Journal of Mathematical Extension
Vol. 17, No. 9, (2023) (4)1-20
URL: https://doi.org/10.30495/JME.2023.2621
ISSN: 1735-8299
Original Research Paper

6-Valent Arc-Transitive Cayley Graphs on Abelian Groups

M. Alaeiyan*
Iran University of Science and Technology

M. Akbarizadeh
Iran University of Science and Technology

Z. Heydari
Iran University of Science and Technology

Abstract. Let G be a finite group and $\mathcal{S}$ be a subset of G such that $1_G \notin \mathcal{S}$ and $\mathcal{S}^{-1} = \mathcal{S}$. The *Cayley graph* $\Sigma = Cay(G, \mathcal{S})$ on G with respect to $\mathcal{S}$ is the graph with the vertex set G such that, for $\S, \dagger \in G$, the pair $(\S, \dagger)$ is an arc in $Cay(G, \mathcal{S})$ if and only if $\dagger\S^{-1} \in \mathcal{S}$. The graph Σ is said to be arc-transitive if its full automorphism group $\mathrm{Aut}(\Sigma)$ is transitive on its arc set. In this paper we give a classification for arc-transitive Cayley graphs with valency six on finite abelian groups which are non-normal. Moreover, we classify all normal Cayley graphs on non-cyclic abelian groups with valency 6.

AMS Subject Classification: 05C25; 20B25
Keywords and Phrases: Cayley graph, normal Cayley graph, arc-transitive graph

Received: July 2023; Accepted: December 2023
*Corresponding Author

1 Introduction

In this paper, the vertex set, edge set and the full automorphism group of a finite, simple and undirected graph Σ are denoted by $V(\Sigma)$, $E(\Sigma)$, and $\mathrm{Aut}(\Sigma)$, respectively. A graph Σ is said to be *vertex-transitive* and *edge-transitive* if $\mathrm{Aut}(\Sigma)$ acts transitively on $V(\Sigma)$ and $E(\Sigma)$, respectively. For a positive integer s, an *s-arc* of Σ is an $(s+1)$-tuple $(v_0, v_1, \ldots, v_s)$ of vertices such that $\{v_{i-1}, v_i\} \in E(\Sigma)$ for $1 \leq i \leq s$ and if $s \geq 2$, then $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s-1$. A graph Σ is called *s-arc-transitive* if $\mathrm{Aut}(\Sigma)$ acts transitively on $V(\Sigma)$ and on the set of s-arcs and also it is called *s-transitive* graph if Σ is an s-arc-transitive but not $(s+1)$-arc-transitive. Note that for $s = 1$, we simply use $A(\Sigma)$ to denote its 1-arc set and 1-arc-transitive graph is called *arc-transitive*. An arc-transitive graph Σ is said to be *s-regular* if for any two s-arcs in Σ, there is a unique automorphism of Σ mapping one to the other. Also, an arc-transitive graph Σ is said to be *one regular* if $|\mathrm{Aut}(\Sigma)| = |A(\Sigma)|$.

Let G be a finite group and $\mathcal{S} \subset G$ such that $1_G \notin \mathcal{S}$. The *Cayley digraph* $\mathcal{CD} = Cay_{\mathcal{D}}(G, \mathcal{S})$ on G with respect to $\mathcal{S}$ is defined by $V(\mathcal{CD}) = G$ and $E(\mathcal{CD}) = \{(g, sg) | g \in G,\ s \in \mathcal{S}\}$. The three obvious results follow immediately from this definition: (1) The automorphism group of $\mathcal{CD}$, $\mathrm{Aut}(\mathcal{CD})$, contains the right regular representation G_R of G, and so $\mathcal{CD}$ is vertex-transitive; (2) $\mathcal{CD}$ is connected if and only if $G = \langle \mathcal{S} \rangle$; (3) $\mathcal{CD}$ is undirected if and only if $\mathcal{S}^{-1} = \mathcal{S}$. In this case, we denote $\mathcal{CD} = Cay_{\mathcal{D}}(G, \mathcal{S})$ by $\Sigma = Cay(G, \mathcal{S})$.
A Cayley graph $\Sigma = Cay(G, \mathcal{S})$ (digraph $\mathcal{CD} = Cay_{\mathcal{D}}(G, \mathcal{S})$) is called *normal* if $G \trianglelefteq \mathrm{Aut}(\Sigma)$ ($G \trianglelefteq \mathrm{Aut}(\mathcal{CD})$).

in [13], Xu and Xu classified all arc-transitive Cayley graphs of valency at most four on abelian groups, and in [14] Xu classified all one-regular circulant graphs of valency four. Xu et al. [15] classified all arc-transitive circulant graphs and digraphs of order p^m, where p is an odd prime. Chao [6], classified symmetric graphs of order a prime number p, and Berggren [5] simplified Chao's proof and then Chao and Wells [7] gave a classification of symmetric digraphs of order a prime number p. A generalization of [14], is the classification of 2-arc-transitive circulant graphs, which was given by Alspach et. al [3]. In [1] the first author classified all arc-transitive Cayley graphs with valency 5 of abelian groups. The aim of this paper is to investigate the arc-transitive Cayley graphs

with valency six on abelian groups. Recent research has classified Cayley graphs of valency 6 and edge-transitive Cayley graphs in [9, 10] and [8], respectively.
The group- and graph-theoretic notations and terminologies are standard; see [3, 4, 12] for example. We will denote the semi-directed product of group H by K with $H \cdot K$.

Theorem 1.1. Let G be an abelian group and let $\mathcal{S}$ be a subset of G such that $1_G \notin \mathcal{S}$ and $\mathcal{S} = \mathcal{S}^{-1}$. Suppose that $\Sigma = Cay(G, \mathcal{S})$ is a connected Cayley graph with valency six on group G with respect to $\mathcal{S}$. Then we have:

(a) If Σ is non-normal, then all arc-transitive Σ are as follows:

1. $G = \mathbb{Z}_4 \times \mathbb{Z}_2^4 = \langle\lambda\rangle \times \langle\mu\rangle \times \langle\sigma\rangle \times \langle\theta\rangle \times \langle\varrho\rangle$
$\mathcal{S} = \{\lambda, \lambda^{-1}, \mu, \sigma, \theta, \varrho\}, \Sigma = C_4 \times Q_4 = Q_6$, $\mathrm{Aut}(\Sigma) = S_2 wr S_6$.
2. $G = \mathbb{Z}_4^2 \times \mathbb{Z}_2^2 = \langle\lambda\rangle \times \langle\mu\rangle \times \langle\sigma\rangle \times \langle\varrho\rangle$, $\mathcal{S} = \{\lambda, \lambda^{-1}, \mu, \mu^{-1}, \sigma, \theta\}$,
$\Sigma = C_4 \times Q_4 = Q_6$, $\mathrm{Aut}(\Sigma) = S_2 wr S_6$.
3. $G = \mathbb{Z}_4 \times \mathbb{Z}_2^3 = \langle\lambda\rangle \times \langle\mu\rangle \times \langle\sigma\rangle \times \langle\theta\rangle$,
$\mathcal{S} = \{\lambda, \lambda^{-1}, \mu, \sigma, \theta, \lambda^2\mu\sigma\theta\}$, $\Sigma = Q_5^\theta$, $\mathrm{Aut}(\Sigma) = S_2^5.S_6$.
4. $G = \mathbb{Z}_4^3 = \langle\lambda\rangle \times \langle\mu\rangle \times \langle\sigma\rangle, \mathcal{S} = \{\lambda, \lambda^{-1}, \mu, \mu^{-1}, \sigma, \sigma^{-1}\}$,
$\Sigma = C_4 \times C_4 \times C_4 = Q_6$, $\mathrm{Aut}(\Sigma) = S_2 wr S_6$.
5. $G = \mathbb{Z}_3 \times \mathbb{Z}_3 = \langle\lambda\rangle \times \langle\mu\rangle$, $\mathcal{S} = \{\lambda, \lambda^{-1}, \mu, \mu^{-1}, \lambda\mu^{-1}, \lambda^{-1}\mu\}$,
$\Sigma = K_{3,3,3}$.
6. $G = \mathbb{Z}_4 \times \mathbb{Z}_2 = \langle\lambda\rangle \times \langle\mu\rangle$, $\mathcal{S}_1 = \{\mu, \lambda, \lambda^{-1}, \lambda\mu, \lambda^2\mu, \lambda^3\mu\}$,
$\mathcal{S}_2 = \{\mu, \lambda, \lambda^{-1}, \lambda\mu, \lambda^2, \lambda^3\mu\}$, $\mathcal{S}_3 = \{\lambda, \lambda^{-1}, \lambda\mu, \lambda^2, \lambda^2\mu, \lambda^3\mu\}$,
$\Sigma = K_8 - 8K_2$.
7. $G = \mathbb{Z}_6 \times \mathbb{Z}_2 = \langle\lambda\rangle \times \langle\mu\rangle$, $\mathcal{S}_1 = \{\mu, \lambda, \lambda^{-1}, \lambda^3, \lambda\mu, \lambda^2\mu, \lambda^4\mu\}$,
$\mathcal{S}_2 = \{\lambda, \lambda^{-1}, \lambda^3, \lambda\mu, \lambda^3\mu, \lambda^5\mu\}$, $\Sigma = K_{6,6}$, $\mathrm{Aut}(\Sigma) = S_6 wr S_2$.
8. $G = \mathbb{Z}_4 \times \mathbb{Z}_2^2 = \langle\lambda\rangle \times \langle\mu\rangle \times \langle\sigma\rangle$, $\mathcal{S} = \{\lambda, \lambda^{-1}, \lambda^2, \mu, \sigma, \mu\sigma\}$,
$\Sigma = K_4 \times K_4$, $\mathrm{Aut}(\Sigma) = S_4 \times S_2$.
9. $G = \mathbb{Z}_4 \times \mathbb{Z}_4 = \langle\lambda\rangle \times \langle\mu\rangle$, $\mathcal{S} = \{\lambda, \lambda^{-1}, \lambda^2, \mu, \mu^{-1}, \mu^2\}$,
$\Sigma = K_4 \times K_4$, $\mathrm{Aut}(\Sigma) = S_4 \times S_2$.

10. $G = \mathbb{Z}_2^3 = \langle\lambda\rangle \times \langle\mu\rangle \times \langle\sigma\rangle,\ \mathcal{S} = \{\lambda, \mu, \sigma, \lambda\mu, \lambda\sigma, \lambda\mu\sigma\}$,
$\Sigma = K_8 - 8K_2$.

11. $G = \mathbb{Z}_2^4 = \langle\lambda\rangle \times \langle\mu\rangle \times \langle\sigma\rangle \times \langle\theta\rangle$,
$\mathcal{S} = \{\lambda, \mu, \sigma, \theta, \lambda\mu\sigma, \lambda\mu\theta\}$.

12. $G = \mathbb{Z}_{14} = \langle\lambda\rangle,\ \mathcal{S} = \{\lambda, \lambda^3, \lambda^5, \lambda^{-1}, \lambda^{-3}, \lambda^{-5}\}$,
$\Sigma = K_{7,7} - 7K_2, \mathrm{Aut}(\Sigma) = S_7 \times S_2$.

13. $G = \mathbb{Z}_{12} = \langle\lambda\rangle,\ \mathcal{S} = \{\lambda, \lambda^2, \lambda^5, \lambda^7, \lambda^{10}, \lambda^{11}\}$,
$\Sigma = K_{4,4,4} - 12K_2$.

14. $G = \mathbb{Z}_{12} = \langle\lambda\rangle,\ \mathcal{S} = \{\lambda, \lambda^3, \lambda^5, \lambda^7, \lambda^9, \lambda^{11}\}$,
$\Sigma = K_{6,6}, \mathrm{Aut}(\Sigma) = S_6 wr S_2$.

15. $G = \mathbb{Z}_9 = \langle\lambda\rangle,\ \mathcal{S} = \{\lambda, \lambda^2, \lambda^4, \lambda^5, \lambda^7, \lambda^8\},\ \Sigma = K_{3,3,3}$.

16. $G = \mathbb{Z}_8 = \langle\lambda\rangle,\ \mathcal{S} = \{\lambda, \lambda^2, \lambda^3, \lambda^5, \lambda^6, \lambda^7\},\ \Sigma = K_8 - 8K_2$.

17. $G = \mathbb{Z}_7 = \langle\lambda\rangle, \mathcal{S} = \{\lambda, \lambda^2, \lambda^3, \lambda^4, \lambda^5, \lambda^6\}$,
$\Sigma = K_7$, $\mathrm{Aut}(\Sigma) = S_7$.

(b) If G is a non-cyclic abelian group and Σ is normal, then Σ is arc-transitive if one of the following happens:

1. $G = \mathbb{Z}_2^6 = \langle\lambda\rangle \times \langle\mu\rangle \times \langle\sigma\rangle \times \langle\theta\rangle \times \langle\varrho\rangle \times \langle\xi\rangle$,
$\mathcal{S} = \{\lambda, \mu, \sigma, \theta, \varrho, \xi\}, \Sigma = Q_6$.

2. $G = \mathbb{Z}_2^5 = \langle\lambda\rangle \times \langle\mu\rangle \times \langle\sigma\rangle \times \langle\theta\rangle \times \langle\varrho\rangle$,
$\mathcal{S} = \{\lambda, \mu, \sigma, \theta, \varrho, \lambda\mu\sigma\theta\varrho\}, \Sigma = Q_5^+$.

3. $G = \mathbb{Z}_2^4 = \langle\lambda\rangle \times \langle\mu\rangle \times \langle\sigma\rangle \times \langle\theta\rangle,\ \mathcal{S} = \{\lambda, \mu, \sigma, \theta, \lambda\mu, \sigma\theta\}$,
$\Sigma = K_4 \times K_4$.

4. $\Sigma = Ac(n, n, n, 0, 0)$ for $n \geq 3$ and $n \neq 4$.

5. $\Sigma = Ac(2m, m, m, 1, 0)$ for $m \geq 3$.

6. $\Sigma = Ac(2m, m, m, 1, 1)$ for $m \geq 3$.

7. $\Sigma = Ac(2m, 2m, m, 0, 1)$ for $m \geq 3$.

8. $\Sigma = Ac(2m, 2m, m, 1, 1)$ for $m \geq 3$.
9. $\Sigma = Ac(2m, 2m, p, 1, w^{'})$ with $k^{'} \geq 3$ and $(w^{'})^2 \equiv \pm 1 \pmod{k^{'}}$.
10. $\Sigma = Ac(m, m, p, 0, w^{'})$ with $k^{'} \geq 3$ and $(w^{'})^2 \equiv \pm 1 \pmod{k^{'}}$.
11. $\Sigma = Ac(n, m, p, w, w^{'})$ with $k^{'} \geq 3$, $k \geq 3$, $(w)^2 \equiv \pm 1 \pmod{k}$ and $(w^{'})^2 \equiv \pm 1 \pmod{k^{'}}$.

2 Primary Analysis

Let $\Sigma = Cay(G, \mathcal{S})$ be a Cayley graph on G with respect to $\mathcal{S}$ and let $\mathrm{Aut}(G, \mathcal{S}) = \{\alpha \in \mathrm{Aut}(G) | \ \mathcal{S}^{\alpha} = \mathcal{S}\}$. Clearly, $G \cdot \mathrm{Aut}(G, \mathcal{S}) \leq \mathrm{Aut}(\Sigma)$. Also, we have the following:

Proposition 2.1. *[13, 15] Let G be a finite group, $\mathcal{S}$ be a subset of G non containing 1_G and $\Sigma = Cay(G, \mathcal{S})$ be a Cayley graph on G with respect to $\mathcal{S}$.*
(1) $N_A(G) = G.Aut(G, \mathcal{S})$.
(2) $A = G.Aut(G, \mathcal{S})$ is equivalent to $G \lhd A$.

Proposition 2.2. *[14] A graph Σ is arc-transitive if and only if it is vertex-transitive and the stabilizer G_u of a vertex u acts transitively on the neighborhood $\Sigma_1(u)$ of u in Σ.*

Proposition 2.3. *Let $\Sigma = Cay(G, \mathcal{S})$ be a normal Cayley graph on G with relative to $\mathcal{S}$. Then Σ is arc-transitive if and only if $Aut(G, \mathcal{S})$ acts transitively on the neighborhood $\Sigma_1(1)$ of 1 in Σ.*

Now we introduce some graph products which are used in the paper. Let $\mathcal{X}$ and $\mathcal{Y}$ be two graphs. The *direct product* $\mathcal{X} \times \mathcal{Y}$ is defined as the graph with vertex set $V(\mathcal{X} \times \mathcal{Y}) = V(\mathcal{X}) \times V(\mathcal{Y})$. Two vertices $u = [\S_1, \dagger_1]$ and $v = [\S_2, \dagger_2]$ are adjacent whenever $\S_1 = \S_2$ and $[\dagger_1, \dagger_2] \in E(\mathcal{Y})$ or $\dagger_1 = \dagger_2$ and $[\S_1, \S_2] \in E(\mathcal{X})$. Two graphs are called *relatively prime* if they have no nontrivial common direct factor. Another graph with vertex set $V(\mathcal{X} \times \mathcal{Y})$ is the *lexicographic product* $\mathcal{X}[\mathcal{Y}]$. Two vertices $u = [\S_1, \dagger_1]$ and $v = [\S_2, \dagger_2]$ in $V(\mathcal{X}[\mathcal{Y}])$, are adjacent, if either $[\S_1, \S_2] \in E(\mathcal{X})$ or $\S_1 = \S_2$ and $[\dagger_1, \dagger_2] \in E(\mathcal{Y})$. Let $\mathcal{V}(Y) = \{\dagger_1, \dagger_2, \ldots, \dagger_n\}$. Then there is a natural embedding of $n\mathcal{X}$ in $\mathcal{X}[\mathcal{Y}]$, where for $1 \leq i \leq n$, the ith copy of $\mathcal{X}$ is the subgraph induced on the vertex subset $\{(\S, \dagger_i) | \ \S \in V(\mathcal{X})\}$ in $\mathcal{X}[\mathcal{Y}]$. The *deleted lexicographic product* $\mathcal{X}[\mathcal{Y}] - n\mathcal{X}$ is the graph obtained by deleting all the edges of (this natural embedding of) $n\mathcal{X}$ from $\mathcal{X}[\mathcal{Y}]$.

Let $\mathcal{X}$ be a graph, α be a permutation on $V(\mathcal{X})$ and C_n be a circuit of length n. The *twisted product* $\mathcal{X} \times_\alpha C_n$ of $\mathcal{X}$ by C_n with respect to α is defined as follows:

$$\begin{aligned} V(\mathcal{X} \times_\alpha C_n) \ &= V(\mathcal{X}) \times V(C_n) = \{ \ (\S, i) \mid \ \S \in V(\mathcal{X}), \\ &\quad i = 0, 1, \ldots, n-1\}, \\ E(\mathcal{X} \times_\alpha C_n) \ &= \{[(\S, i), (\S, i+1)] \mid \ \S \in V(\mathcal{X}), \ i = 0, 1, \ldots, n-2\} \\ &\cup \{[(\S, n-1), \ (\S^\alpha, 0)] \mid \ \S \in V(\mathcal{X})\} \\ &\cup \{[(\S, i), (y, i)] \mid \ [\S, \dagger] \in E(\mathcal{X}), \ i = 0, 1, \ldots, n-1\}. \end{aligned}$$

Finally, we introduce some new graphs used in this paper. A circulant graph $C(n; n_1, \ldots, n_d)$ is a graph with vertex set $VC = \{0, 1, \ldots, n-1\}$ and edge set $EC = \{(i, j) \mid |j - i| = n_1, \ldots, n_{d-1} \text{ or } n_d \pmod{n}\}$, which has order n and valency $2d$ or $2d - 1$. Thus $C_n = C(n; 1)$. If n is even then the graph $C(n; 1, n/2)$ is of valency 3, denoted by M_n. The graph Q_d^+ for $d = 4, 5$, denotes the graph obtained by connecting all the long diagonal of d-cube Q_d, that is connecting all vertices u and v in Q_d such that $d(u, v) = d$. The graph $K_{m,m} \times_c C_n$ is the twisted product of $K_{m,m}$ by C_n such that c is a cycle permutation on each part of the complete bipartite graph $K_{m,m}$. The graph $Q_3 \times_d C_n$ is the twisted product of Q_3 by C_n such that d transposes each pair of elements on the long diagonals of Q_3. The graph $C^d_{2m}[2K_1]$ is defined as the following:

$$\begin{aligned} V(C^d_{2m}[2K_1]) \ &= V(C_{2m}[2K_1]), \\ E(C^d_{2m}[2K_1]) \ &= E(C_{2m}[2K_1]) \cup \{[(\S_i, \dagger_j), (\S_{i+m}, \dagger_j)] | \\ &\quad i = 0, 1, \ldots, m-1, \ j = 1, 2\} \end{aligned}$$

where $V(C_{2m}) = \{\S_0, \S_1, \ldots, \S_{2m-1}\}$ and $V(2K_1) = \{\dagger_1, \dagger_2\}$.

In the following theorem, all the non-normal Cayley graphs of valency six on abelian groups are classified.

Theorem 2.4. *[2] Let G be an abelian group and $\Sigma = Cay(G, \mathcal{S})$ be a connected Cayley graph on G with respect to $\mathcal{S}$ of degree 6. Then Σ is normal unless one of the following cases holds:*

1. $G = \mathbb{Z}_2^3 \times \mathbb{Z}_m = \langle\lambda\rangle \times \langle\mu\rangle \times \langle\sigma\rangle \times \langle\varrho\rangle$ $(m \geq 3)$,
$\mathcal{S} = \{\lambda, \mu, \sigma, \lambda\mu\sigma\theta, \theta^{-1}\}, \Sigma = K_{4,4} \times C_m$.

2. $G = \mathbb{Z}_2^5 = \langle\lambda\rangle \times \langle\mu\rangle \times \langle\sigma\rangle \times \langle\theta\rangle \times \langle\varrho\rangle$,
$\mathcal{S} = \{\lambda, \mu, \sigma, \lambda\mu\sigma, \theta, \varrho\}, \Sigma = C_4 \times K_{4,4}$.

3. $G = \mathbb{Z}_2^2 \times \mathbb{Z}_4 = \langle\lambda\rangle \times \langle\mu\rangle \times \langle\sigma\rangle$,
$\mathcal{S} = \{\lambda, \mu, \lambda\mu, \sigma^2, \sigma, \sigma^{-1}\}, \Sigma = K_4 \times K_4$.

4. $G = \mathbb{Z}_2^4 \times \mathbb{Z}_4 = \langle\lambda\rangle \times \langle\mu\rangle \times \langle\sigma\rangle \times \langle\theta\rangle \times \langle\varrho\rangle$,
$\mathcal{S} = \{\lambda, \mu, \sigma, \theta, \varrho, \varrho^{-1}\}, \Sigma = C_4 \times Q_4 = Q_6$.

5. $G = \mathbb{Z}_2^3 \times \mathbb{Z}_4 = \langle\lambda\rangle \times \langle\mu\rangle \times \langle\sigma\rangle \times \langle\theta\rangle$, $\mathcal{S}_1 = \{\lambda, \mu, \sigma, \theta^2, \theta, \theta^{-1}\}$,
$\Sigma = Q_3 \times K_4$; $\mathcal{S}_2 = \{\lambda, \mu, \lambda\mu, \sigma, \theta, \theta^{-1}\}$, $\Sigma = K_4 \times K_2 \times C_4$;
$\mathcal{S}_3 = \{\lambda, \mu, \sigma, \lambda\theta^2, \theta, \theta^{-1}\}$, $\Sigma = K_{4,4} \times C_4$.

6. $G = \mathbb{Z}_2^2 \times \mathbb{Z}_6 = \langle\lambda\rangle \times \langle\mu\rangle \times \langle\sigma\rangle$, $\mathcal{S} = \{\lambda, \mu, \lambda\mu, \sigma^3, \sigma, \sigma^{-1}\}$,
$\Sigma = K_4 \times K_{3,3}$.

7. $G = \mathbb{Z}_2^3 \times \mathbb{Z}_6 = \langle\lambda\rangle \times \langle\mu\rangle \times \langle\sigma\rangle \times \langle\theta\rangle$,
$\mathcal{S} = \{\lambda, \mu, \sigma, \theta^3, \theta, \theta^{-1}\}, \Sigma = Q_3 \times K_{3,3}$.

8. $G = \mathbb{Z}_2 \times \mathbb{Z}_6 \times \mathbb{Z}_m = \langle\lambda\rangle \times \langle\mu\rangle \times \langle\sigma\rangle$ $(m \geq 3)$,
$\mathcal{S} = \{\lambda, \mu^3, \mu, \mu^{-1}, \sigma, \sigma^{-1}\}, \Sigma = K_2 \times K_{3,3} \times C_m$.

9. $G = \mathbb{Z}_6 \times \mathbb{Z}_{2m} = \langle\lambda\rangle \times \langle\mu\rangle$ $(m \geq 2)$,
$\mathcal{S} = \{\lambda^3, \mu^m, \lambda, \lambda^{-1}, \mu, \mu^{-1}\}, \Sigma = K_{3,3} \times M_{2m}$.

10. $G = \mathbb{Z}_4 \times \mathbb{Z}_{2m} = \langle\lambda\rangle \times \langle\mu\rangle$ $(m \geq 2)$,
$\mathcal{S} = \{\lambda, \lambda^{-1}, \lambda^2, \mu, \mu^{-1}, \mu^m\}, \Sigma = K_4 \times M_{2m}$.

11. $G = \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_m = \langle\lambda\rangle \times \langle\mu\rangle \times \langle\sigma\rangle$ $(m \geq 3)$,
$\mathcal{S}_1 = \{\lambda, \mu, \mu^{-1}, \mu^2, \sigma, \sigma^{-1}\}, \Sigma = K_2 \times K_4 \times C_m$;
$\mathcal{S}_2 = \{\lambda, \mu, \mu^{-1}, \lambda\mu^2, \sigma, \sigma^{-1}\}, \Sigma = K_{4,4} \times C_m$.

12. $G = \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_{2m} = \langle\lambda\rangle \times \langle\mu\rangle \times \langle\sigma\rangle$ $(m \geq 2)$,
$\mathcal{S} = \{\lambda, \mu, \mu^{-1}, \sigma, \sigma^{-1}, \sigma^m\}, \Sigma = K_2 \times C_4 \times M_{2m}$.

13. $G = \mathbb{Z}_2^2 \times \mathbb{Z}_4 \times \mathbb{Z}_m = \langle\lambda\rangle \times \langle\mu\rangle \times \langle\sigma\rangle \times \langle\theta\rangle$ $(m \geq 3)$,
$\mathcal{S} = \{\lambda, \mu, \sigma, \sigma^{-1}, \theta, \theta^{-1}\}$, $C_4 \times C_4 \times C_m = Q_4 \times C_m$.

14. $G = \mathbb{Z}_2^3 \times \mathbb{Z}_m = \langle\lambda\rangle \times \langle\mu\rangle \times \langle\sigma\rangle \times \langle\theta\rangle$ $(m \geq 3)$,
$\mathcal{S} = \{\lambda, \mu, \sigma\theta, \sigma\theta^{-1}, \theta, \theta^{-1}\}, \Sigma = C_4 \times C_m[2K_1]$.

15. $G = \mathbb{Z}_2^2 \times \mathbb{Z}_m = \langle\lambda\rangle \times \langle\mu\rangle \times \langle\sigma\rangle$ $(m = 5, 10)$,
$\mathcal{S} = \{\lambda, \mu, \sigma, \sigma^{-1}, \sigma^3, \sigma^{-3}\}, \Sigma = C_4 \times K_5$ *if* $m = 5$ *and*
$\Sigma = C_4 \times K_{5,5} - 5K_2$ *if* $m = 10$.

16. $G = \mathbb{Z}_2^2 \times \mathbb{Z}_{4m} = \langle\lambda\rangle \times \langle\mu\rangle \times \langle\sigma\rangle$ $(m \geq 2)$,
$\mathcal{S} = \{\lambda, \mu, \sigma, \sigma^{-1}, \sigma^{2m+1}, \sigma^{2m-1}\}, \Sigma = C_4 \times C_m[2K_1]$.

17. $G = \mathbb{Z}_4 \times \mathbb{Z}_{2m} = \langle\lambda\rangle \times \langle\mu\rangle$ $(m \geq 3,\ m$ *is odd*$)$,
$\mathcal{S} = \{\lambda, \lambda^3, \mu, \mu^{-1}, \mu^{m+1}, \mu^{m-1}\}, \Sigma = C_4 \times C_m[2K_1]$.

18. $G = \mathbb{Z}_4^2 \times \mathbb{Z}_m = \langle\lambda\rangle \times \langle\mu\rangle \times \langle\sigma\rangle$ $(m \geq 3)$,
$\mathcal{S} = \{\lambda, \lambda^3, \mu, \mu^3, \sigma, \sigma^{-1}\}, \Sigma = C_4 \times C_4 \times C_m = Q_4 \times C_m$.

19. $G = \mathbb{Z}_2 \times \mathbb{Z}_m \times \mathbb{Z}_n = \langle\lambda\rangle \times \langle\mu\rangle \times \langle\sigma\rangle$ $(m \geq 3, n \geq 3)$,
$\mathcal{S} = \{\lambda\mu, \lambda\mu^{-1}, \mu, \mu^{-1}, \sigma, \sigma^{-1}\},\ \Sigma = C_m[2K_1]$.

20. $G = \mathbb{Z}_m \times \mathbb{Z}_n = \langle\lambda\rangle \times \langle\mu\rangle$ $(m = 5, 10,\ n \geq 3)$,
$\mathcal{S} = \{\lambda, \lambda^{-1}, \lambda^3, \lambda^{-3}, \mu, \mu^{-1}\}, \Sigma = K_5 \times C_n$ *if* $m = 5$
and $\Sigma = K_{5,5} - 5K_2 \times C_n$ *if* $m = 10$.

21. $G = \mathbb{Z}_{4m} \times \mathbb{Z}_n = \langle\lambda\rangle \times \langle\mu\rangle$ $(m \geq 2, n \geq 3)$,
$\mathcal{S} = \{\lambda, \lambda^{-1}, \lambda^{2m+1}, \lambda^{2m-1}, \mu, \mu^{-1}\}, \Sigma = C_{2m}[2K_1] \times C_n$.

22. $G = \mathbb{Z}_2^4 = \langle\lambda\rangle \times \langle\mu\rangle \times \langle\sigma\rangle \times \langle\theta\rangle, \mathcal{S} = \{\lambda, \mu, \lambda\mu, \sigma, \lambda\mu\sigma, \theta\}$,
$\Sigma = K_2 \times K_2[2K_2]$.

23. $G = \mathbb{Z}_2^2 \times \mathbb{Z}_4 = \langle\lambda\rangle \times \langle\mu\rangle \times \langle\sigma\rangle$,
$\mathcal{S} = \{\lambda, \mu, \lambda\sigma^2, \sigma, \sigma^{-1}, \sigma^2\}, \Sigma = K_2 \times K_2[2K_2]$.

24. $G = \mathbb{Z}_2^3 \times \mathbb{Z}_4 = \langle\lambda\rangle \times \langle\mu\rangle \times \langle\sigma\rangle \times \langle\theta\rangle$,
$\mathcal{S} = \{\lambda, \mu, \sigma, \theta, \theta^{-1}, \lambda\mu\theta^2\}, \Sigma = K_2 \times Q_4^+$.

25. $G = \mathbb{Z}_2^2 \times \mathbb{Z}_{3m} = \langle\lambda\rangle \times \langle\mu\rangle \times \langle\sigma\rangle$ $(m \geq 1)$,
$\mathcal{S} = \{\lambda, \mu, \lambda\sigma^m, \lambda\sigma^{2m}, \sigma, \sigma^{-1}\}$.

26. $G = \mathbb{Z}_2 \times \mathbb{Z}_{10} = \langle\lambda\rangle \times \langle\mu\rangle,\ \mathcal{S} = \{\lambda, \mu, \mu^3, \mu^5, \mu^7, \mu^9\}$,
$\Sigma = K_2 \times K_{5,5}$.

27. $G = \mathbb{Z}_2^2 \times \mathbb{Z}_{2m} = \langle\lambda\rangle \times \langle\mu\rangle \times \langle\sigma\rangle$ $(m \geq 2)$,
$\mathcal{S} = \{\lambda\sigma, \lambda\sigma^{-1}, \mu, \sigma^m, \sigma, \sigma^{-1}\}, \Sigma = C_{2m}^d[2K_1] \times K_2$.

28. $G = \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_{2m} = \langle\lambda\rangle \times \langle\mu\rangle \times \langle\sigma\rangle$ $(m \geq 2)$,
$\mathcal{S} = \{\lambda, \mu^2\sigma^m, \mu, \mu^{-1}, \sigma, \sigma^{-1}\}, \Sigma = K_2 \times Q_3 \times C_m = Q_4 \times C_m$.

29. $G = \mathbb{Z}_2 \times \mathbb{Z}_{2m} = \langle\lambda\rangle \times \langle\mu\rangle$ $(m \geq 3)$,
$\mathcal{S} = \{\lambda, \mu^m, \mu, \mu^{-1}, \mu^{m+1}, \mu^{m-1}\}, \Sigma = K_2 \times C_m[K_2]$.

30. $G = \mathbb{Z}_2^2 \times \mathbb{Z}_{2m} = \langle\lambda\rangle \times \langle\mu\rangle \times \langle\sigma\rangle$ $(m \geq 2)$,
$\mathcal{S} = \{\lambda, \mu, \lambda\sigma, \lambda\sigma^{-1}, \sigma, \sigma^{-1}\}, \Sigma = K_2 \times C_{2m}[K_2]$.

31. $G = \mathbb{Z}_2 \times \mathbb{Z}_{6m} = \langle\lambda\rangle \times \langle\mu\rangle$ $(m \geq 3,\ m \text{ is odd})$,
$\mathcal{S} = \{\lambda, \mu^2, \mu^{-2}, \mu^m, \mu^{5m}, \mu^{3m}\}, \Sigma = K_2 \times K_{3,3} \times_c C_m$.

32. $G = \mathbb{Z}_2^2 \times \mathbb{Z}_{6m} = \langle\lambda\rangle \times \langle\mu\rangle \times \langle\sigma\rangle$ $(m \geq 2)$,
$\mathcal{S} = \{\lambda, \mu\sigma^m, \mu\sigma^{3m}, \mu\sigma^{5m}, \sigma, \sigma^{-1}\}, \Sigma = K_2 \times K_{3,3} \times_c C_{2m}$.

33. $G = \mathbb{Z}_2^3 = \langle\lambda\rangle \times \langle\mu\rangle \times \langle\sigma\rangle,\ \mathcal{S} = \{\lambda, \mu, \sigma, \lambda\mu, \lambda\sigma, \lambda\mu\sigma\}$,
$\Sigma = K_8 - 8K_2$.

34. $G = \mathbb{Z}_2^4 = \langle\lambda\rangle \times \langle\mu\rangle \times \langle\sigma\rangle \times \langle\theta\rangle,\ \mathcal{S} = \{\lambda, \mu, \sigma, \theta, \lambda\mu\sigma, \lambda\mu\theta\}$.

35. $G = \mathbb{Z}_2^2 \times \mathbb{Z}_{2m} = \langle\lambda\rangle \times \langle\mu\rangle \times \langle\sigma\rangle$ $(m \geq 2)$,
$\mathcal{S} = \{\lambda, \mu, \lambda\sigma^m, \mu\sigma^m, \sigma, \sigma^{-1}\}$.

36. $G = \mathbb{Z}_2^2 \times \mathbb{Z}_4 = \langle\lambda\rangle \times \langle\mu\rangle \times \langle\sigma\rangle$,
$\mathcal{S}_1 = \{\lambda, \mu, \lambda\mu, \lambda\sigma^2, \sigma, \sigma^{-1}\}, \mathcal{S}_2 = \{a, b, ac^2, abc^2, c, c^{-1}\}$.

37. $G = \mathbb{Z}_2^3 \times \mathbb{Z}_4 = \langle\lambda\rangle \times \langle\mu\rangle \times \langle\sigma\rangle \times \langle\theta\rangle$,
$\mathcal{S} = \{\lambda, \mu, \sigma, \theta, \theta^{-1}, \lambda\mu\sigma\theta^2\}, \Sigma = Q_5^+$.

38. $G = \mathbb{Z}_2 \times \mathbb{Z}_{6m} = \langle\lambda\rangle \times \langle\mu\rangle$ $(m \geq 2)$,
$\mathcal{S} = \{\lambda, \mu^{3m}, \lambda\mu^{2m}, \lambda\mu^{4m}, \mu, \mu^{-1}\}$.

39. $G = \mathbb{Z}_2 \times \mathbb{Z}_{4m} = \langle\lambda\rangle \times \langle\mu\rangle$ $(m \geq 1)$,
$S = \{\lambda, \lambda\mu^m, \lambda\mu^{2m}, \lambda\mu^{3m}, \mu, \mu^{-1}\}$.

40. $G = \mathbb{Z}_4 \times \mathbb{Z}_{2m} = \langle\lambda\rangle \times \langle\mu\rangle$ $(m \geq 2)$,
$\mathcal{S} = \{\lambda, \lambda^{-1}, \mu^m, \lambda^2\mu^m, \mu, \mu^{-1}\}$.

41. $G = \mathbb{Z}_2^2 \times \mathbb{Z}_{4m} = \langle\lambda\rangle \times \langle\mu\rangle \times \langle\sigma\rangle$ $(m \geq 1)$,
$\mathcal{S} = \{\lambda, \lambda\sigma^{2m}, \mu\sigma^m, \mu\sigma^{3m}, \sigma, \sigma^{-1}\}$.

42. $G = \mathbb{Z}_2 \times \mathbb{Z}_{10} = \langle\lambda\rangle \times \langle\mu\rangle$, $\mathcal{S} = \{\lambda, \lambda\mu^5, \mu, \mu^9, \mu^3, \mu^7\}$.

43. $G = \mathbb{Z}_2 \times \mathbb{Z}_{2m} = \langle\lambda\rangle \times \langle\mu\rangle$,
$\mathcal{S}_1 = \{\lambda, \mu, \mu^{-1}, \mu^m, \lambda\mu, \lambda\mu^{-1}\}$ $(m \geq 2)$,
$\mathcal{S}_2 = \{\lambda, \lambda\mu^m, \mu, \mu^{-1}, \lambda\mu, \lambda\mu^{-1}\}$ $(m \geq 2)$,
$\mathcal{S}_3 = \{\lambda\mu^m, \mu^m, \mu, \mu^{-1}, \lambda\mu, \lambda\mu^{-1}\}$ $(m \geq 2)$,
$\mathcal{S}_4 = \{\lambda, \lambda\mu^m, \mu, \mu^{-1}, \mu^{m+1}, \mu^{m-1}\}$ $(m \geq 3)$,
$\mathcal{S}_5 = \{\lambda, \mu, \mu^{-1}, \mu^m, \lambda\mu^{m+1}, \lambda\mu^{m-1}\}$ $(m \geq 3)$,
$\mathcal{S}_6 = \{\lambda, \lambda\mu^m, \mu, \mu^{-1}, \lambda\mu^{m+1}, \lambda\mu^{m-1}\}$ $(m \geq 3)$,
$\mathcal{S}_7 = \{\lambda\mu^m, \mu, \mu^{-1}, \mu^m, \lambda\mu^{m+1}, \lambda\mu^{m-1}\}$ $(m \geq 3)$.

44. $G = \mathbb{Z}_2^2 \times \mathbb{Z}_m = \langle\lambda\rangle \times \langle\mu\rangle \times \langle\sigma\rangle$,
$\mathcal{S}_1 = \{\lambda, \mu, \sigma, \sigma^{-1}, \lambda\mu\sigma, \lambda\mu\sigma^{-1}\}(m \geq 3)$,
$\mathcal{S}_2 = \{\lambda, \mu, \sigma, \sigma^{-1}, \lambda\sigma^{k+1}, \lambda\sigma^{k-1}\}$ $(m = 2k,\ k \geq 3)$,
$\mathcal{S}_3 = \{\lambda, \mu, \sigma, \sigma^{-1}, \lambda\mu\sigma^{k+1}, \lambda\mu\sigma^{k-1}\}$ $(m = 2k,\ k \geq 3)$,
$\mathcal{S}_4 = \{\lambda, \mu\sigma, \mu\sigma^{-1}, \lambda\sigma^k, \sigma, \sigma^{-1}\}$ $(m = 2k,\ k \geq 2)$,
$\mathcal{S}_5 = \{\lambda, \mu\sigma^{k+1}, \mu\sigma^{k-1}, \sigma^k, \sigma, \sigma^{-1}\}$ $(m = 2k,\ k \geq 3)$,
$\mathcal{S}_6 = \{\lambda, \mu\sigma^{k+1}, \mu\sigma^{k-1}, \lambda\sigma^k, \sigma, \sigma^{-1}\}$ $(m = 2k,\ k \geq 3)$,
$\mathcal{S}_7 = \{\lambda, \mu, \sigma, \sigma^{-1}, \lambda\sigma, \lambda\sigma^{-1}\}$ $(m = 2k-1,\ k \geq 2)$.

45. $G = \mathbb{Z}_{4m} = \langle\lambda\rangle$ $(m \geq 2)$, $\mathcal{S} = \{\lambda, \lambda^{-1}, \lambda^m, \lambda^{-m}, \lambda^{2m+1}, \lambda^{2m-1}\}$.

46. $G = \mathbb{Z}_{2m} = \langle\lambda\rangle$ $(m \geq 4)$,
$\mathcal{S} = \{\lambda, \lambda^{-1}, \lambda^{m+1}, \lambda^{m-1}, \lambda^k, \lambda^{-k}\}$ $(2 \leq k \leq m-2)$, $(m,k) = l$, *if* $l > 2$ *or* $l = 2$ *for* $m = 4i+2$; ($k = 2i$, *with* i *odd or* $k = 2i+2$, *with* i *even).*

47. $G = \mathbb{Z}_2 \times \mathbb{Z}_m = \langle\lambda\rangle \times \langle\mu\rangle$ $(m \geq 5)$,
$\mathcal{S}_1 = \{\lambda\mu, \lambda\mu^{-1}, \mu, \mu^{-1}, \mu^j, \mu^{-j}\}(2 \leq j < \frac{m}{2})$, $(m,j) = p > 2$,
$m = (t+1)p$,
$\mathcal{S}_2 = \{\lambda\mu, \lambda\mu^{-1}, \mu, \mu^{-1}, \lambda\mu^j, \lambda\mu^{-j}\}(2 \leq j < \frac{m}{2})$, $(m,j) = p > 2$,
$m = (t+1)p$.

48. $G = \mathbb{Z}_2 \times \mathbb{Z}_8 = \langle\lambda\rangle \times \langle\mu\rangle$, $\mathcal{S}_1 = \{\lambda\mu, \lambda\mu^{-1}, \mu, \mu^{-1}, \mu^3, \mu^{-3}\}$,
$\mathcal{S}_2 = \{\lambda\mu, \lambda\mu^{-1}, \mu, \mu^{-1}, \lambda\mu^3, \lambda\mu^{-3}\}$.

49. $G = \mathbb{Z}_{2m} \times \mathbb{Z}_n = \langle\lambda\rangle \times \langle\mu\rangle$ $(m \geq 2,\ n \geq 3)$,
$\mathcal{S} = \{\lambda, \lambda^{-1}, \lambda^m\mu, \lambda^m\mu^{-1}, \mu, \mu^{-1}\}$.

50. $G = \mathbb{Z}_{2m} \times \mathbb{Z}_{2n} = \langle\lambda\rangle \times \langle\mu\rangle$ $(m \geq 3,\ n \geq 2)$,
$\mathcal{S} = \{\lambda, \lambda^{-1}, \lambda^{m+1}\mu^n, \lambda^{m-1}\mu^n, \mu, \mu^{-1}\}$.

51. $G = \mathbb{Z}_{6m} = \langle\lambda\rangle$ $(m \geq 2)$, $\mathcal{S}_1 = \{\lambda, \lambda^{-1}, \lambda^3, \lambda^{-3}, \lambda^{3m+1}, \lambda^{3m-1}\}$,
$\mathcal{S}_2 = \{\lambda, \lambda^{-1}, \lambda^{3m+1}, \lambda^{3m-1}, \lambda^{3m+3}, \lambda^{3m-3}\}$.

52. $G = \mathbb{Z}_m = \langle\lambda\rangle$ $(m = 7, 14)$, $\mathcal{S} = \{\lambda, \lambda^{-1}, \lambda^3, \lambda^{-3}, \lambda^5, \lambda^{-5}\}$,
$\Sigma = K_7$ *if* $m = 7$ *and* $\Sigma = K_{7,7} - 7K_2$ *if* $m = 14$.

53. $G = \mathbb{Z}_{3m} = \langle\lambda\rangle$ $(m \geq 3)$,
$\mathcal{S} = \{\lambda, \lambda^{-1}, \lambda^{m-1}, \lambda^{m+1}, \lambda^{2m-1}, \lambda^{2m+1}\}$.

54. $G = \mathbb{Z}_{16m-4} = \langle\lambda\rangle$ $(m \geq 1)$,
$\mathcal{S} = \{\lambda, \lambda^{-1}, \lambda^{4m-2}, \lambda^{12m-2}, \lambda^{8m-3}, \lambda^{8m-1}\}$.

55. $G = \mathbb{Z}_{16m+4} = \langle\lambda\rangle$ $(m \geq 1)$,
$\mathcal{S} = \{\lambda, \lambda^{-1}, \lambda^{4m+2}, \lambda^{12m+2}, \lambda^{8m+1}, \lambda^{8m+3}\}$.

56. $G = \mathbb{Z}_3 \times \mathbb{Z}_3 = \langle\lambda\rangle \times \langle\mu\rangle$, $\mathcal{S} = \{\lambda, \lambda^2, \mu, \mu^2, \lambda^2\mu, \lambda\mu^2\}$,
$\Sigma = K_{3,3,3}$.

57. $G = \mathbb{Z}_4^2 \times \mathbb{Z}_2 = \langle\lambda\rangle \times \langle\mu\rangle \times \langle\sigma\rangle$ $(m \geq 3)$,
$\mathcal{S} = \{\lambda, \lambda^{-1}, \mu, \mu^{-1}, \sigma, \lambda^2\mu^2 c\}$.

3 The Proof of Theorem 1.1

Here, we will give all non-normal arc-transitive Cayley graphs on abelian groups of degree six. Moreover, we will characterize all normal arc-transitive Cayley graphs on the non-cyclic abelian groups. First, we will introduce a family of graphs of valency 6, the Cayley graph $Cay(G, \mathcal{S}_{ww'})$, on a non-cyclic abelian group G.

Lemma 3.1. *Let* n, m, p, k, k', w *and* w' *be positive integers with* $m|n$, $n = mk$, $p|m$, $m = pk'$, $n \geq 3$, $m \geq 3$, $p \geq 1$, $\gcd(w, k) = 1$, $\gcd(w', k') = 1$, $0 \leq w \leq k-1$ *and* $0 \leq w' \leq k'-1$*. Let* $G = \mathbb{Z}_n \times \mathbb{Z}_m \times \mathbb{Z}_p = \langle\lambda\rangle \times \langle\mu\rangle \times \langle\sigma\rangle$*, and* $\mathcal{S}_{ww'} = \{\lambda, \lambda^{-1}, \lambda^w\mu, \lambda^{-w}\mu^{-1}, \lambda^w\mu^{w'}\sigma,$

$\lambda^{-w}\mu^{-w'}\sigma^{-1}\}$. *The Cayley graph* $Cay(G, \mathcal{S}_{ww'}) := Ac(n, m, p, w, w')$ *is a regular graph of degree 6 and we have:*

(1) $Ac(n, m, p, w, w')$ *is non-normal if and only if one of the following happens:*

(i) $(n, m, p, w, w') = (4, 4, 4, 0, 0)$.

(ii) $n, m(\geq 4)$ *are even,* $p = 2$ *and* $w' = \pm 1$.

(2) *Suppose that* $Ac(n, m, p, w, w')$ *is normal. Then,* $Ac(n, m, p, w, w')$ *is arc-transitive if and only if one of the following holds:*

(i) $k \leq 2$ *and* $k' \leq 2$.

(ii) $k \leq 2$, $k' \geq 3$ *and* $(w')^2 \equiv \pm 1 (\mathrm{mod}\ k')$.

(iii) $k \geq 3$, $k' \geq 3$, $w^2 \equiv \pm 1 (\mathrm{mod}\ k)$ *and* $(w')^2 \equiv \pm 1 (\mathrm{mod}\ k)$.

Proof. (1) This is a straightforward result of Theorem 2.4.
(2) Since $G = \langle \lambda, \lambda^w \mu, \lambda^w \mu^{w'} \sigma \rangle$, $\mathrm{Aut}(G, \mathcal{S}_{ww'})$ acts on $\mathcal{S}_{ww'}$ faithfully. Thus $\mathrm{Aut}(G, \mathcal{S}_{ww'})$ is isomorphic to a subgroup of S_6. Now by Proposition 2.3, $Ac(n, m, p, w, w')$ is arc-transitive if and only if $\mathrm{Aut}(G, \mathcal{S}_{ww'})$ acts transitively on $\mathcal{S}_{ww'}$. So, all elements of $\mathcal{S}_{ww'}$ have the same order. □

Now we are ready to prove the Theorem 1.1. Set $A = \mathrm{Aut}(\Sigma)$.

Proof. (a) All non-normal Cayley graphs with valency six are classified in Theorem 2.4 Now we investigate which of them are arc-transitive. In the cases (1), (2), (5) for $\mathcal{S} = S_3$ and (11) for $\mathcal{S} = S_2$, we have $\Sigma = C_m \times K_{4,4}$. Let $V(C_m) = \{1, \ldots, m\}$ and $V(K_{4,4}) = \{\S_1, \S_2, \S_3, \S_4, \S_1', \S_2', \S_3', \S_4'\}$ such that $(\S_i, \S_j') \in E(K_{4,4})$ for $1 \leq i, j \leq 4$. One can see that there is no $f \in A_{(1,\S_1)}$ such that $f(1, \S_1') = (4, \S_1)$, which implies that Σ is not arc-transitive.
In (5) for $\mathcal{S} = S_1$, let $V(K_4) = \{\dagger_1, \dagger_2, \dagger_3, \dagger_4\}$ and Q_3 contain two circuits C_4, C_4' with $V(C_4) = \{\S_1, \S_2, \S_3, \S_4\}$ and $V(C_4') = \{\S_1', \S_2', \S_3', \S_4'\}$ such that $(\S_i, \S_i') \in E(Q_3)$ for $1 \leq i \leq 4$. Note that the edge $[(\S_i, \dagger_j)(\S_i, \dagger_{j+1})]$ is contained in a cycle of length 3 in Σ, but the edge $[(\S_i, \dagger_j)(\S_{i+1}, \dagger_j)]$ is not containedin any cycle, for $1 \leq i, j \leq 3$. Therefore, Σ is not edge transitive and then is not

arc-transitive. In (6), let $V(K_4) = \{\dagger_1, \dagger_2, \dagger_3, \dagger_4\}$ and $V(K_{3,3}) = \{\S_1, \S_2, \S_3, \S_1^{'}, \S_2^{'}, \S_3^{'}\}$ such that $(\S_i, \S_j^{'}) \in E(K_{3,3})$ for $1 \leq i, j \leq 3$. Note that the edge $[(\dagger_j, \S_i)(\dagger_{j+1}, \S_i)]$ is contained in any cycle of length 3 in Σ, but $[(\dagger_j, \S_i)(\dagger_j, \S_k^{'})]$ is not contained in any cycle, for $1 \leq j \leq 3$ and for any $1 \leq i, k \leq 4$. Therefore, Σ is not edge transitive and then is not arc-transitive. In (7), let Q_3 contain two circuits $C_4, C_4^{'}$ respectively with the set of vertices $V(C_4) = \{\S_1, \S_2, \S_3, \S_4\}$ and $V(C_4^{'}) = \{\S_1^{'}, \S_2^{'}, \S_3^{'}, \S_4^{'}\}$ such that $(\S_i, \S_i^{'}) \in E(Q_3)$ for $1 \leq i \leq 4$ and $V(K_{3,3}) = \{\dagger_1, \dagger_2, \dagger_3, \dagger_1^{'}, \dagger_2^{'}, \dagger_3^{'}\}$ such that $(\dagger_i, \dagger_j^{'}) \in E(K_{3,3})$ for $1 \leq i, j \leq 3$. One can see that there is no $f \in A_{(\S_1, \dagger_1)}$ such that $f(\S_1, \dagger_1^{'}) = (\S_2, \dagger_1)$. Thus Σ is not arc-transitive. In (8), let $V(K_{3,3}) = \{\S_1, \S_2, \S_3, \S_1^{'}, \S_2^{'}, \S_3^{'}\}$ such that $(\S_i, \S_j^{'}) \in E(K_{3,3})$ for $1 \leq i, j \leq 3$ and $V(M_{2m}) = \{1, \ldots, 2m\}$. One can see that there is no $f \in A_{(\S_1, 1)}$ such that $f(\S_1^{'}, 1) = (\S_1, 2)$. So, Σ is not arc-transitive.
In (9), let $V(K_2) = \{\S_1, \S_2\}$, $V(K_{3,3}) = \{\dagger_1, \dagger_2, \dagger_3, \dagger_1^{'}, \dagger_2^{'}, \dagger_3^{'}\}$ such that $(\dagger_i, \dagger_j^{'}) \in E(K_{3,3})$ for $1 \leq i, j \leq 3$ and $V(C_m) = \{1, \ldots, m\}$. One can see that there is no $f \in A_{(\S_1, \dagger_1, 1)}$ such that $f(\S_1, \dagger_1^{'}, 1) = (\S_2, \dagger_1, 1)$. Thus from Proposition 2.2, we conclude that Σ is not arc-transitive.
In (10), let $V(K_4) = \{\dagger_1, \dagger_2, \dagger_3, \dagger_4\}$ and $V(M_{2m}) = \{1, \ldots, 2m\}$ for $m \neq 2$. Note that the edge $[(\dagger_i, j)(\dagger_{i+1}, j)]$ is contained in a cycle of length 3 in Σ, but the edge $[(\dagger_i, j)(\dagger_i, j{+}m)]$ is not contained in any cycle, for $1 \leq i \leq 4$ and $1 \leq j \leq 2m-1$. Therefore, Σ is not edge transitive and then is not arc-transitive. In (11) for $\mathcal{S} = S_1$ and (5) for $\mathcal{S} = S_2$, we have $\Sigma = K_2 \times K_4 \times C_n$. Let $V(K_2) = \{\S_1, \S_2\}$, $V(K_4) = \{\dagger_1, \dagger_2, \dagger_3, \dagger_4\}$ and $V(C_n) = \{1, \ldots, n\}$. Note that the edge $[(\S_i, \dagger_j, k)(\S_i, \dagger_{j+1}, k)]$ is contained in a cycle of length 3 but the edge $[(\S_i, \dagger_j, k)(\S_i, \dagger_j, k+1)]$ is not, for $i = \{1, 2\}$, $1 \leq j \leq 4$ and $1 \leq k \leq n$, $n \neq 4$. Now, if $n = 4$, the edge $[(\S_1, \dagger_j, k)(\S_2, \dagger_j, k)]$ is contained in a cycle of length 3 but the edge $[(\S_i, \dagger_i, k)(\S_i, \dagger_i, k+1)]$ is not contained in any cycle, for $i = \{1, 2\}$, $1 \leq j \leq 4$ and $1 \leq k \leq 4$. Then, in both cases, Σ is not arc-transitive. In (12), let $V(K_2) = \{\S_1, \S_2\}$, $V(C_4) = \{\dagger_1, \ldots, \dagger_4\}$ and $V(M_{2m}) = \{1, 2, \ldots, 2m\}$ for $m \geq 3$. One can see that there is no $f \in A_{(\S_1, \dagger_1, 1)}$ such that $f(\S_1, \dagger_1^{'}, 1) = (\S_2, \dagger_1, 1)$, which implies that Σ is not arc-transitive. In (13), (18) for $m \neq 4$ and (28), let Q_4 contain two graphs $Q_3, Q_3^{'}$ with set of vertices $V(Q_3) = \{\S_1, \ldots, \S_4, \S_1^{'}, \ldots, \S_4^{'}\}$ such that $(\S_i, \S_i^{'}) \in E(Q_3)$ for

$1 \leq i \leq 4$ and $V(Q_3^{'}) = \{\dagger_1, \ldots, \dagger_4, \dagger_1^{'}, \ldots, \dagger_1^{'}\}$ such that $(\dagger_i, \dagger_i^{'}) \in E(Q_3^{'})$ for $1 \leq i \leq 4$. One can see that there is no $f \in A_{(\S_1,1)}$ such that $f(\S_2, 1) = (\S_1, m)$. So, by Proposition 2.2, Σ is not arc-transitive.
In (14), (16), (17), (19) and (20), we have $\Sigma = C_n \times C_m[2k1]$. Let $V(C_n) = \{1, \ldots, n\}$, $V(C_m) = \{1, \ldots, m\}$ and $V(2k1) = \{\dagger_1, \dagger_2\}$ such that $[(\S_i, \dagger_j)(\S_{i+1}, \dagger_k)] \in E(C_m[2k1])$ for $k, j = \{1, 2\}$ and $1 \leq i \leq m$. Note that there is no $f \in A_{(1,\S_1,\dagger_1)}$ such that $f(2, \S_1, \dagger_1) = (1, \S_2, y_2)$. So by the note on Proposition 2.2, Σ is not arc-transitive.
In (15) for $m = 10$ and (21) for $[m = 10, n \geq 4]$, let $V(C_n) = \{1, \ldots, n\}$ and $V(K_{5,5} - 5K_2) = \{\S_1, \S_2, \ldots, \S_5, \S_1^{'}, \S_2^{'}, \ldots, \S_5^{'}\}$ such that $(\S_i, \S_j^{'}) \in E(K_{5,5} - 5K_2)$ for $i \neq j, 1 \leq i, j \leq 5$. One can see that there is no $f \in A_{(1,\S_1)}$ such that $f(2, \S_1) = (1, y_2)$, which means Σ is not arc-transitive. Now suppose that $[m = 10$ and $n = 3]$. Note that the edge $[(i, \S_j)(i + 1, \S_j)]$ is contained in a cycle of length 3 in Σ, but the edge $[(i, \S_j)(i, \S_k^{'})]$ is not, for $1 \leq i \leq 3$ and $1 \leq j, k \leq 5$. Therefore, Σ is not arc-transitive.
In (15) for $m = 5$ and (21) for $[m = 5, n \geq 4]$, we have $\Sigma = C_n \times K_5$. Let $V(C_n) = \{1, \ldots, n\}$ and $V(K_5) = \{\S_1, \ldots, \S_5\}$. Note that the edge $[(i, \S_j)(i, \S_{j+1})]$ is contained in a cycle of length 3 in Σ , but the edge $[(i, \S_j)(i+1, \S_j)]$ is not, for $1 \leq i \leq 4$ and $1 \leq j \leq 5$. Therefore, Σ is not arc-transitive.
In (22), the edge $(\lambda, \lambda\mu)$ is contained in a cycle of length 3, but the edge $(\lambda, \lambda\mu)$ is not. Therefore, Σ is not arc-transitive.
In (23), the edge (λ, σ^2) is contained in a cycle of length 3, but the edge $(\lambda, \lambda\mu)$ is not. Therefore, Σ is not arc-transitive.
In (25), one can see there is no $f \in A_\lambda$ such that $f(\lambda\mu) = (\sigma^m)$. So, Σ is not arc-transitive.
In (26), let $V(K_2) = \{\S_1, \S_2\}$ and $V(K_{5,5}) = \{\S_1, \S_2, \ldots, \S_5, \S_1^{'}, \S_2^{'}, \ldots, \S_5^{'}\}$, such that $(\S_i, \S_j^{'}) \in E(K_{5,5})$ for $1 \leq i, j \leq 5$. One can see that there is no $f \in A_{(\S_1,\dagger_1)}$ such that $f(\S_1, \dagger_1^{'}) = (\S_2, \dagger_1)$. It follows that Σ is not arc-transitive.
In (27), we have $\Sigma = C_{2m}^d[2k_1] \times K_2$. Let $V(C_{2m}) = \{1, \ldots, 2m\}$, $V(2K_1) = \{\S_1, \S_2\}$ and $V(K_2) = \{\dagger_1, \dagger_2\}$. One can see that there is no $f \in A_{1,\S_1,\dagger_1}$ such that $f(1, \S_1, \dagger_2) = (2m, \S_2, \dagger_1)$. So, by Proposition 2.2, Σ is not arc-transitive.
In (29), note that the edge (μ^m, μ^{m+1}) is contained in a cycle of length

3, but the edge $(1, \lambda)$ is not. Then, Σ is not arc-transitive.
In (30) and (43) for $\mathcal{S} = S_7$, note that the edge (λ, σ) is contained in a cycle of length 3, but the edge $(\lambda, \lambda\mu)$ is not. Then Σ is not arc-transitive.
In (31) and (32), one can see that there is no $f \in A_{\mu_2}$ such that $f(\lambda\mu_2) = (\mu^{m+2})$. Hence Σ is not arc-transitive.
In (34), Γ is a bipartite graph of diameter three and girth four. Therefore by [4, Proposition 17.2], Γ is at most 3-transitive. Hence by [11], there are 4 symmetric graphs of order 16.
In (35), one can see that there is no $f \in A_\lambda$ such that $f(\lambda\mu) = (\lambda\sigma)$. So, by Proposition 2.2, Σ is not arc-transitive.
In (36) for $[\mathcal{S} = S_1, S_2]$, note that the edge (λ, μ) is contained in a cycle of length 3, but the edges $(\lambda, \lambda\mu)$ and (σ, σ_2) are not contained in a cycle of length 3. Then Σ is not arc-transitive.
In (38) and (39), one can see that there is no $f \in A_\lambda$ such that $f(\lambda\mu) = (\mu^{2m})$ and also in the cases (40), (41) and (42), one can see that there is no $f \in A_\lambda$ such that $f(\lambda^2) = (\lambda\mu)$, $f(\lambda\sigma) = (\sigma^{2m})$ and $f(\lambda\mu) = (\mu^5)$, respectively. So, by Proposition 2.2, Σ is not arc-transitive.
In (43) for $[\mathcal{S} = S_1, m \geq 3]$ and $[\mathcal{S} = S_5, m \geq 4]$, one can see that there is no $f \in A_\lambda$ such that $f(\lambda\mu) = (\lambda\mu^m)$. For $[\mathcal{S} = S_2, m \geq 4]$ and $[\mathcal{S} = S_4, S_3, m \geq 3]$, there is no $f \in A_\lambda$ such that $f(\lambda\mu) = (\mu^m)$. Also, for $[\mathcal{S} = S_3, m \geq 4]$ there is no $f \in A_\lambda$ such that $f(\lambda) = (\mu^{m+1})$. Finally, for $[\mathcal{S} = S_7, m \geq 3]$ there is no $f \in A_\lambda$ such that $f(\lambda\mu^{m+1}) = (\mu^{m+1})$. So, by Proposition 2.2, Σ is not arc-transitive.
In (44) for $[\mathcal{S} = S_1, S_2, S_3, m \geq 3]$, one can see that there is no $f \in A_\lambda$ such that $f(\lambda\mu) = (\lambda\sigma)$. Also, for $[\mathcal{S} = S_4, m \geq 2]$ there is no $f \in A_\lambda$ such that $f(\lambda\mu) = (\mu\sigma)$. For $[\mathcal{S} = S_5, m \geq 3, m = 2k]$, there is no $f \in A_\lambda$ such that $f(\lambda\mu\sigma^{k+1}) = (\lambda\sigma^k)$. Finally, for $[\mathcal{S} = S_6, m \geq 3, m = 2k]$, there is no $f \in G_\lambda$ such that $f(\lambda\mu\sigma^{k+1}) = (\sigma^k)$. So, by Proposition 2.2, Σ is not arc-transitive. In (45), one can see that there is no $f \in G_\lambda$ such that $f(\lambda^2) = (\lambda^{m+1})$. Thus, by Proposition 2.2, Σ is not arc-transitive.
In (46), there is no $f \in A_\lambda$ such that $f(\lambda^m) = (\lambda^{m+2})$. So, Proposition 2.2 implies that Σ is not arc-transitive.
In (47), for $\mathcal{S} = S_1$, there is no $f \in A_\mu$ such that $f(\mu) = (\lambda\mu^j)$ and for $\mathcal{S} = S_2$, $f \notin A_\mu$ such that $f(\mu) = (\mu^j)$. Therefore, by Proposition 2.2 Σ is not arc-transitive.

In (48) for $\mathcal{S} = S_1$ and $\mathcal{S} = S_2$, there is no $f \in A_\lambda$ such that $f(\mu) = (\lambda\mu^3)$ and $f(\mu) = (\lambda\mu)$, respectively, which implies Σ is not arc-transitive.
In (49) and (50), there is no $f \in A_\lambda$ such that $f(\lambda^2) = (\lambda\mu)$. So, by Proposition 2.2, Σ is not arc-transitive.
In (51), (53), (54) and (55), for $[\mathcal{S} = S_1, S_2]$, there is no $f \in A_\lambda$ such that $f(\lambda^2) = (\lambda^{3m})$, (λ^{2m}), (λ^{4m-1}) and (λ^{4m+3}), repectively. So Σ is not arc-transitive.
In (57), since there is no $f \in A_\lambda$ such that $f(\lambda\mu) = (\lambda\sigma)$, Σ is not arc-transitive.
In (4), we have $\Sigma = K_2 \times Q_5 \simeq C_4 \times Q_4$. Since Q_4 is arc-transitive, Σ is arc-transitive.
The cases (13) and (18) for $m = 4$ are similarly as the case (4).
In (24), we have $\Sigma = K_2 \times Q_4^+$. Note that [4, Proposition 17.2] tells us that the Cayley graph is at most 3-transitive. Let $[\alpha]$ be a 3-arc in Σ. Then there are automorphisms $g_1, \ldots, g_5$ such that $g_i[\alpha] = [\beta^{(i)}]$ $(1 \leq i \leq 5)$, so that each $[\beta^{(i)}]$ is a successor of $[\alpha]$. Then $\mathrm{Aut}(\Sigma)$ is transitive on 3-arcs and Σ is vertex-transitive. So, Σ is 2-transitive and 1-transitive. Therefore, the graph $\Sigma = K_2 \times Q_4^+$ is arc-transitive.
In (37), we have the graph $\Sigma = Q_5^+$, which is arc-transitive.
In (52) for $m = 7$ and $m = 14$, we have $\Sigma = K_7$ and $\Sigma = K_{7,7} - 7K_2$ respectively, which are arc-transitive.
In (51) for $m = 2$, (53) for $m = 4$, (54) for $m = 1$, (45) for $m = 3$ and (43) for $[\mathcal{S} = S_3, m = 3]$ and $[\mathcal{S} = S_5, m = 3]$, we have $\Sigma = K_{6,6}$, which is arc-transitive.
In (45) for $m = 2$, (46) for $m = 4$, (39) for $m = 1$, (33) and (43) for $[\mathcal{S} = S_1, S_2, S_3, m = 2]$, we have $\Sigma = K_8 - 8K_2$, which is arc-transitive.
In (53) for $m = 3$ and (56), we have $\Sigma = K_{3,3,3}$, which is arc-transitive.
Now the proof of Theorem 1.1 (a) is completed.
(b) Assume G is a non-cyclic group, and $\Sigma = Cay(G, \mathcal{S})$ is a normal Cayley graph of valency six. Since the order of all elements of $\mathcal{S}$ is equal to n, we investigate two deferent cases $n = 2$ and $n > 2$. If $n = 2$, then S contains six involutions and up to an isomorphism, one of the following cases happens:

1. $G = \mathbb{Z}_2^3 = \langle\lambda\rangle \times \langle\mu\rangle \times \langle\sigma\rangle, \mathcal{S} = \{\lambda, \mu, \sigma, \lambda\mu, \lambda\sigma, \lambda\mu\sigma\}, \Sigma = K_8 - 8K_2.$
2. $G = \mathbb{Z}_2^4 = \langle\lambda\rangle \times \langle\mu\rangle \times \langle\sigma\rangle \times \langle\theta\rangle,$

$\mathcal{S}_1 = \{\lambda, \mu, \sigma, \theta, \lambda\mu, \lambda\mu\sigma\}, \Sigma = K_2 \times K_2[2K_2]$,
$\mathcal{S}_2 = \{\lambda, \mu, \sigma, \theta, \lambda\mu, \sigma\theta\},\ \Sigma = K_4 \times K_4$,
$\mathcal{S}_3 = \{\lambda, \mu, \sigma, \theta, \lambda\mu\sigma, \lambda\mu\theta\},\ \mathcal{S}_4 = \{\lambda, \mu, \sigma, \theta, \lambda\mu, \lambda\mu\sigma\theta\}$,
$\mathcal{S}_5 = \{\lambda, \mu, \sigma, \theta, \lambda\mu\sigma, \lambda\mu\sigma\theta\},\ \mathcal{S}_6 = \{\lambda, \mu, \sigma, \theta, \lambda\theta, \lambda\mu\sigma\}$.

3. $G = \mathbb{Z}_2^5 = \langle\lambda\rangle \times \langle\mu\rangle \times \langle\sigma\rangle \times \langle\theta\rangle \times \langle\varrho\rangle$,
$\mathcal{S}_1 = \{\lambda, \mu, \sigma, \theta, \varrho, \lambda\mu\},\ \Sigma = K_4 \times Q_3$,
$\mathcal{S}_2 = \{\lambda, \mu, \sigma, \theta, \varrho, \lambda\mu\sigma\},\ \Sigma = C_4 \times Q_3^+$,
$\mathcal{S}_3 = \{\lambda, \mu, \sigma, \theta, \varrho, \lambda\mu\sigma\theta\},\ \Sigma = K_2 \times Q_4^+$,
$\mathcal{S}_4 = \{lambda, \mu, \sigma, \theta, \varrho, \lambda\mu\sigma\theta\varrho\},\ \Sigma = Q_5^+$.

4. $G = \mathbb{Z}_2^6 = \langle\lambda\rangle \times \langle\mu\rangle \times \langle\sigma\rangle \times \langle\theta\rangle \times \langle\varrho\rangle \times \langle\xi\rangle$,
$\mathcal{S} = \{\lambda, \mu, \sigma, \theta, \varrho, \xi\}, \Sigma = Q_6$.

Note that by part (a) of Theorem 1.1, the graphs of the cases (1), (2) for $[\mathcal{S} = \mathcal{S}_1, \mathcal{S}_3]$, (3) for $[\mathcal{S} = \mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3]$ are non-normal. Also, the graphs Q_6, $K_4 \times K_4$ and Q_5^+ are arc-transitive.
If $n > 2$, we suppose that $\mathcal{S} = \{\S, \S^{-1}, \dagger, \dagger^{-1}, \ddagger, \ddagger^{-1}\}$, where $o(\S) = o(\dagger) = o(\ddagger) = n \geq 3$. Then, G is an abelian group generated by $\S$, $\dagger$ and $\ddagger$, so $G \cong \mathbb{Z}_n \times \mathbb{Z}_m \times \mathbb{Z}_p = \langle\lambda\rangle \times \langle\mu\rangle \times \langle\sigma\rangle$, where $m|n$ and $p|m$ (i.e., $n = mk$, $m = pk^{'}$). Note that $\mathrm{Aut}(G)$ acts transitively on the set of elements of G with the highest order. So, we can take $\S = \lambda$, $\dagger = \lambda^w\mu^j$, and $\ddagger = \lambda^w\mu^{w^{'}}\sigma^i$ such that $\mu \in \langle\mu^j\rangle$ and $\sigma \in \langle\sigma^i\rangle$. One can see that the orders of $\lambda^w\mu^j$ and $\lambda^w\mu^{w^{'}}\sigma^i$ are n. Therefore, $\gcd(j, m) = 1$ and $\gcd(p, i) = 1$. So, we may also take $\dagger = \lambda^w\mu$ and $\ddagger = \lambda^w\mu^{w^{'}}\sigma$, under the action of a suitable automorphism of G. Since the mapping $\lambda \mapsto \lambda$, $\mu \mapsto \lambda^k\mu$ and $\sigma \mapsto \lambda^k\mu^{k^{'}}\sigma$ is an automorphism of G, without loss of generality, we can assume that $0 \leq w \leq k-1$ and $0 \leq w^{'} \leq k^{'} - 1$. Now, since $o(\dagger) = o(\ddagger) = n$, we have $\gcd(w, k) = 1$ and $\gcd(w^{'}, k^{'}) = 1$. However, G is not cyclic and then $m \geq 2$ and $p \geq 2$. Thus $\Sigma \cong Ac(n, m, p, w, w^{'})$. Now, by Lemma 3.1, the proof of Theorem 1.1 (b) is complete. □

4 Conclusion

In this paper, we have studied the arc-transitive Cayley graphs with valency six on finite abelian groups. We have shown that there are only finitely many such graphs that are non-normal, and we have classified them completely. We have also classified all normal Cayley graphs on non-cyclic abelian groups with valency six, and we have given some examples of such graphs. Our results extend and generalize some previous works on arc-transitive Cayley graphs of low valency.

References

[1] M. Alaeiyan, Arc-transiyive and s-regular Cayley graph of valency five on abelian groups, *Discussiones Mathematica Graph Theory*, 26, 2006, 359-368.

[2] M. Alaeiyan, Normal 6-valent Cayley graph of abelian groups, *International Journal of Engineering Science,* 19(1-2), 2008, 1-11.

[3] B. Alspach, M. Conder, D. Marusic and Ming-Yao Xu, A classification of 2-arc-transitive circulant, *Journal of Algebraic Combinatorics,* 5, 1996, 83-86.

[4] N. Biggs, *Algebraic Graph Theory*, Cambridge University Press, 1974.

[5] J. L. Berggren, An algebraic characterization of symmetric graph with p point, *Bulletin of the Australian Mathematical Society*, 158, 1971, 247-256.

[6] C. Y. Chao, On the classification of symmetric graph with a prime number of vertices, *Transactions of the American Mathematical Society*, 158, 1971, 247-256.

[7] C. Y. Chao, and J. G. Wells, A class of vertex-transitive di-graphs, *Journal of Combinatorial Theory, Series B*, 14, 1973, 246-255.

[8] M. Conder, J. X. Zhou, Y. Q. Feng and M.M. Zhang, Edge-transitive bi-Cayley graphs, *Journal of Combinatorial Theory, Series B*, 145, 2020, 264-306.

[9] X. Fang, P. Niu, and J. Wang, On Automorphism Groups of Symmetric Cayley Graphs of Finite Simple Groups with Valency Six, *Algebra Colloquium*, 17, 1, 2010, . 161-172.

[10] M. Hashemian and M. Alaeiyan, Novel Conditions on the Non-Normal Cayley Graphs of Valency Six, *Indian Journal of Science and Technology*, 8, 2015.

[11] N. J. A. Sloane, *Sequences A286280 in The On-Line Encyclopedia of Integer Sequences.*

[12] H. Wielandt, *Finit Permutation Group*, Academic Press, New York, 1964.

[13] M. Y. Xu and J. Xu, Arc-transitive Cayley graph of valency at most four on abelian groups, *Southeast Asian Bulletin of Mathematics*, 25, 2001, 355-363.

[14] M. Y. Xu, A note on one-regular graphs of valency four, *Chinese Science Bulletin*, 45, 2000, 2160-2162.

[15] M. Y. Xu, H. S. Sim and Y. G. Baik, Arc-transitive circulant digraphs of odd prime-power order, *Discrete Mathematics*, 287(1-3), 2004, 113-119.

[16] M. Y. Xu, Automorphism groups and isomorphisms of Cayley digraphs, *Discrete Mathematics*, 182, 1998, 309-319.

Mehdi Alaeiyan
Professor of Mathematics
School of Mathematics
Iran University of Science and Technology
Tehran, Iran
E-mail: alaeiyan@iust.ac.ir

Masoumeh Akbarizadeh
PhD of Mathematics
School of Mathematics
Iran University of Science and Technology
Tehran, Iran
E-mail: masoumeh.akbarizadeh@gmail.com

Zahra Heydari
M.S. in Mathematics

School of Mathematics
Iran University of Science and Technology
Tehran, Iran