1}{2}\Big[\hat{u}_{\theta_1}\Big(\hat{L}_{\theta_1\theta_1\theta_1}\hat{\sigma}_{\theta_1\theta_1}\hat{\sigma}_{\theta_1\theta_1} + \hat{L}_{\theta_2\theta_2\theta_1}\hat{\sigma}_{\theta_2\theta_2}\hat{\sigma}_{\theta_1\theta_1} + \hat{L}_{\theta_3\theta_3\theta_1}\hat{\sigma}_{\theta_3\theta_3}\hat{\sigma}_{\theta_1\theta_1}\Big) \\ &+ \hat{u}_{\theta_2}\Big(\hat{L}_{\theta_1\theta_1\theta_2}\hat{\sigma}_{\theta_1\theta_1}\hat{\sigma}_{\theta_2\theta_2} + \hat{L}_{\theta_2\theta_2\theta_2}\hat{\sigma}_{\theta_2\theta_2}\hat{\sigma}_{\theta_2\theta_2} + \hat{L}_{\theta_3\theta_3\theta_2}\hat{\sigma}_{\theta_3\theta_3}\hat{\sigma}_{\theta_2\theta_2}\Big) \\ &+ \hat{u}_{\theta_3}\Big(\hat{L}_{\theta_1\theta_1\theta_3}\hat{\sigma}_{\theta_1\theta_1}\hat{\sigma}_{\theta_3\theta_3} + \hat{L}_{\theta_2\theta_2\theta_3}\hat{\sigma}_{\theta_2\theta_2}\hat{\sigma}_{\theta_3\theta_3} + \hat{L}_{\theta_3\theta_3\theta_3}\hat{\sigma}_{\theta_3\theta_3}\hat{\sigma}_{\theta_3\theta_3}\Big)\Big] \end{aligned}$$

m=4

$$
\begin{aligned}
I(\boldsymbol{x}) \quad = \quad & u(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3, \hat{\theta}_4) \\
& \quad + \frac{1}{2}\Bigg[(\hat{u}_{\theta_1\theta_1} + 2\hat{u}_{\theta_1}\hat{\rho}_{\theta_1})\hat{\sigma}_{\theta_1\theta_1} + (\hat{u}_{\theta_2\theta_2} + 2\hat{u}_{\theta_2}\hat{\rho}_{\theta_2})\hat{\sigma}_{\theta_2\theta_2} \\
& \quad + (\hat{u}_{\theta_3\theta_3} + \hat{u}_{\theta_3}\hat{\rho}_{\theta_3})\hat{\sigma}_{\theta_3\theta_3} + (\hat{u}_{\theta_4\theta_4} + 2\hat{u}_{\theta_4}\hat{\rho}_{\theta_4})\hat{\sigma}_{\theta_4\theta_4}\Bigg] \\
& + \frac{1}{2}\Bigg[\hat{u}_{\theta_1}\Bigg(\hat{L}_{\theta_1\theta_1\theta_1}\hat{\sigma}_{\theta_1\theta_1}\hat{\sigma}_{\theta_1\theta_1} + \hat{L}_{\theta_2\theta_2\theta_1}\hat{\sigma}_{\theta_2\theta_2}\hat{\sigma}_{\theta_1\theta_1} + \hat{L}_{\theta_3\theta_3\theta_1}\hat{\sigma}_{\theta_3\theta_3}\hat{\sigma}_{\theta_1\theta_1} \\
& \quad + \hat{L}_{\theta_4\theta_4\theta_1}\hat{\sigma}_{\theta_4\theta_4}\hat{\sigma}_{\theta_1\theta_1}\Bigg) \\
& \quad + \hat{u}_{\theta_2}\Bigg(\hat{L}_{\theta_1\theta_1\theta_2}\hat{\sigma}_{\theta_1\theta_1}\hat{\sigma}_{\theta_2\theta_2} + \hat{L}_{\theta_2\theta_2\theta_2}\hat{\sigma}_{\theta_2\theta_2}\hat{\sigma}_{\theta_2\theta_2} + \hat{L}_{\theta_3\theta_3\theta_2}\hat{\sigma}_{\theta_3\theta_3}\hat{\sigma}_{\theta_2\theta_2} \\
& \quad + \hat{L}_{\theta_4\theta_4\theta_2}\hat{\sigma}_{\theta_4\theta_4}\hat{\sigma}_{\theta_2\theta_2}\Bigg) \\
& \quad + \hat{u}_{\theta_3}\Bigg(\hat{L}_{\theta_1\theta_1\theta_3}\hat{\sigma}_{\theta_1\theta_1}\hat{\sigma}_{\theta_3\theta_3} + \hat{L}_{\theta_2\theta_2\theta_3}\hat{\sigma}_{\theta_2\theta_2}\hat{\sigma}_{\theta_3\theta_3} + \hat{L}_{\theta_3\theta_3\theta_3}\hat{\sigma}_{\theta_3\theta_3}\hat{\sigma}_{\theta_3\theta_3} \\
& \quad + \hat{L}_{\theta_4\theta_4\theta_3}\hat{\sigma}_{\theta_4\theta_4}\hat{\sigma}_{\theta_3\theta_3}\Bigg) \\
& \quad + \hat{u}_{\theta_4}\Bigg(\hat{L}_{\theta_1\theta_1\theta_4}\hat{\sigma}_{\theta_1\theta_1}\hat{\sigma}_{\theta_4\theta_4} + \hat{L}_{\theta_2\theta_2\theta_4}\hat{\sigma}_{\theta_2\theta_2}\hat{\sigma}_{\theta_4\theta_4} + \hat{L}_{\theta_3\theta_3\theta_4}\hat{\sigma}_{\theta_3\theta_3}\hat{\sigma}_{\theta_4\theta_4} \\
& \quad + \hat{L}_{\theta_4\theta_4\theta_4}\hat{\sigma}_{\theta_4\theta_4}\hat{\sigma}_{\theta_4\theta_4}\Bigg)\Bigg]
\end{aligned}
$$

m=5

$$
\begin{aligned}
I(\boldsymbol{x}) \quad = \quad & u(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3, \hat{\theta}_4, \hat{\theta}_5) \\
& + \frac{1}{2}\Big[(\hat{u}_{\theta_1\theta_1} + 2\hat{u}_{\theta_1}\hat{\rho}_{\theta_1})\hat{\sigma}_{\theta_1\theta_1} + (\hat{u}_{\theta_2\theta_2} + 2\hat{u}_{\theta_2}\hat{\rho}_{\theta_2})\hat{\sigma}_{\theta_2\theta_2} \\
& + (\hat{u}_{\theta_3\theta_3} + 2\hat{u}_{\theta_3}\hat{\rho}_{\theta_3})\hat{\sigma}_{\theta_3\theta_3} + (\hat{u}_{\theta_4\theta_4} + 2\hat{u}_{\theta_4}\hat{\rho}_{\theta_4})\hat{\sigma}_{\theta_4\theta_4} + (\hat{u}_{\theta_5\theta_5} + 2\hat{u}_{\theta_5}\hat{\rho}_{\theta_5})\hat{\sigma}_{\theta_5\theta_5}\Big] \\
& + \frac{1}{2}\Big[\hat{u}_{\theta_1}\Big(\hat{L}_{\theta_1\theta_1\theta_1}\hat{\sigma}_{\theta_1\theta_1}\hat{\sigma}_{\theta_1\theta_1} + \hat{L}_{\theta_2\theta_2\theta_1}\hat{\sigma}_{\theta_2\theta_2}\hat{\sigma}_{\theta_1\theta_1} + \hat{L}_{\theta_3\theta_3\theta_1}\hat{\sigma}_{\theta_3\theta_3}\hat{\sigma}_{\theta_1\theta_1} \\
& + \hat{L}_{\theta_4\theta_4\theta_1}\hat{\sigma}_{\theta_4\theta_4}\hat{\sigma}_{\theta_1\theta_1} + \hat{L}_{\theta_5\theta_5\theta_1}\hat{\sigma}_{\theta_5\theta_5}\hat{\sigma}_{\theta_1\theta_1}\Big) \\
& + \hat{u}_{\theta_2}\Big(\hat{L}_{\theta_1\theta_1\theta_2}\hat{\sigma}_{\theta_1\theta_1}\hat{\sigma}_{\theta_2\theta_2} + \hat{L}_{\theta_2\theta_2\theta_2}\hat{\sigma}_{\theta_2\theta_2}\hat{\sigma}_{\theta_2\theta_2} + \hat{L}_{\theta_3\theta_3\theta_2}\hat{\sigma}_{\theta_3\theta_3}\hat{\sigma}_{\theta_2\theta_2} \\
& + \hat{L}_{\theta_4\theta_4\theta_2}\hat{\sigma}_{\theta_4\theta_4}\hat{\sigma}_{\theta_2\theta_2} + \hat{L}_{\theta_5\theta_5\theta_2}\hat{\sigma}_{\theta_5\theta_5}\hat{\sigma}_{\theta_2\theta_2}\Big) \\
& + \hat{u}_{\theta_3}\Big(\hat{L}_{\theta_1\theta_1\theta_3}\hat{\sigma}_{\theta_1\theta_1}\hat{\sigma}_{\theta_3\theta_3} + \hat{L}_{\theta_2\theta_2\theta_3}\hat{\sigma}_{\theta_2\theta_2}\hat{\sigma}_{\theta_3\theta_3} + \hat{L}_{\theta_3\theta_3\theta_3}\hat{\sigma}_{\theta_3\theta_3}\hat{\sigma}_{\theta_3\theta_3} \\
& + \hat{L}_{\theta_4\theta_4\theta_3}\hat{\sigma}_{\theta_4\theta_4}\hat{\sigma}_{\theta_3\theta_3} + \hat{L}_{\theta_5\theta_5\theta_3}\hat{\sigma}_{\theta_5\theta_5}\hat{\sigma}_{\theta_3\theta_3}\Big) \\
& + \hat{u}_{\theta_4}\Big(\hat{L}_{\theta_1\theta_1\theta_4}\hat{\sigma}_{\theta_1\theta_1}\hat{\sigma}_{\theta_4\theta_4} + \hat{L}_{\theta_2\theta_2\theta_4}\hat{\sigma}_{\theta_2\theta_2}\hat{\sigma}_{\theta_4\theta_4} + \hat{L}_{\theta_3\theta_3\theta_4}\hat{\sigma}_{\theta_3\theta_3}\hat{\sigma}_{\theta_4\theta_4} \\
& + \hat{L}_{\theta_4\theta_4\theta_4}\hat{\sigma}_{\theta_4\theta_4}\hat{\sigma}_{\theta_4\theta_4} + \hat{L}_{\theta_5\theta_5\theta_4}\hat{\sigma}_{\theta_5\theta_5}\hat{\sigma}_{\theta_4\theta_4}\Big) \\
& + \hat{u}_{\theta_5}\Big(\hat{L}_{\theta_1\theta_1\theta_5}\hat{\sigma}_{\theta_1\theta_1}\hat{\sigma}_{\theta_5\theta_5} + \hat{L}_{\theta_2\theta_2\theta_5}\hat{\sigma}_{\theta_2\theta_2}\hat{\sigma}_{\theta_5\theta_5} + \hat{L}_{\theta_3\theta_3\theta_5}\hat{\sigma}_{\theta_3\theta_3}\hat{\sigma}_{\theta_5\theta_5} \\
& + \hat{L}_{\theta_4\theta_4\theta_5}\hat{\sigma}_{\theta_4\theta_4}\hat{\sigma}_{\theta_5\theta_5} + \hat{L}_{\theta_5\theta_5\theta_5}\hat{\sigma}_{\theta_5\theta_5}\hat{\sigma}_{\theta_5\theta_5}\Big)\Big]
\end{aligned}
$$