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# The Stability of Functional Equations in Quasi-normed Quasilinear Spaces

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**Abstract.** In this article, we define quasi-normed quasilinear spaces and show that the space of all bounded quasilinear q-operators on this space is an $\Omega$-space. Due to the importance of the problem of stability of functional equations in different spaces, many authors have studied the stability of functional equations in different spaces. Tabor proved the stability of the Cauchy functional equation in quasi-Banach spaces. We prove the stability of functional equations in quasi-normed quasilinear spaces.

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**Keywords and Phrases:** Quasilinear spaces, Quasilinear $q$-operators, Quasi-normed quasilinear spaces, Stability of functional equations.

## 1 Introduction

Some authors have introduced different kinds of normed space. Aseev [1] presented the concept of normed quasilinear spaces which is a gen-

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eralization of classical linear spaces. Rano [5] introduced quasi-normed linear spaces.

In Section 3, we define the quasi-normed quasilinear spaces. In Section 4, we introduce the quasilinear $q$-operators on quasi-normed quasilinear spaces and show that the space of all bounded quasilinear $q$-operators is an $\Omega$-space.

The problem of the stability of functional equations arose from a question about group homomorphisms [8]. Numerous authors have solved the problem of stability of different functional equations in various spaces. For example, see [3, 4, 7]. In Section 5, we prove the stability of the Cauchy functional equation in quasi-normed quasilinear spaces.

## 2 Preliminaries

**Definition 2.1.** *[1, 9], A set $X$ is called a quasilinear space (QLS, for short), if a partial order relation $\preceq$, an algebraic sum operation and an operation of multiplication by real numbers are defined on it, such that the following conditions hold for any elements $x, y, z, v \in X$ and any $\alpha, \beta \in \mathbb{R}$:*

1. $x \preceq x$;
2. $x \preceq z$ *if* $x \preceq y$ *and* $y \preceq z$;
3. $x = y$ *if* $x \preceq y$ *and* $y \preceq x$;
4. $x + y = y + x$;
5. $x + (y + z) = (x + y) + z$;
6. *There exists an element* $\theta \in X$ *such that* $x + \theta = x$;
7. $\alpha.(\beta.x) = (\alpha.\beta).x$;
8. $\alpha.(x + y) = \alpha.x + \alpha.y$;
9. $1.x = x$;
10. $0.x = \theta$;
11. $(\alpha + \beta).x \preceq \alpha.x + \beta.x$;
12. $x + z \preceq y + v$ *if* $x \preceq y$ *and* $z \preceq v$;
13. $\alpha.x \preceq \alpha.y$ *if* $x \preceq y$.

A QLS $X$ with the partial order relation $\preceq$ is denoted by $(X, \preceq)$ where $\theta$ is the zero element of $X$.

An element $x^{'} \in X$ is called an inverse of $x \in X$ if $x + x^{'} = \theta$. If an inverse element exists, then it is unique.

**Definition 2.2.** *Let $X$ be a QLS. A real function $\| \ . \ \|_X : X \to \mathbb{R}$ is called a norm if the following conditions hold:*

1. $\|x\|_X > 0$ *if* $x \neq \theta$*;*
2. $\|x + y\|_X \leq \|x\|_X + \|y\|_X$*;*
3. $\|\alpha.x\|_X = |\alpha|.\|x\|_X$*;*
4. *If* $x \preceq y$*, then* $\|x\|_X \leq \|y\|_X$*;*
5. *If for any $\varepsilon > 0$ there exists an element $x_\varepsilon \in X$ such that $x \preceq y + x_\varepsilon$ and $\|x_\varepsilon\|_X \leq \varepsilon$, then $x \preceq y$.*

*A QLS $X$ with a norm defined on it is called normed quasilinear space (normed QLS, for short).*

Let $X$ be a normed QLS. The Hausdorff metric on $X$ is defined by

$$h_X(x, y) = inf\{r \geq 0 : x \preceq y + a_1^r, \ y \preceq x + a_2^r, \ \|a_i^r\|_X \leq r, \ i = 1, 2\}.$$

**Definition 2.3.** *Let $X$ be an abstract set, and let $K \geq 1$ be a given real number. A functional $d : X \times X \to [0, \infty)$ is called a b-metric if for all $x, y, z \in X$ the following conditions hold:*

1. $d(x, y) = 0$ *if and only if* $x = y$*;*
2. $d(x, y) = d(y, x)$*;*
3. $d(x, z) \leq K(d(x, y) + d(y, z))$*.*

Then, the ordered pair $(X, d)$ is called a $b$-meric space.

**Definition 2.4.** *[5, 6] Let $X$ be a linear space and $\theta$ be the origin of $X$. Let $|.|_q : X \to [0.\infty)$ satisfying the following conditions:*

1. $|x|_q > 0$ *if* $x \neq \theta$*;*
2. *There is a constant $K \geq 1$ such that $|x + y|_q \leq K(|x|_q + |y|_q)$;*

*3.* $|\alpha x|_q = |\alpha||x|_q$.

*Then* $(X, |.|_q)$ *is called a quasi-normed linear space and the least value of constant* $K \geq 1$ *is called the index of quasi-norm* $|.|_q$.

**Example 2.5.** *By defining* $|x|_q = (\sqrt{|x_1|} + \sqrt{|x_2|})^2$ *for* $x = (x_1, x_2) \in \mathbb{R}^2$, $(\mathbb{R}^2, |x|_q)$ *is a quasi-normed linear space but it is not a normed linear space.*

## 3 Quasi-normed Quasilinear Spaces

**Definition 3.1.** *Let* $X$ *be a quasilinear space. A real function* $\|.\|_q : X \to \mathbb{R}$ *is called a quasi-norm if for all* $x, y \in X$ *the following conditions hold:*

1. $\|x\|_q > 0$ *if* $x \neq \theta$;
2. *There is a constant* $K \geq 1$ *such that* $\|x + y\|_q \leq K(\|x\|_q + \|y\|_q)$;
3. $\|\alpha x\|_q = |\alpha|\|x\|_q$;
4. *If* $x \preceq y$, *then* $\|x\|_q \leq \|x\|_q$;
5. *If for any* $\varepsilon > 0$ *there exists an element* $x_\varepsilon \in X$ *such that* $x \preceq y + x_\varepsilon$ *and* $\|x_\varepsilon\|_q \leq \varepsilon$, *then* $x \preceq y$.

A quasilinear space $X$ with a quasi-norm defined on it is called a quasi-normed quasilinear space.

If any element in the quasi-normed quasilinear space $X$ has an inverse element, then $X$ is a quasi-normed linear space.

A quasi-Banach quasilinear space is a complete quasi-normed quasilinear space.

A quasi-norm $\|.\|_q$ is called a $p$-norm $(0 < p \leq 1)$ if for all $x, y \in X$,

$$\|x + y\|_q^p \leq \|x\|_q^p + \|y\|_q^p.$$

In this case, a quasi-Banach quasilinear space is called a $q$-$p$-Banach space. Each quasi-norm is equivalent to some $p$-norm ([2]).

Let $X$ be a quasi-normed quasilinear space with quasi-norm $\|.\|_q$. We defined for all $x, y \in X$,

$$h_q(x, y) := \inf\{r \geq 0 : x \leq y + a_1^r,\ y \leq x + a_2^r,\ \|a_i^r\|_q \leq r,\ i = 1, 2\}.$$

Then $(X, h_q)$ is a $b$-metric space and $h_q(x,y) \leq \|x-y\|_q$.

It is easy to see that $h_q(x,y)$ satisfies all the conditions of a $b$-metric. We only prove the part 3 of Definition 2.3.

Let $\varepsilon > 0$ and $x, y, z \in X$. There are $a_i^r, b_i^r \in X$ such that,

$$x \preceq y + a_1^r,\ y \preceq x + a_2^r,\ \|a_i^r\|_q \leq h_q(x,y) + \frac{\varepsilon}{2},$$

$$y \preceq z + b_1^r,\ z \preceq y + b_2^r,\ \|b_i^r\|_q \leq h_q(y,z) + \frac{\varepsilon}{2}.$$

Hence,

$$x \preceq z + a_1^r + b_1^r,\ z \preceq x + a_2^r + b_2^r,\ \|a_i^r + b_i^r\|_q \leq K(h_q(x,y) + h_q(y,z) + \varepsilon).$$

Consequently, $h_q(x,z) \leq K(h_q(x,y) + h_q(y,z))$.

**Example 3.2.** *The set $\Omega(\mathbb{R})$ is the set of all nonempty closed bounded subsets of real numbers with the inclusion relation $\subseteq$, the algebraic sum operation*

$$A + B = \overline{\{a + b : a \in A, b \in B\}}$$

*and the real scalar multiplication $\alpha.A = \{\alpha.a : a \in A\}$ where $\alpha \in \mathbb{R}$, with norm is defined by*

$$\|A\| := \sup_{a \in A} a^2,$$

*is a quasi-normed quasilinear space.*

**Definition 3.3.** *The sequence $\{x_n\}$ in quasi-normed quasilinear space $X$ converges to an element $x \in X$ if and only if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $h_q(x_n, x) < \varepsilon$.*

**Lemma 3.4.** *The following conditions hold with respect to the $b$-metric $h_q$:*

*a) The operations of algebraic sum and the real scaler multiplication are continuous,*

*b) If $x_n \to x$, $y_n \to y$ and for all $n$, $x_n \preceq y_n$, then $x \preceq y$,*

*c) If $x_n, z_n \to x$ and for all $n$, $x_n \preceq y_n \preceq z_n$, then $y_n \to x$,*

*d) If $x_n + y_n \to x$ and $y_n \to 0$, then $x_n \to x$.*

**Proof.** Suppose that $x_n \to x$ and $y_n \to y$. Then for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$x_n \preceq x + a_1,\ x \preceq x_n + a_2,\ \|a_i\|_q \leq \frac{\varepsilon}{2K},$$

$$y_n \preceq y + b_1,\ y \preceq y_n + b_2,\ \|b_i\|_q \leq \frac{\varepsilon}{2K}.$$

Hence,

$$x_n + y_n \preceq x + y + a_1 + b_1,\ x + y \preceq x_n + y_n + a_2 + b_2,$$

and

$$\|a_i + b_i\|_q \leq K(\|a_i\|_q + \|b_i\|_q) \leq \varepsilon.$$

Consequently, $h_q(x_n + y_n, x + y) < \varepsilon$ and operation of sum is continuous. Now, we prove that the operation of real scaler multiplication is continuous. Suppose $h_q(x_n, x) \to 0$. If $\alpha \neq 0$, then for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$x_n \preceq x + a_1,\ x \preceq x_n + a_2,\ \|a_i\|_q \leq \frac{\varepsilon}{|\alpha|}.$$

Concequently,

$$\alpha.x_n \preceq \alpha.x + \alpha.a_1,\ \alpha x \preceq \alpha.x_n + \alpha.a_2,\ \|\alpha.a_i\|_q \leq \varepsilon,$$

and so $h_q(\alpha.x_n, \alpha.x) \to 0$.

The proof of conditions $b, c$ and $d$ of lemma are analogous. □

**Lemma 3.5.** *The $b$-metric $h_q$ for all $x, y, z, v$ in the quasi-normed quasilinear space $X$ and $\alpha \in \mathbb{R}$ satisfies in the following conditions:*

*a)* $h_q(\alpha.x, \alpha.y) = |\alpha| h_q(x, y)$,

*b)* $h_q(x + y, z + v) \leq K(h_q(x, z) + h_q(y, v))$,

*c)* $h_q(x, \theta) = \|x\|_q$.

**Lemma 3.6.** *Let $X$ be a quasi-normed quasilinear space. If sequence $\{x_n\} \subset X$ converges to $x \in X$, then $\|x\|_q \leq K \lim_{n\to\infty} \|x_n\|_q$.*

**Definition 3.7.** *A quasi-normed quasilinear space $X$ is called an $\Omega$-space if there exists an element $B_X \neq \theta$ such that if $\|x\|_q \leq \|B_X\|_q$, then $x \preceq B_X$.*

It can be assumed that $\|B_X\|_q = 1$ (see [1]). If $X$ is an $\Omega$-space, then the mapping $B_X : [0,\infty) \to X$ defined by $B_X(t) = t.B_X$ satisfies the conditions:

$$\begin{aligned} x &\preceq B_X(\|x\|_q); \\ t \le s &\Longrightarrow B_X(t) \preceq B_X(s); \\ B_X(t+s) &\preceq B_X(t) + B_X(s). \end{aligned}$$

For example, the space $\Omega(\mathbb{R})$ is an $\Omega$-space with the element $B_\Omega = [-1, 1]$.

## 4 Quasilinear $q$-operators

Throughout this section, assume that $X$ with $\|.\|_X$ and $Y$ with $\|.\|_Y$ be quasi-normed quasilinear spaces with the indexes $K_X$ and $K_Y$, respectively.

**Definition 4.1.** *A mapping $\Lambda : X \to Y$ is called a quasilinear $q$-operator if for all $x, x_1, x_2 \in X$ satisfies the following conditions:*

1. $\Lambda(\alpha.x) = \alpha.\Lambda(x)$, *for any* $\alpha \in \mathbb{R}$;
2. $\Lambda(x_1 + x_2) \preceq K(\Lambda(x_1) + \Lambda(x_2))$, $K = \max\{K_X, K_Y\}$;
3. *If* $x_1 \preceq x_2$, *then* $\Lambda(x_1) \preceq \Lambda(x_2)$.

A quasilinear $q$-operator $\Lambda : X \to Y$ is said to be bounded if there exists a real number $M > 0$ such that $\|\Lambda(x)\|_Y \le M\|x\|_X$ for all $x \in X$.

**Lemma 4.2.** *A quasilinear $q$-operator $\Lambda : X \to Y$ is bounded if and only if it is continuous at $\theta \in X$.*

**Proof.** If $q$-operator $\Lambda$ is bounded, clearly it is continuous at $\theta \in X$. Suppose that $q$-operator $\Lambda$ is continuous at $\theta \in X$. Then, for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x \in X$ condition $\|x\|_X \le \delta$ implies $\|\Lambda(x)\|_Y \le \varepsilon$. Consequently, for all $x \ne \theta$,

$$\|\Lambda(\frac{\delta x}{\|x\|_X})\|_Y \le \varepsilon.$$

Therefore, $\|\Lambda(x)\|_Y \leq \frac{\varepsilon}{\delta}\|x\|_X$. □

The space of all bounded quasilinear $q$-operators denote by $\Lambda_q(X,Y)$. The partial order on $\Lambda_q(X,Y)$ is given by $\Lambda_1 \preceq \Lambda_2$ if for all $x \in X$, $\Lambda_1(x) \preceq \Lambda_2(x)$. The sum operation on $\Lambda_q(X,Y)$ is defined by the equality $(\Lambda_1+\Lambda_2)(x) = \Lambda_1(x)+\Lambda_2(x)$ and the scaler multiplication is defined by $(\alpha.\Lambda)(x) = \alpha.(\Lambda(x))$. Then $\Lambda_q(X,Y)$ is a quasi-normed quasilinear space with norm is defined by

$$\|\Lambda\|_\Lambda = \sup_{x\neq 0} \frac{\|\Lambda(x)\|_Y}{\|x\|_X}.$$

**Lemma 4.3.** *Suppose that the sequence $\{\Lambda_n\} \subset \Lambda_q(X,Y)$ converges at each point $x \in X$. Then, the operator $\Lambda(x) = \lim_{n\to\infty} \Lambda_n(x)$ is quasilinear $q$-operator.*

**Lemma 4.4.** *Suppose $Y$ is an $\Omega$-space. Then, the operator $\Lambda(x) = B_Y(\|x\|_X) \in \Lambda_q(X,Y)$.*

**Proof.** We prove $\Lambda(x_1+x_2) \preceq K(\Lambda(x_1)+\Lambda(x_2))$:

$$\begin{aligned}
\Lambda(x_1+x_2) &= B_Y(\|x_1+x_2\|_X) \\
&\preceq B_Y(K_X(\|x_1\|_X + \|x_2\|_X)) \\
&\preceq K_X B_Y(\|x_1\|_X) + K_X B_Y(\|x_2\|_X) \\
&\preceq K(\Lambda(x_1)+\Lambda(x_2)).
\end{aligned}$$

See [1] for the rest of the prove. □

**Theorem 4.5.** *If $Y$ is an $\Omega$-space, then $\Lambda_q(X,Y)$ is $\Omega$-space.*

**Proof.** We prove that $\Lambda_q(X,Y)$ is complete. For the rest of the prove, see [1]. Suppose $\{\Lambda_n\} \subset \Lambda_q(X,Y)$ is a Cauchy sequence. Then for all $\varepsilon > 0$,

$$\exists N \in \mathbb{N}\ \forall n,m \geq N,\ \exists \Lambda^{n,m}_{i,\varepsilon},\ \Lambda_n \preceq \Lambda_m + \Lambda^{n,m}_{1,\varepsilon},\ \Lambda_m \preceq \Lambda_n + \Lambda^{n,m}_{2,\varepsilon},\ \|\Lambda^{n,m}_{i,\varepsilon}\|_\Lambda \leq \varepsilon.$$

Consequently, $h_Y(\Lambda_n(x),\Lambda_m(x)) \leq \varepsilon\|x\|_X$ and so $\{\Lambda_n(x)\} \subset Y$ is Cauchy and there exists an element $\Lambda(x) \in Y$ such that $\Lambda(x) = \lim_{n\to\infty} \Lambda_n(x)$. By

Lemma 4.3, $\Lambda : X \to Y$ is a quasilinear $q$-operator. Now, we show it is bounded.

$$\begin{aligned} \|\Lambda_n(x)\|_Y &\leq K_Y(\|\Lambda_m(x)\|_Y + \|\Lambda_{1,\varepsilon}^{n,m}(x)\|_Y) \\ &\leq K_Y(\|\Lambda_m(x)\|_Y + \varepsilon)\|x\|_X. \end{aligned}$$

By taking the limit as $n \to \infty$,

$$\|\Lambda(x)\|_Y \leq K_Y^2(\|\Lambda_m(x)\|_Y + \varepsilon)\|x\|_X.$$

Hence, $\Lambda$ is bounded.

$$h_Y(\Lambda_n(x), \Lambda(x)) \leq K_Y(h_Y(\Lambda_n(x), \Lambda_m(x)) + h_Y(\Lambda_m(x), \Lambda(x)),$$

Take the limit as $m \to \infty$, we get $h_Y(\Lambda_n(x), \Lambda(x)) \leq \varepsilon\|x\|_X$. By Lemma 4.4 operator $\Lambda_\varepsilon(x) := \varepsilon B_Y(\|x\|_Y)$ is a quasilinear $q$-operator. □

## 5 Stability of Cauchy Functional Equation

Throughout this section, assume that $X$ with $\| \, . \, \|_X$ is a quasi-normed linear space and $Y$ with $\| \, . \, \|_Y$ is a $q$-$p$-Banach space with zero element $\theta$. In this section we prove the stability of Cauchy functional equation $f(x+y) = f(x) + f(y)$ in the quasi-normed spaces.

**Theorem 5.1.** *Let $f : X \to Y$ with $f(0) = \theta$ and symmetric function $\varphi : X \times X \to [0, \infty)$ for all $x, y \in X$ satisfy the following conditions:*

$$\|f(x+y) - f(x) - f(y)\|_Y \leq \varphi(x, y), \tag{1}$$

$$\lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y)}{2^n} = 0, \tag{2}$$

$$\sum_{l=0}^{\infty} \frac{1}{2^{pl}} \varphi^p(2^l x, 2^l x) < \infty. \tag{3}$$

*Then, there is a unique additive mapping $g : X \to Y$ such that,*

$$g(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n},$$

$$\|f(x)-g(x))\|_Y \leq \frac{1}{2}(\sum_{l=0}^{\infty}\frac{1}{2^{pl}}\varphi^p(2^lx,2^lx))^{\frac{1}{p}},$$

*for all* $x \in X$.

**Proof.** Letting $y = x$ in condition (1) then for all $x \in X$,

$$\|f(2x)-f(x)-f(x)\|_Y \leq \varphi(x,x).$$

Since $2f(x) \preceq f(x)+f(x)$, we have

$$\|f(2x)-2f(x)\|_Y \leq \|f(2x)-f(x)-f(x)\|_Y.$$

Therefore,

$$\|f(2x)-2f(x)\|_Y \leq \varphi(x,x).$$

Replacing $x$ by $2^nx$ in the above inequality and dividing by $2^{n+1}$, we get

$$\|\frac{f(2^{n+1}x)}{2^{n+1}}-\frac{f(2^nx)}{2^n}\|_Y \leq \frac{1}{2^{n+1}}\varphi(2^nx,2^nx).$$

for all $x \in X$ and $n,m \in \mathbb{N}$ with $n \geq m$,

$$\begin{aligned}\|\frac{f(2^{n+1}x)}{2^{n+1}}-\frac{f(2^mx)}{2^m}\|_Y^p &\leq \sum_{l=m}^{n}\|\frac{f(2^{l+1}x)}{2^{l+1}}-\frac{f(2^lx)}{2^l}\|_Y^p \\ &\leq \frac{1}{2^p}\sum_{l=m}^{n}\frac{1}{2^{pl}}\varphi^p(2^lx,2^lx). \qquad (4)\end{aligned}$$

From the above inequality and condition (3) of the theorem we conclude that $\{\frac{f(2^nx)}{2^n}\}$ is a Cauchy sequence and since $Y$ is a $q$-$p$-Banach space, therefore this sequence converges in $Y$. We define the mapping $g : X \to Y$ by

$$g(x) = \lim_{n\to\infty}\frac{f(2^nx)}{2^n},$$

for all $x \in X$. Letting $m = 0$ and by limiting $n \to \infty$ in inequality (4), we get

$$\|f(x)-g(x)\|_Y^p \leq \frac{1}{2^p}\sum_{l=0}^{\infty}\frac{1}{2^{pl}}\varphi^p(2^lx,2^lx).$$

According to the condition (1) and (2) of theorem and by Lemma 3.6,

$$\begin{aligned} g(x+y)-g(x)-g(y)\|_Y &\leq K\lim_{n\to\infty}\frac{\|f(2^n(x+y))-f(2^nx)-f(2^ny)\|_Y}{2^n} \\ &\leq K\lim_{n\to\infty}\frac{\varphi(2^nx,2^ny)}{2^n}=0. \end{aligned}$$

Then, according to

$$h_Y(g(x+y),g(x)+g(y))\leq\|g(x+y)-g(x)-g(y)\|=0,$$

we conclude $g$ is an additive mapping.

Now we show that $g$ is unique. Assume that $T$ is another additive mapping such that

$$\|f(x)-T(x))\|_Y\leq\frac{1}{2}(\sum_{l=0}^{\infty}\frac{1}{2^{pl}}\varphi^p(2^lx,2^lx))^{\frac{1}{p}},$$

for all $x\in X$. Therefore, according to Lemma 3.5, for all $n\in\mathbb{N}$,

$$\begin{aligned} h_Y(\frac{g(2^nx)}{2^n},\frac{T(2^nx)}{2^n}) &\leq K(h_Y(\frac{g(2^nx)}{2^n},\frac{f(2^nx)}{2^n})+h_Y(\frac{f(2^nx)}{2^n},\frac{T(2^nx)}{2^n})) \\ &\leq K(\sum_{l=0}^{\infty}\frac{1}{2^{pl}}(\frac{\varphi(2^l2^nx,2^l2^nx)}{2^n})^p)^{\frac{1}{p}}. \end{aligned}$$

By taking the limit, it follows that $T(x)=g(x)$. $\square$

**Corollary 5.2.** *Let $r<1$ and $\varepsilon$ be nonnegative real numbers and $f: X\to Y$ be a mapping with $f(0)=\theta$ and for all $x,y\in X$,*

$$\|f(x+y)-f(x)-f(y)\|_Y\leq\varepsilon(\|x\|^r+\|y\|^r).$$

*Then, there is a unique additive mapping $g:X\to Y$ such that,*

$$g(x)=\lim_{n\to\infty}\frac{f(2^nx)}{2^n},$$

$$\|f(x)-g(x))\|_Y\leq\frac{\varepsilon}{(1-2^{(r-1)p})^{\frac{1}{p}}}\|x\|^r,$$

*for all $x\in X$.*

**Proof.** By taking $\varphi(x,y)=\varepsilon(\|x\|^r+\|y\|^r)$ in previous Theorem, we get the desired result. $\square$

**Theorem 5.3.** *Let function $f: X\to Y$ with $f(0)=\theta$ and symmetric function $\varphi: X\times X\to[0,\infty)$ for all $x,y\in X$ satisfy the following conditions:*

$$\|f(x+y)-f(x)-f(y)\|_Y\leq\varphi(x,y),$$

$$\lim_{n\to\infty}2^n\varphi(\frac{x}{2^n},\frac{y}{2^n})=0, \tag{5}$$

$$\sum_{l=0}^{\infty}2^{pl}(\varphi(\frac{x}{2^{l+1}},\frac{y}{2^{l+1}}))^p<\infty.$$

*Then, there is a unique additive mapping $g: X\to Y$ such that,*

$$g(x)=\lim_{n\to\infty}2^n f(\frac{x}{2^n}),$$

$$\|f(x)-g(x))\|_Y\leq(\sum_{l=0}^{\infty}2^{pl}\varphi^p(\frac{x}{2^l},\frac{y}{2^l}))^{\frac{1}{p}},$$

*for all $x\in X$.*

**Proof.** Letting $y=x$ in condition (5) then for all $x\in X$,

$$\|f(2x)-f(x)-f(x)\|_Y\leq\varphi(x,x).$$

Since $2f(x)\preceq f(x)+f(x)$, we have

$$\|f(2x)-2f(x)\|_Y\leq\|f(2x)-f(x)-f(x)\|_Y.$$

Therefore,

$$\|f(2x)-2f(x)\|_Y\leq\varphi(x,x).$$

Replacing $x$ by $\frac{x}{2^{n+1}}$ in the above inequality and multipling by $2^n$ we get

$$\|2^n f(\frac{x}{2^n})-2^{n+1}f(\frac{x}{2^{n+1}})\|\leq 2^n\varphi(\frac{x}{2^{n+1}},\frac{y}{2^{n+1}}).$$

for all $x \in X$ and $n, m \in \mathbb{N}$ with $n \geq m$,

$$\|2^{n+1} f(\frac{x}{2^{n+1}}) - 2^m f(\frac{x}{2^m})\|_Y^p \leq \sum_{l=0}^{\infty} 2^{pl} (\varphi(\frac{x}{2^{l+1}}, \frac{y}{2^{l+1}}))^p.$$

We conclude $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence in $q$-$p$-Banach space $Y$ therefore converges in $Y$. We define the mapping $g : X \to Y$ by

$$g(x) = \lim_{n\to\infty} 2^n f(\frac{x}{2^n}),$$

for all $x \in X$. The rest of the proof is similar to the proof of Theorem 5.1. $\square$

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