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Discussions in Integral Closure of Polynomial Ideals

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Abstract. The main purpose of this paper is the comparison between an ideal and its integral closure and how to obtain the integral closure of ideals in different ways. We know that integral closure of a monomial ideal is again a monomial ideal and monomial ideals are associated to graphs. As graphs are widely used in computer science and cryptography in which there are only binary numbers, we introduce the associated matrix to a graph only with digits of 0 and 1.
Moreover, we compare and compute the index of stability of associated prime ideals with index of stability of the integral closure of associated prime ideals.

AMS Subject Classification: MSC code1; MSC code 2; more
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1 Introduction

Comparison between an ideal and its integral closure and the methods of obtaining integral closure of monomial ideals have been studied by several people, for example see [5], [8], [9] and [10]. In section 2, we present the primary concepts related to integral closure of monomial ideals. Throughout this paper K is an arbitrary finite field and $R = K[x_0, \ldots, x_m]$, where the indeterminates $x_0, \ldots, x_m$ have weights $A_0, \ldots, A_m$, respectively. Also, let A be the least common multiple of $A_0, \ldots, A_m$. In section 3, we show that for every positive integer n, $(\overline{I_{\geq mA}})^n = \overline{(I^n_{\geq mA})}$, where $I_{\geq mA}$ refers to the ideal I generated by the elements of degree at least mA in the graded ring R. Let R be a polynomial (localized) ring or a power series ring. Let M be the maximal (irrelevant) ideal of R. If $r \in R$, then $o(r) = \max\{l | r \in M^l\}$. The order of an ideal $I \subset R$ is defined as $o(I) = \min\{o(r) | r \in I\}$. The least number of generators of I is denoted by $\mu(I)$. We show that $\mu(I^n_{\geq mA}) = o(I^n_{\geq mA}) + 1$.

Since the integral closure of a monomial ideal is again a monomial ideal and monomial ideals are associated to graphs, one of our goals in section 3 is to study integral closure of monomial ideals produced by monomials of degree two. For any terminology or unexplained notion we refer the reader to [9]. We also look at the relationship between integral closure of monomial ideals and graph theory. Let $I(G)$ be the edge ideal associated to a complete bipartite graph (or the edge ideal associated to a simple graph) and $J(G)$ be the ideal associated to a graph with loops. We show that $(I(G) : J(G))$ is integrally closed.

Since graphs are widely used in computer science and cryptography and there are only two digits of 0 and 1 in binary numbers, we introduce the associated matrix to a graph (with digits of 0 and 1).

Let n and t be positive integers and $n \geq 3$. Also, let $R = K[x_1, \ldots, x_n] = K[X]$. For any positive integer j, let $[j] = \{1, \ldots, j\}$. For $i \in [n]$, let $e_i = (t, \ldots, t, 0, t, \ldots, t) \in \mathbb{N}^n$, where 0 is in the i-th coordinate. Thus, the corresponding monomial with exponent e_i is $X^{e_i} = x_1^t \ldots x_{i-1}^t x_{i+1}^t \ldots x_n^t$. Consider the monomial ideal $M_{n,t} = \langle X^{e_1}, \ldots, X^{e_n} \rangle$. Jarrah [6] showed that $M_{n,t}$ is a Cohen-Macaulay monomial ideal. In section 4, we show in a simpler and in different way that $M_{n,t}$ is Cohen-Macaulay. Also,

we show that $\overline{M}_{n,t} = (M_{n,1})^t$ and $\overline{\langle (X^{e_1})^p, \ldots, (X^{e_n})^p \rangle} = (M_{n,1})^{pt(n-1)}$, where p is a positive integer.
Let $(R, \mathfrak{m})$ be a commutative Noetherian ring and I be an ideal of R. Brodmann [1] proved that the set of associated prime ideals $Ass(I^k)$ stabilizes. In other words there exists an integer k_0 such that $Ass(I^k) = Ass(I^{k_0})$ for all $k \geq k_0$. The smallest such integer k_0 is called the index of Ass-stability of I, and denoted by $astab(I)$. Moreover, $Ass(I^{k_0})$ is called the stable set of associated prime ideals of I. It is denoted by $Ass^{\infty}(I)$. For the integral closures $\overline{I^k}$ of the powers of I, Mc Adam and Eakin [7] showed that $Ass(\overline{I^k})$ stabilizes as well. We denote the index of stability for the integral closures of the powers of I by $\overline{astab}(I)$, and denote its stable set of associated prime ideals by $\overline{Ass}^{\infty}(I)$. Brodmann [2], also showed that $depth\dfrac{R}{I^k}$ stabilizes. The smallest power of I for which $depth$ stabilizes is denoted by $dstab(I)$. We compare the index of Ass-stability of ideals with the index of stability for their integral closures by examples of $M_{n,t}$.

2 Preliminaries

In this section K is an arbitrary field and $R = K[x_0, \ldots, x_n]$ is a polynomial ring over field K. Let I be an ideal of R. An element $r \in R$ is said to be integral over I if there exists an integer n and elements $a_i \in I^i$, $i = 1, 2, \ldots, n$, such that

$$r^n + a_1 r^{n-1} + a_2 r^{n-2} + \ldots + a_{n-1} r + a_n = 0$$

Such an equation is called an equation of integral dependence of r over I. The set of all elements that are integral over I is called the integral closure of I, and is denoted by $\overline{I}$. If $I = \overline{I}$, then I is called integrally closed. If $I \subseteq J$ are ideals, we say that J is integral over I if $J \subseteq \overline{I}$. The integral closure of a monomial ideal I in a polynomial ring $K[x_1, \ldots, x_n]$ is a monomial ideal. In this case by Proposition 1.3 of [8] we have:

$$\overline{I} = \langle m \in R | m \text{ is a monomial and } m^l \in I^l \text{ for some } l > 0 \rangle.$$

Example 2.1. In general, if m is a monomial and $m \in \overline{I}$, then we cannot say that $m^l \in I^l$ for all l. Let $I = \langle x^3, y^3 \rangle \subset K[x, y]$. Then, we have

$\overline{I} = \langle x^3, x^2y, xy^2, y^3\rangle$. In this case $x^2y \in \overline{I}$; but $x^2y \notin I$. Also, $(x^2y)^2 \notin I^2 = \langle x^6, x^3y^3, y^6\rangle$. Moreover, $(x^2y)^3 \in I^3 = \langle x^9, x^6y^3, x^3y^6, y^9\rangle$; but $(x^2y)^4 \notin I^4 = \langle x^{12}, x^9y^3, x^6y^6, x^3y^9, y^{12}\rangle$.

If a polynomial $p \in \overline{I}$, then we cannot say that $p^l \in I^l$, for all $l \geq 2$.

Example 2.2. In the previous example, $p_1 = x^2y + xy^2 \in \overline{I} \setminus I$. For which l does $p_1^l \in I^l$? In other words, we want to determine an integer number l such that for all $0 \leq i \leq l$ there is some j such that

$$(x^3)^j(y^3)^{l-j}|(x^2y)^i(xy^2)^{l-i},$$

therefore $l \leq 3j - i$. It is clear that

$$(x^2y + xy^2)^3 = x^6y^3 + 3x^5y^4 + 3x^4y^5 + x^3y^6 \equiv x^6y^3 + x^3y^6 \in I^3.$$

A monomial $m \in \overline{I} \setminus I$ may satisfy in the condition $m^{l_1} \in I^{l_2}$, where $l_1 < l_2$; see the following example.

Example 2.3. Let $I = \langle x^4, y^5\rangle \subset K[x, y]$, then $\overline{I} = \langle y^5, y^4x, y^3x^2, y^2x^3, x^4\rangle$ and $(x^2y^3)^{20} = (x^4)^{10}(y^5)^{12} \in I^{22}$. Also, $(x^3y^2)^{20} = (x^4)^{15}(y^5)^8 \in I^{23}$.

Let R be the polynomial ring $K[x_1, x_2, ..., x_d]$. For any monomial $m = X_1^{a_1}X_2^{a_2}\dots X_d^{a_d}$, its exponent vector is $(a_1, a_2, \dots, a_d) \in \mathbb{N}^d$. For any monomial ideal I, the set of all exponent vectors of all the monomials in I is called the exponent set of I. By Proposition 1.4.6 of [10], we know that the exponent set of the integral closure of a monomial ideal I equals to all the integer lattice points in the convex hull of the exponent set of I.

Example 2.4. Let $I = (x^3, x^2y, y^4)$ be a monomial ideal of $\mathbb{C}[x, y]$. Then integral closure of I can be read off from the convex hull of the exponent set below, proving that $\overline{I}$ equals to (x^3, x^2y, xy^3, y^4), see Figure 1 . Its exponent set consists of all integer lattice points touching or in the shaded gray area below, see Figure 2 .

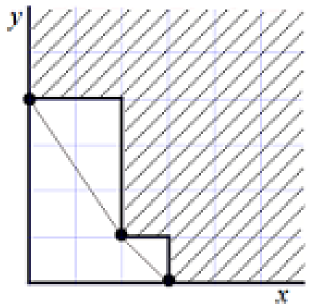

Figure 1: The exponent set of I

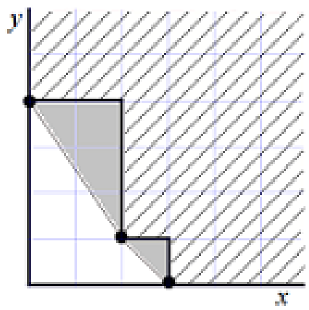

Figure 2: The convex hull of the exponent set of I

3 Integral Closure of Polynomial Ideals

3.1 Normal ideals

Now, we are going to study normal ideals, where by normal ideal we mean an ideal all of whose positive powers are integrally closed. In this section K is an arbitrary field and $R = K[x_0, \ldots, x_m]$ is a polynomial ring over field K. First, we present the following theorem:

Theorem 3.1. (See [3]) *Let R be a graded domain, which is a quotient of a polynomial ring $K[x_0, x_1, \ldots, x_m]$ modulo a homogeneous ideal J, where K is an arbitrary domain and $x_0, \ldots, x_m$ are indeterminates of positive weights $A_0, \ldots, A_m$. Let A be the least common multiple of $A_0, \ldots, A_m$. Then the ideal $I = R_{\geq mA}$ is a normal ideal, where $R_{\geq \alpha}$ refers to the ideal of R generated by the elements of degree at least α in* *the graded ring R.*

Example 3.2. Let K be a field, and $R = \dfrac{K[x,y,z]}{(x^2+y^3+z)}$ be a domain, where the indeterminates x, y, z have weights 3, 2 and 6, respectively. Then ideal $I = R_{\geq 12}$ is a normal ideal of R.

$$I = R_{\geq 12} = (x^4, x^3y^2, x^2y^3, x^2z, xy^5, xy^2z, y^6, y^3z, z^2).$$

Let R, A and m be as in Theorem 3.1. Then we have the following results:

Corollary 3.3. *For any positive integer n, we have $(\overline{R_{\geq mA}})^n = \overline{R_{\geq mnA}}$.*

Proof. By the proof of Theorem 3.1 of [3], for any positive integer n, we have:

$$(R_{\geq mA})^n = R_{\geq nmA}. \tag{1}$$

Since $R_{\geq mA}$ is a normal ideal, we have $R_{\geq mA} = \overline{R_{\geq mA}}$ and $R^n_{\geq mA} = \overline{R^n_{\geq mA}}$, therefore $(\overline{R_{\geq mA}})^n = \overline{R^n_{\geq mA}}$. Now, by (1) we have $(\overline{R_{\geq mA}})^n = \overline{R_{\geq mnA}}$. $\square$

Proposition 3.4. *Let K be a field and $R = K[x_0, x_1, \ldots, x_m]$, where indeterminates $x_0, \ldots, x_m$ have weights $A_0, \ldots, A_m$, respectively. Then, for every positive integer n, we have*

$$(\overline{R_{\geq mA}})^n = \overline{(R^n_{\geq mA})},$$

and

$$\mu(R^n_{\geq mA}) = o(R^n_{\geq mA}) + 1.$$

Also, for any $n > m$ we have

$$\overline{R_{\geq nmA}} = R_{\geq mA}\overline{R_{\geq (n-1)mA}}.$$

Proof. By Theorem 3.1 and Example 3 in [3], we can say that if K is a field and $R = K[x_0, x_1, \ldots, x_m]$, then $I = R_{\geq mA}$ is a normal ideal, where $degx_i = a_i$ for all i that $0 \leq i \leq m$ and A is the least common multiple of $x_0, x_1, \ldots, x_m$. On the other hand, by Corollary 3.2 in [5], integrally closed ideals are contracted. Also, by Proposition 2.3 in [5], an ideal I is contracted if and only if $\mu(I) = o(I) + 1$. Therefore, it is clear that $(\overline{R_{\geq mA}})^n = \overline{(R^n_{\geq mA})}$, and $\mu(R^n_{\geq mA}) = o(R^n_{\geq mA}) + 1$. Also, according to our prior conclusion and by Exercise 1.6 in [10], we have:

$$\overline{R_{\geq nmA}} = \overline{R^n_{\geq mA}} = R_{\geq mA}\overline{R^{n-1}_{\geq mA}} = R_{\geq mA}\overline{R_{\geq (n-1)mA}}$$

□

Corollary 3.5. *Let K be a field and $R = K[x, y]$. Then ideal $R_{\geq\alpha}$ is normal. Therefore, $(\overline{R_{\geq\alpha}})^n = \overline{(R_{\geq\alpha})^n}$. In particular, $\mu(R^n_{\geq\alpha}) = o(R^n_{\geq\alpha}) + 1$.*

Proof. All integrally closed ideals of $R = K[x, y]$ are normal. In particular, any ideal $I = R_{\geq\alpha}$ is normal, where α is a positive integer. Therefore, each power is integrally closed. Also, since $R_{\geq\alpha}$ is normal, for every positive integer n we have $(\overline{R_{\geq\alpha}})^n = \overline{(R_{\geq\alpha})^n}$ and $\mu(R^n_{\geq\alpha}) = o(R^n_{\geq\alpha}) + 1$. □

3.2 Relationship between integral closure of monomial ideals and graphs

If I is an ideal of R generated by squarefree monomials, then I is integrally closed. The aim of this section is to study the integral closure of monomial ideals generated by monomials of degree two in the indeterminates x_i, $i = 1, 2, \dots, n$, not necessarily squarefree. This type of ideals deals with graph theory. In particular if we consider a graph G with loops, we can associate to G a polynomial ring with one indeterminate x_i for each of its vertex, and we can consider the edge ideal associated to G. To get acquainted with the introductory concepts of this section, see [9]. Let G be a graph with vertex set $V = \{x_1, \dots, x_n\}$ and $R = K[x_1, \dots, x_n]$ a polynomial ring on field K with one indeterminate x_i for each of its vertex. We represent the edge ideal associated to graph G by $I(G)$.

Example 3.6. Let G be the following graph with loops:

Then $I(G) = (x_1x_2, x_1x_3, x_2x_4, x_2^2, x_3^2)$. Also, $\overline{I(G)} = I(G) + (x_2x_3)$.

Proposition 3.7. *Let $I(G)$ be the edge ideal associated to a complete bipartite graph (or edge ideal associated to a simple graph) and $J(G)$*

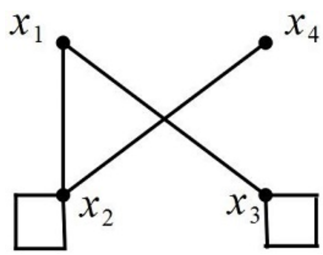

Figure 3: Graph G with loops

be an ideal associated to a graph with loops. Then $(I(G) : J(G))$ is integrally closed.

Proof. We know that if $I(G)$ be the edge ideal associated to a simple graph G or associated to a complete graph with loops G or associated to a complete bipartite graph G, then $I(G)$ is integrally closed. Let r be integral over $(I(G) : J(G))$. Then r satisfies in an equation of integral dependence over $(I(G) : J(G))$. Suppose the degree of this equation is n. For any $x_ix_j \in J(G)$, multiply the equation by $x_i^nx_j^n$ to get an equation of integral dependence of rx_ix_j over $I(G)$. It follows that $rx_ix_j \in I(G)$. Hence $rJ(G) \subset I(G)$, which means that $(I(G) : J(G))$ is integrally closed. $\square$

3.2.1 Adding edges

Now, we consider graphs whose edge ideals are not integrally closed and we compute the integral closure. Since the integral closure of an edge ideal $I(G)$ is again a monomial ideal of degree 2, we can associate $\overline{I(G)}$ to a graph, denoted by $\overline{G}$.

Example 3.8. Let $R = K[x_1, x_2, x_3, x_4]$ be a polynomial ring over an arbitrary field K and G be the following graph: (See Figure 4)
Then $I(G) = (x_1^2, x_1x_2, x_2x_3, x_3^2, x_3x_4, x_4^2) \subset K[x_1, x_2, x_3, x_4]$. Also,

$$\overline{I(G)} = (x_1^2, x_1x_2, x_2x_3, x_3^2, x_3x_4, x_1x_3, x_4^2, x_1x_4).$$

For monomial ideals $I(G)$ and $\overline{I(G)}$ we have $deg(I(G)) = deg(\overline{I(G)})$. Graph $\overline{G}$ associated to $\overline{I(G)}$ is the following: (See Figure 5)

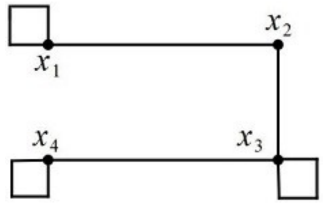

Figure 4: Graph G

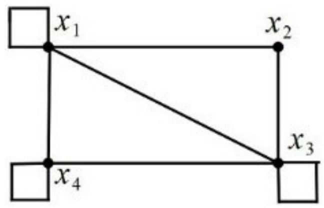

Figure 5: Graph $\overline{G}$

Example 3.9. Let $R = K[x_1, x_2, x_3; y_1, y_2, y_3]$ be a polynomial ring over an arbitrary field K and G be a strong quasi-bipartite graph as following: (See Figure 6)

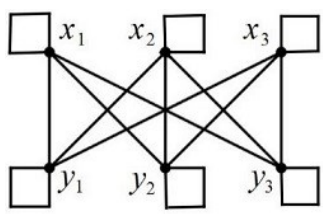

Figure 6: Strong quasi-bipartite graph

$I(G) = (x_1^2, x_1y_1, x_1y_2, x_1y_3, x_2^2, x_2y_1, x_2y_2, x_2y_3, x_3^2, x_3y_1, x_3y_2, x_3y_3, y_1^2, y_2^2y_3^2)$. Therefore, $\overline{I(G)} = I(G)+(y_2^2, y_3^2, x_1x_2, x_1x_3, x_2x_3, y_1y_2, y_1y_3, y_2y_3)$. We see that $I(G) \neq \overline{I(G)}$. Therefore, if $I(G)$ be the edge ideal associated to a strong quasi-bipartite graph, in general $I(G)$ is not integrally closed.

3.3 Associated matrix to a graph

Let G be a graph with vertices set $V = \{x_1, \dots, x_n\}$. We consider the associated matrix to the graph as following:

$$\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}$$

If there is a path between vertices x_i and x_j, we set the a_{ij} and a_{ji} of associated matrix equal to 1. (Obviously, if vertex x_i has a loop, then we set element a_{ii} equal to 1.) We set the other elements of this matrix equal to 0.

Remark 3.10. Based on what we have said, we have the following:
(1) A graph is simple when all the elements on the principal diagonal of its associated matrix are equal to 0.
(2) A graph is complete when the out of principal diagonal elements of its associated matrix are equal to 1.
(3) A graph has loops if at least one of the principal diagonal elements of its associated matrix is equal to 1.
(4) A graph is a complete graph with loops when all elements of its associated matrix are equal to 1.

Remark 3.11. To obtain a complementary graph by the associated matrix, it is sufficient to put the number 0 outside the principal diagonal of the associated matrix instead of any member that is equal to 1, and vice versa.

Now, we are going to prove the existing results in different and simpler ways according to what we have said.

Corollary 3.12. *If all elements on the principal diagonal of the associated matrix to the graph G are equal to* 0*, then edge ideal $I(G)$ is integrally closed.*

Proof. If all elements on the principal diagonal of the associated matrix to the graph G are equal to 0, then it is clear that $I(G) = \overline{I(G)}$. Therefore, edge ideal $I(G)$ is integrally closed. □

Corollary 3.13. *If $I(G)$ be the edge ideal associated to a simple graph, then it is integrally closed.*

Proof. By Remark 3.10 all the elements on the principal diagonal of its associated matrix are equal to 0, therefore by Corollary 3.12 it is clear that $I(G)$ is integrally closed. $\square$

Corollary 3.14. *If $I(G)$ be the edge ideal associated to a complete bipartite graph, then it is integrally closed.*

Proof. Since the edge ideal associated to a complete bipartite graph has no loop, all the elements on the principal diagonal of the associated matrix of the graph G are equal to 0, therefore by Corollary 3.12 it is clear that $I(G)$ is integrally closed. $\square$

Remark 3.15. In general, the edge ideal $I(G)$ is integrally closed when for both different elements a_{ii} and a_{jj} of the principal diagonal of the associated matrix that both values are equal to 1, there must be $a_{ij} = a_{ji} = 1$, where $i \neq j$. In other words, in order for any edge ideal $I(G)$ to be integrally closed, there must be a path between the two distinct vertices of the graph G, that both have loops.

Corollary 3.16. *If G be a complete graph with loops, then edge ideal associated to this graph is integrally closed.*

Proof. By Remark 3.15 it is clear, since all elements of associated matrix of graph G are equal to 1. $\square$

Example 3.17. Let $R = K[x_1, x_2, x_3]$ and $I(G) = (x_1^2, x_1x_2, x_2x_3, x_3^2)$ be edge ideal associated to graph G. Then its associated matrix is as following:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Given that a_{11} and a_{33} of the principal diagonal of associated matrix to graph $I(G)$ are equal to 1, to get associated matrix to $\overline{I(G)}$, it is enough to set element a_{13} and a_{31} equal to 1.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

According to the above matrix we have

$$\overline{I(G)} = I(G) + (x_1x_3).$$

Corollary 3.18. *If x_i^2 and x_j^2 be two generators of the edge ideal of the graph G, then x_ix_j is a generator of the integral closure of $I(G)$.*

Proof. If x_i^2 and x_j^2 be two generators of the edge ideal of the graph G then a_{ij} and a_{ji} of its associated matrix are equal to 1. Hence, according to what we have said, a_{ij} and a_{ji} of its associated matrix are equal to 1. $\square$

4 Integral Closures of Cohen-Macaulay Monomial Ideals

In this section K is an arbitrary field and $R = K[x_0, \ldots, x_n]$ is a polynomial ring over field K. The purpose of this section is to present a family of Cohen-Macaulay monomial ideals such that their integral closures have embedded components and hence are not Cohen-Macaulay. Considering the conditions and definition of $M_{n,t}$ in the introduction of this paper, Jarrah [6] showed that $M_{n,t}$ is a Cohen-Macaulay monomial ideal.

Integral closure $\overline{M}_{n,t}$ of $M_{n,t}$ is a monomial ideal and we have:

$$\overline{M}_{n,t} = \langle x_1^{a_1} \ldots x_n^{a_n} : \ a_i \in [t] \cup \{0\}, \forall i \in [n], \ a_1 + \ldots + a_n = t(n-1)\rangle.$$

In Figure 7, the vertices of the triangle correspond to the generators of the ideal $M_{3,4}$. Also, all the dots inside and on the boundary of the triangle correspond to the generators of the integral closure $\overline{M}_{3,4}$ of $M_{3,4}$. (See [6])

By [6], for $t \geq 2$, the integral closure of $M_{n,t}$ has $\langle x_{i_1}, x_{i_2}, x_{i_3}\rangle$ as an embedded associated prime, for all i_1, i_2, i_3 such that $1 \leq i_1 < i_2 < i_3 \leq n$. Therefore, for all $t \geq 2$, the ideal $\overline{M}_{n,t}$ is not Cohen-Macaulay.

Example 4.1. Ideal $M_{4,2} = \langle x_2^2x_3^2x_4^2, x_1^2x_3^2x_4^2, x_1^2x_2^2x_4^2, x_1^2x_2^2x_3^2\rangle$, is Cohen-Macaulay. Moreover,

$$M_{4,2} = \langle x_1^2x_2^2\rangle \cap \langle x_1^2x_3^2\rangle \cap \langle x_1^2x_4^2\rangle \cap \langle x_2^2x_3^2\rangle \cap \langle x_2^2x_4^2\rangle \cap \langle x_3^2x_4^2\rangle.$$

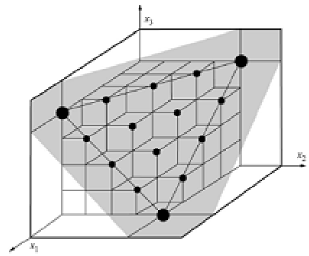

Figure 7: The convex hull of $M_{3,4}$ and the generators of $\overline{M}_{3,4}$.

Now, we have

$$\overline{M}_{4,2} = \langle x_1^2x_2^2x_3^2, x_1^2x_2^2x_4^2, x_1^2x_3^2x_4^2, x_2^2x_3^2x_4^2, x_1^2x_2^2x_3x_4, x_1^2x_2x_3^2x_4, x_1^2x_2x_3x_4^2,$$
$$x_1x_2^2x_3^2x_4, x_1x_2^2x_3x_4^2, x_1x_2x_3^2x_4^2\rangle,$$

that has embedded associated prime and hence is not Cohen-Macaulay.

Remark 4.2. We know that $I \subseteq \overline{I} \subseteq \sqrt{I}$. Therefore, $M_{3,2} \subseteq \overline{M}_{3,2} \subseteq \sqrt{M_{3,2}} = M_{3,1}$. On the other hand, ideals $M_{3,2}$ and $M_{3,1}$ are Cohen-Macaulay; but $\overline{M}_{3,2}$ is not Cohen-Macaulay. Hence, if I and $\sqrt{I}$ are Cohen-Macaulay, then $\overline{I}$ is not necessarily Cohen-Macaulay. Also, if I, J and K are ideals of R such that $I \subset J \subset K$ and I, K are Cohen-Macaulay, then J is not necessarily Cohen-Macaulay.

The following example shows that in general $\overline{\cap_i I_i}$ is not equal to $\cap_i \overline{I_i}$.

Example 4.3. We have $\overline{\cap\langle x_i^2x_j^2\rangle} = \overline{M}_{3,2} = \langle x_2^2, x_2x_3, x_3^2\rangle \cap \langle x_1^2, x_1x_3, x_3^2\rangle$ $\cap\langle x_1^2, x_1x_2, x_2^2\rangle \cap \langle x_1^2, x_2^2, x_3^2\rangle$. Also, we have

$$\cap_{1\le i<j\le 3}\overline{\langle x_i^2, x_j^2\rangle} = \langle x_2^2, x_2x_3, x_3^2\rangle \cap \langle x_1^2, x_1x_3, x_3^2\rangle \cap \langle x_1^2, x_1x_2, x_3^2\rangle.$$

It follows that, $\overline{\cap\langle x_i^2x_j^2\rangle} \neq \cap_{1\le i<j\le 3}\overline{\langle x_i^2, x_j^2\rangle}$.

Let I, J and M be ideals of R. In general, if $\overline{MI} = \overline{MJ}$, then we cannot say that $\overline{I} = \overline{J}$.

Proposition 4.4. *Let I and J be ideals and $\overline{M_{n,t}.I} = \overline{M_{n,t}.J}$, then $\overline{I} = \overline{J}$.*

Proof. Let $\overline{M_{n,t}.I} = \overline{M_{n,t}.J}$. We know that $ht(M_{n,t}) = ht(\sqrt{M_{n,t}}) = ht(I_{n-1}; 1, \dots, 1) = 2$, therefore $ht(M_{n,t}) > 0$. By Cancellation theorem (Exercise 1.2 of [10]), we have $\overline{I} = \overline{J}$.

Remark 4.5. Let $R = K[X] = K[x_1, \dots, x_n]$, then for any $m \geq n$ we have $\overline{M^m_{n,t}} = M_{n,t}\overline{M^{m-1}_{n,t}}$. In particular, $(\overline{M}_{n,t})^m = M_{n,t}(\overline{M}_{n,t})^{m-1}$.

Proposition 4.6. *Let n and t be positive integers and $n \geq 3$. We have $\overline{M}_{n,t} = (M_{n,1})^t$.*

Proof. Ideal $\overline{M}_{n,t}$ is an ideal of Veronese type. In fact

$$\overline{M}_{n,t} = I_{t(n-1);t,\dots,t} = (I_{n-1}; 1, \dots, 1)^t = (M_{n,1})^t.$$

□

Corollary 4.7. *For any positive integer t, we have:*

$$\overline{\cap_{1\leq i<j\leq n}(x_i^t, x_j^t)} = (\cap_{1\leq i<j\leq}(x_i, x_j))^t.$$

Proof. It is obvious that $M_{n,t} = \cap_{1\leq i<j\leq n}(x_i^t, x_j^t)$. Therefore, by Proposition 4.6, we have $\overline{\cap_{1\leq i<j\leq n}(x_i^t, x_j^t)} = (\cap_{1\leq i<j\leq n}(x_i, x_j))^t$. □

Remark 4.8. Let n and t be positive integers and $n \geq 3$. We have:

$$deg(M_{n,t}) = deg(\overline{M}_{n,t}) = t(n-1).$$

Theorem 4.9. *For any positive integer p we have:*

$$\overline{\langle (X^{e_1})^p, \dots, (X^{e_n})^p \rangle} = (M_{n,1})^{pt(n-1)}.$$

Proof. We know that

$$\begin{aligned}
\sqrt{\langle (X^{e_1})^p, \dots, (X^{e_n})^p \rangle} &= \sqrt{\langle (x_2^t \dots x_n^t)^p, \dots, (x_1^t \dots x_{n-1}^t)^p \rangle} \\
&= \sqrt{\langle x_2^{pt} \dots x_n^{pt}, \dots, x_1^{pt} \dots x_{n-1}^{pt} \rangle} \\
&= M_{n,1}.
\end{aligned}$$

Therefore, by Exercise 1.16 of [10], we have:

$$\overline{\langle (X^{e_1})^p, \dots, (X^{e_n})^p \rangle} = (M_{n,1})^{pt(n-1)}.$$

□

Now, we intend to show that the ideal $M_{n,t}$ is Cohen-Macaulay in a simpler and different way.

Theorem 4.10. *For any positive integers n and t, where $n \geq 3$, $M_{n,t}$ is Cohen-Macaulay.*

Proof. For any positive integers n and t, where $n \geq 3$, we have $\sqrt{M_{n,t}}$ $= \sqrt{(M_{n,1})^2} = M_{n,1} = I_{n-1;1,\dots,1}$. Let $J = I_{n-1;1,\dots,1}$. By flat homomorphism definition of $J \to M_{n,t}$ with $x_i \mapsto x_i^t$ and given that J is Cohen-Macaulay, we can conclude that $M_{n,t}$ is Cohen-Macaulay as well. □

4.1 Compare the value of $astab(I)$ with $\overline{astab}(I)$ and $dstab(I)$

If $(R, \mathfrak{m})$ is a regular local ring with $\dim R \leq 2$, then all 3 stability indices are equal; i.e., $astab(I) = dstab(I) = \overline{astab}$. But if $\dim R = 3$, then we still have $astab(I) = dstab(I)$, while $astab(I)$ and $\overline{astab}(I)$ may differ by any amount, see [4]. The following examples show that for an ideal I in a 3-dimensional polynomial ring the invariants $astab(I)$ and $\overline{astab}(I)$ may differ.

Example 4.11. Let $R = K[x_1, x_2, x_3]$ be a polynomial ring over an arbitrary field K. Then $astab(M_{3,2}) = dstab(M_{3,2}) = 2$ and $\overline{astab}(M_{3,2}) = 1$.

Example 4.12. Let $R = K[x_1, \dots, x_n]$ be a polynomial ring over an arbitrary field K. Also, let $1 \leq i < j \leq n$ and $X^{e_{ii+1}} = x_1^t \dots x_{i-1}^t x_{i+2}^t \dots x_n^t$. If $I = \langle X^{e_{12}}, X^{e_{23}}, \dots, X^{e_{n1}} \rangle$, then $astab(I) = dstab(I) = 2$.

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