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Some Achievements on Two Variable σ-Derivations

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Abstract. Let $\mathcal{B}$ and $\mathcal{A}$ be two Banach algebras and $\mathcal{M}$ be a Banach $\mathcal{B}$-bimodule. Suppose that $\sigma : \mathcal{A} \to \mathcal{B}$ is a linear mapping. A bilinear map $\phi : \mathcal{A}\times\mathcal{A} \to \mathcal{M}$ is called a two variable σ-derivation whenever $\phi(ab, c) = \phi(a, c)\sigma(b) + \sigma(a)\phi(b, c)$ and $\phi(a, bc) = \phi(a, b)\sigma(c) + \sigma(b)\phi(a, c)$ for all $a, b, c \in \mathcal{A}$. In this paper, we prove that if $\mathcal{A}$ and $\mathcal{B}$ are unital and $\phi : \mathcal{A} \times \mathcal{A} \to \mathcal{B}$ is a two variable σ-derivation such that $\phi(\mathbf{1}, a_0) = \mathbf{1}$ for some $a_0 \in \mathcal{A}$ then ϕ is symmetric, i.e. $\phi(a, b) = \phi(b, a)$ and there exists a unital homomorphism $\theta : \mathcal{A} \to \mathcal{B}$ such that $\phi(a, b) = \theta(ab)\theta(a_0)^{-1}$.

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1. Introduction

Throughout this paper, $\mathcal{A}$ will represent an algebra. If $\mathcal{A}$ is unital then $\mathbf{1}$, will show its unit element. We shall write $[a, b]$ for $ab - ba$. An algebra $\mathcal{A}$ is said to be a domain, if whenever $ab = 0$, with $a, b \in \mathcal{A}$, then $a = 0$ or $b = 0$. A mapping $D : \mathcal{A} \to \mathcal{A}$ is called a *derivation* if $D(ab) = D(a)b + aD(b)$ holds for all $a, b \in \mathcal{A}$. Let $\sigma : \mathcal{A} \to \mathcal{A}$ be a linear mapping. A linear mapping $d : \mathcal{A} \to \mathcal{A}$ is called a σ-derivation if $d(ab) = d(a)\sigma(b) + \sigma(a)d(b)$ holds for all $a, b \in \mathcal{A}$. Clearly, if $\sigma = id$,

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if $\sigma = id$, the identity mapping on $\mathcal{A}$, then d is an ordinary derivation. As another example, every homomorphism $\theta : \mathcal{A} \to \mathcal{A}$ is a $\frac{\theta}{2}$-derivation. Hence, the theory of σ-derivations combines the theory of derivations and homomorphisms (for more details see [9, 11-13]. The notion of generalized derivation was introduced by Hvala [10]. An additive mapping $f : \mathcal{A} \to \mathcal{A}$ is called a *generalized derivation* if there exists a derivation $D : \mathcal{A} \to \mathcal{A}$ such that $f(ab) = f(a)b + aD(b)$ holds for all $a, b \in \mathcal{A}$. For convenience, such a derivation f is said to be a D-derivation. By getting idea from this definition, Hosseini et al [7], [8] defined a generalized σ-derivation as follows: A linear mapping $\delta : \mathcal{A} \to \mathcal{A}$ is called a *generalized σ-derivation* if there exists a σ-derivation $d : \mathcal{A} \to \mathcal{A}$ such that $\delta(ab) = \delta(a)\sigma(b) + \sigma(a)d(b)$ holds for all $a, b \in \mathcal{A}$. For convenience, we say that such a generalized σ-derivation δ is a (σ, d)-derivation. An additive mapping H from $\mathcal{A}$ into itself is called a *left* (*right*) *centralizer* if $H(ab) = H(a)b$ ($H(ab) = aH(b)$) for all $a, b \in \mathcal{A}$ (see [1] and the references therein). A *centralizer* of $\mathcal{A}$ is an additive mapping which is both a left and a right centralizer. Suppose that $f : \mathcal{A} \to \mathcal{A}$ is a D-derivation. By putting $H = f - D$, we get $H(ab) = f(a)b + aD(b) - D(a)b - aD(b) = H(a)b$ for all $a, b \in \mathcal{A}$. It means that H is a left centralizer. Similarly, if $\delta : \mathcal{A} \to \mathcal{A}$ is a (σ, d)-derivation then $T(ab) = T(a)\sigma(b)$ for all $a, b \in \mathcal{A}$, where $T = \delta - d$. The linear mapping T is called a σ-algebraic map (the reader is referred to [7]). M. Hassani and A. Hosseini [6] defined a two variable $(\sigma, \tau) - derivation$ as follows:

Let $\mathcal{A}$ be a Banach algebra and $\mathcal{M}$ be a Banach $\mathcal{A} - bimodule$. Suppose that $\sigma, \tau : \mathcal{A} \to \mathcal{A}$ are two linear mappings. A bilinear mapping $\phi : \mathcal{A} \times \mathcal{A} \to \mathcal{M}$ is called a left two variable $\sigma - derivation$ if $\phi(ab, c) = \phi(a, c)\sigma(b) + \sigma(a)\phi(b, c)$ for all $a, b, c \in \mathcal{A}$. Similarly, ϕ is called a right two variable $\tau - derivation$ if $\phi(a, bc) = \phi(a, b)\tau(c) + \tau(b)\phi(a, c)$ for all $a, b, c \in \mathcal{A}$. A bilinear mapping $\phi : \mathcal{A} \times \mathcal{A} \to \mathcal{M}$ is said to be a two variable $(\sigma, \tau) - derivation$ if it is a left two variable $\sigma - derivation$ as well as a right two variable $\tau - derivation$. A two variable $(\sigma, \sigma) - derivation$ is called a two variable $\sigma - derivation$. If $\sigma = \tau = id$, the identity mapping on $\mathcal{A}$, then the bilinear map $\phi : \mathcal{A} \times \mathcal{A} \to \mathcal{M}$ is called a two variable derivation. For example, a bilinear map $\phi : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ defined by

$\phi(a,b) = [a,b] = ab - ba$ is a two variable derivation. Furthermore, they showed that if $\mathcal{A}$ is a commutative Banach algebra and $\phi : \mathcal{A}\times\mathcal{A} \to \mathcal{A}$ is a continuous two variable derivation then $\phi(e^{za}, e^{\omega b}) = z\omega\ e^{za+\omega b}\phi(a,b)$ for all $a, b \in \mathcal{A}$ and $z \in \mathbb{C}$. Also the following formula has been proved: $\phi(a^n, b^m) = \sum_{k=0}^{n-1}\sum_{j=0}^{m-1} a^k b^j \phi(a,b) b^{m-1-j} a^{n-1-k}$ for every $a, b \in \mathcal{A}$. Thus if $\phi(a,b) = [a,b]$, then
$[a^n, b^m] = \sum_{k=0}^{n-1}\sum_{j=0}^{m-1} a^k b^j [a,b] b^{m-1-j} a^{n-1-k}$, and
$[e^{za}, e^{\omega b}] = z\omega \int_0^1\int_0^1 e^{s\omega b} e^{tza}[a,b] e^{(1-t)za} e^{(1-s)\omega b} dt\ ds$ (for more details see [6]). Moreover, as an application of a two variable σ-derivations, under certain conditions, it has been proved that a simple Banach algebra is a field [[6], Theorem 2.8]. In this research the following main result is proved:
Suppose that $\mathcal{A}$ and $\mathcal{B}$ are unital and $\phi : \mathcal{A}\times\mathcal{A} \to \mathcal{B}$ is a two variable σ-derivation. If $\phi(\mathbf{1}, a_0) = \mathbf{1}$ for some $a_0 \in \mathcal{A}$, then ϕ is symmetric and there exists a unital homomorphism $\theta : \mathcal{A} \to \mathcal{B}$ such that $\phi(a,b) = \theta(ab)(\theta(a_0))^{-1}$ for all $a, b \in \mathcal{A}$. Moreover, $\frac{\mathcal{A}}{ker(\theta)}$ is a commutative algebra.

2. Main Results

Throughout this paper, $\mathcal{A}$ and $\mathcal{B}$ denote two Banach algebras. Moreover, $\mathcal{M}$ denotes a Banach $\mathcal{B}$-bimodule. $\mathcal{M}$ is called *symmetric* if $bx = xb$ for all $b \in \mathcal{B}, x \in \mathcal{M}$. Furthermore, if an algebra is unital then $\mathbf{1}$ will show its unit element.
Suppose $\sigma : \mathcal{A} \to \mathcal{B}$ is a linear operator. We know that a linear operator $d : \mathcal{A} \to \mathcal{M}$ is called a σ-derivation if $d(ab) = d(a)\sigma(b) + \sigma(a)d(b)$ for all $a,b \in \mathcal{A}$. It is clear that if $\mathcal{A}$ is a subalgebra of $\mathcal{B}$ and $\sigma = id$, the inclusion map, then a σ-derivation is an ordinary derivation.

Let $\sigma : \mathcal{A} \to \mathcal{B}$ be a linear operator. A bilinear map $\phi : \mathcal{A}\times\mathcal{A} \to \mathcal{M}$ is called a left two variable σ-derivation whenever,

$$\phi(ab, c) = \phi(a, c)\sigma(b) + \sigma(a)\phi(b, c), \quad (a, b, c \in \mathcal{A}).$$

ϕ is called a right two variable σ-derivation if

$$\phi(a, bc) = \phi(a, b)\sigma(c) + \sigma(b)\phi(a, c), \quad (a, b, c \in \mathcal{A}).$$

If $\mathcal{A}$ is a subalgebra of $\mathcal{B}$ and $\sigma : \mathcal{A} \to \mathcal{B}$ is the inclusion map, then ϕ is called left(right) two variable derivation. Moreover a bilinear map $\phi : \mathcal{A} \times \mathcal{A} \to \mathcal{M}$ is called two variable σ-derivation if it is both a left and a right two variable σ-derivation (see [6]).

For example, suppose that $\mathcal{A}$ is a subalgebra of $\mathcal{B}$ and $\sigma : \mathcal{A} \to \mathcal{B}$ is a homomorphism. Then $\phi : \mathcal{A} \times \mathcal{A} \to \mathcal{B}$ defined by $\phi(a, b) = \sigma(a)x - x\sigma(a) + \sigma(ab - ba)$ $(a, b \in \mathcal{A},\ x \in \mathcal{B})$ is a left two variable σ-derivation. Furthermore, if $\phi(a, b) = [\sigma(a), \sigma(b)] = \sigma(a)\sigma(b) - \sigma(b)\sigma(a)$, then ϕ is a two variable $\sigma - derivation$.

Let $\phi : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ be a continuous two variable derivation. For a, b in $\mathcal{A}$, we define a function $f_{a,b} : \mathbb{R}^2 \to \mathcal{A}$ by the following form:

$$f_{a,b}(r, s) = \phi(e^{ra}, e^{sb}).$$

We have

$$\begin{aligned}
\frac{\partial f_{a,b}}{\partial r}(r, s) &= \lim_{h\to 0} \frac{f_{a,b}(r + h, s) - f_{a,b}(r, s)}{h} \\
&= \lim_{h\to 0} \frac{\phi(e^{(r+h)a}, e^{sb}) - \phi(e^{ra}, e^{sb})}{h} \\
&= \lim_{h\to 0} \frac{\phi(e^{ra}e^{ha} - e^{ra}, e^{sb})}{h} \\
&= \lim_{h\to 0} \frac{\phi(e^{ra}(e^{ha} - \mathbf{1}), e^{sb})}{h} \\
&= \phi(e^{ra} \lim_{h\to 0} \frac{e^{ha} - \mathbf{1}}{h}, e^{sb}) \\
&= \phi(e^{ra} \lim_{h\to 0} ae^{ha}, e^{sb}) \\
&= \phi(e^{ra}a, e^{sb}).
\end{aligned}$$

Similarly,

$$\frac{\partial f_{a,b}}{\partial s}(r, s) = \phi(e^{ra}, e^{sb}b).$$

If $\mathcal{A}$ is commutative, then we have $f_{a,b}(r,s) = rs\ e^{ra+sb}\phi(a,b)$. Hence

$$\frac{\partial f_{a,b}}{\partial r}(r,s) = se^{ra+sb}\phi(a,b) + rsae^{ra+sb}\phi(a,b)$$
$$\frac{\partial f_{a,b}}{\partial s}(r,s) = re^{ra+sb}\phi(a,b) + rsbe^{ra+sb}\phi(a,b).$$

It is clear that $\frac{\partial f_{a,b}}{\partial r}$ and $\frac{\partial f_{a,b}}{\partial s}$ are continuous functions on $\mathbb{R}^2$ and so $f_{a,b}(r,s)$ is differentiable. Thus there exist two functions $\varepsilon_1, \varepsilon_2 : \mathbb{R}^2 \to \mathcal{A}$ such that

$$\lim_{(h,k)\to(0,0)} \varepsilon_i(h,k) = 0 \ \ i = 1,2$$

and furthermore,

$$f_{a,b}(r+h, s+k) - f_{a,b}(r,s) = h\frac{\partial f_{a,b}}{\partial r}(r,s) + k\frac{\partial f_{a,b}}{\partial s}(r,s) + h\varepsilon_1(h,k) + k\varepsilon_2(h,k).$$

Hence

$$\phi(e^{(r+h)a}, e^{(s+k)b}) - \phi(e^{ra}, e^{sb}) = h\phi(ae^{ra}, e^{sb}) + k\phi(e^{ra}, be^{sb}) + h\varepsilon_1(h,k) + k\varepsilon_2(h,k).$$

Putting $r = s = 0$ in the previous equation and using the fact that $\phi(\mathbf{1}, a) = \phi(a, \mathbf{1}) = 0$ for all $a \in \mathcal{A}$, we get $\phi(e^{ha}, e^{kb}) = h\varepsilon_1(h,k) + k\varepsilon_2(h,k)$. If ε_1 and ε_2 have partial derivatives with respect to h, k, then $\frac{\partial}{\partial h}\phi(e^{ha}, e^{kb}) = \frac{\partial}{\partial h}[h\varepsilon_1(h,k) + k\varepsilon_2(h,k)]$ and we have

$$\phi(ae^{ha}, e^{kb}) = \varepsilon_1(h,k) + h\frac{\partial \varepsilon_1}{\partial h}(h,k) + k\frac{\partial \varepsilon_2}{\partial h}(h,k).$$

Thus

$$\frac{\partial}{\partial k}\phi(ae^{ha}, e^{kb}) = \frac{\partial}{\partial k}[\varepsilon_1(h,k) + h\frac{\partial \varepsilon_1}{\partial h}(h,k) + k\frac{\partial \varepsilon_2}{\partial h}(h,k)].$$

It implies that

$$\phi(ae^{ha}, be^{kb}) = \frac{\partial \varepsilon_1}{\partial k}(h,k) + h\frac{\partial^2 \varepsilon_1}{\partial k \partial h}(h,k) + \frac{\partial \varepsilon_2}{\partial h}(h,k) + k\frac{\partial^2 \varepsilon_2}{\partial k \partial h}(h,k).$$

Putting $h = k = 0$ in the previous relation, we arrive at

$$\phi(a,b) = \frac{\partial \varepsilon_1}{\partial k}(0,0) + \frac{\partial \varepsilon_2}{\partial h}(0,0).$$

It is a characterization of a continuous two variable derivation that we conjecture to be applied in differential calculus.

Definition 2.1. *Suppose that $\sigma : \mathcal{A} \to \mathcal{B}$ is a linear operator. If for each $b \in \mathcal{A}$ there exists an element $x_b \in \mathcal{M}$ such that $x_b(\sigma(ac) - \sigma(a)\sigma(c)) = (\sigma(ac) - \sigma(a)\sigma(c))x_b$ for all $a, c \in \mathcal{A}$, then the bilinear mapping $\phi : \mathcal{A} \times \mathcal{A} \to \mathcal{M}$ defined by $\phi(a,b) = \sigma(a)x_b - x_b\sigma(a)$ is called an inner left two variable σ-derivation. Similarly, if for each $a \in \mathcal{A}$ there exists an element $x_a \in \mathcal{M}$ such that $x_a(\sigma(bc) - \sigma(b)\sigma(c)) = (\sigma(bc) - \sigma(b)\sigma(c))x_a$, then the bilinear mapping $\phi(a,b) = \sigma(b)x_a - x_a\sigma(b)$ is called an inner right two variable σ-derivation. ϕ is called an inner two variable σ-derivation if it is an inner left two variable σ-derivation as well as an inner right two variable σ-derivation.*

Remark 2.2. *Suppose that $\phi : \mathcal{A} \times \mathcal{A} \to \mathcal{M}$ is a left(right) two variable σ-derivation and $\{\lambda_i\}_{i\in I}$ is a net in $\mathcal{A}$ such that $\{\phi(a,\lambda_i)\}_{i\in I}(\{\phi(\lambda_i,a)\}_{i\in I})$ is a convergent net for all $a \in \mathcal{A}$. We define $d : \mathcal{A} \to \mathcal{M}$ by $d(a) = \lim_{i\in I}\phi(a,\lambda_i)$*
$(d(a) = \lim_{i\in I}\phi(\lambda_i,a))$. Then d *is a σ-derivation.*
The following proposition and remark have been proved in [6].

Proposition 2.3. *Suppose that $\phi : \mathcal{A} \times \mathcal{A} \to \mathcal{M}$ is a continuous two variable σ-derivation and $\mathcal{A}$ has an approximate identity $\{e_i\}_{i\in I}$ such that $\{\sigma(e_i)\}_{i\in I}$ is a convergent net in $\mathcal{B}$. Then*

$$\lim_{i\in I}\phi(e_i,b)\sigma(a) = \lim_{i\in I}\phi(a,e_i)\sigma(b)$$
$$\lim_{i\in I}\sigma(a)\phi(e_i,b) = \lim_{i\in I}\sigma(b)\phi(a,e_i)$$

for all $a, b \in \mathcal{A}$.

Remark 2.4. *If $\mathcal{A}$ is unital and $\phi : \mathcal{A} \times \mathcal{A} \to \mathcal{M}$ is a two variable*

σ-derivation then

$$\sigma(a)\phi(\mathbf{1}, b) = \sigma(b)\phi(a, \mathbf{1}),$$
$$\phi(\mathbf{1}, b)\sigma(a) = \phi(a, \mathbf{1})\sigma(b),$$

for all $a, b \in \mathcal{A}$. If $\phi(\mathbf{1}, \mathbf{1}) = 0$ and $\phi(a, b) = \phi(\mathbf{1}, ab)$, then ϕ is identically zero. Because $\phi(\mathbf{1}, a) = \phi(\mathbf{1}, \mathbf{1}a) = \phi(\mathbf{1}, \mathbf{1})\sigma(a) + \sigma(\mathbf{1})\phi(\mathbf{1}, a) = \phi(\mathbf{1}, \mathbf{1})\sigma(a) + \sigma(a)\phi(\mathbf{1}, \mathbf{1}) = 0 + 0 = 0$, for all $a \in \mathcal{A}$.

Proposition 2.5. *Suppose that $\phi : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is a continuous two variable derivation. If $\mathcal{A}$ has a bounded approximate identity $\{e_i\}_{i\in I}$, then $\lim_{i\in I} \phi(a, e_i) = \lim_{i\in I} \phi(e_i, a) = 0$ for all $a \in \mathcal{A}$.*

Proof. It is clear that

$$\lim_{i\in I} \phi(e_i, b)a = \lim_{i\in I} a\phi(e_i, b) = \lim_{i\in I} b\phi(a, e_i) = \lim \phi(a, e_i)b = 0$$

for all $a, b \in \mathcal{A}$. Let a be an arbitrary element of $\mathcal{A}$. Since $\mathcal{A}$ has a bounded approximate identity, there exist two elements b and c in $\mathcal{A}$ such that $a = cb$. Then

$$\begin{aligned}
\lim_{i\in I} \phi(a, e_i) &= \lim_{i\in I} \phi(cb, e_i) \\
&= \lim_{i\in I}(\phi(c, e_i)b + c\phi(b, e_i)) \\
&= \lim_{i\in I} \phi(c, e_i)b + \lim_{i\in I} c\phi(b, e_i) \\
&= 0 + 0 = 0
\end{aligned}$$

Similarly, we can get $\lim_{i\in I} \phi(e_i, a) = 0$ for all $a \in \mathcal{A}$. $\square$

Remark 2.6. *Suppose that $\phi : \mathcal{A} \times \mathcal{A} \to \mathcal{M}$ is a two variable derivation such that $\phi(ab, c) = \phi(a, bc)$ for all $a, b, c \in \mathcal{A}$. If $\mathcal{A}$ is unital or ϕ is continuous and $\mathcal{A}$ has a bounded approximate identity, then ϕ is identically zero.*

Theorem 2.7. *Suppose that $\phi : \mathcal{A} \times \mathcal{A} \to \mathcal{M}$ is a continuous two variable σ-derivation and $\mathcal{A}$ has a bounded approximate identity $\{e_i\}_{i\in I}$ such that $\{\sigma(e_i)\}_{i\in I}$ is a convergent net in $\mathcal{B}$. Then there is $a_0 \in \mathcal{B}$*

such that the linear operator $d : \mathcal{A} \to \mathcal{M}$ defined by $d(c) = d(ab) = \phi(a,b) - a_0\phi(a,b) + \phi(b,a) - \phi(b,a)a_0$ (for $c = ab$) is a σ-derivation.

Proof. Let $\lim_{i\in I} \sigma(e_i) = a_0$. Then we have

$$\begin{aligned}\phi(b,a) &= \lim_{i\in I} \phi(be_i, a)\\ &= \lim_{i\in I} \phi(b,a)\sigma(e_i) + \sigma(b)\phi(e_i,a)\\ &= \phi(b,a)a_0 + \lim_{i\in I} \sigma(b)\phi(e_i,a).\end{aligned}$$

So $\phi(b,a) - \phi(b,a)a_0 = \lim_{i\in I} \sigma(b)\phi(e_i,a)$. Similarly, we have $\phi(a,b) = \lim_{i\in I} \phi(e_i a, b) = \lim_{i\in I} \sigma(e_i)\phi(a,b) + \phi(e_i,b)\sigma(a) = a_0\phi(a,b) + \lim_{i\in I} \phi(e_i,b)\sigma(a)$. Hence $\phi(a,b) - a_0\phi(a,b) = \lim_{i\in I} \phi(e_i,b)\sigma(a)$. Consequently,

$$\begin{aligned}\phi(a,b) - a_0\phi(a,b) + \phi(b,a) - \phi(b,a)a_0 &= \lim_{i\in I}(\phi(e_i,b)\sigma(a) + \sigma(b)\phi(e_i,a))\\ &= \lim_{i\in I} \phi(e_i, ba)\end{aligned}$$

for all $a, b \in \mathcal{A}$. Since $\mathcal{A}$ has a bounded approximate identity, $\mathcal{A}^2 = \mathcal{A}$ and so the net $\{\phi(e_i,a)\}_{i\in I}$ is convergent for all $a \in \mathcal{A}^2 = \mathcal{A}$. Now, we define a linear operator $d : \mathcal{A} \to \mathcal{M}$ by $d(a) = \lim_{i\in I} \phi(e_i, a)$. It follows from Remark 2.2 that d is a σ-derivation. □

Suppose that $\mathcal{M}$ is a $\mathcal{B}$-bimodule. We say that $\mathcal{M}$ has no zero divisors if whenever $bx = 0$ or $xb = 0$, with $b \in \mathcal{B}, x \in \mathcal{M}$, then $b = 0$ or $x = 0$. If $\phi : \mathcal{A} \times \mathcal{A} \to \mathcal{M}$ is a two variable σ-derivation then we know that $\phi(a, \mathbf{1}) = \phi(\mathbf{1}, a)$ for all $a \in \mathcal{A}$ (see the proof of Theorem 2.9 in [6]). Suppose that $D, d : \mathcal{A} \to \mathcal{B}$ are two σ-derivations such that $d(a)\sigma(b) = \sigma(b)d(a)$ and $D(a)\sigma(b) = \sigma(b)D(a)$ for all $a, b \in \mathcal{A}$. Then the bilinear map $\phi : \mathcal{A} \times \mathcal{A} \to \mathcal{B}$ defined by $\phi(a,b) = d(a)D(b)$ is a two variable σ-derivation. Using the fact that $\phi(a, \mathbf{1}) = \phi(\mathbf{1}, a)$ for all $a \in \mathcal{A}$, we obtain $d(a)D(\mathbf{1}) = d(\mathbf{1})D(a)$.

Theorem 2.8. *Let $\mathcal{B}, \mathcal{A}$ be unital, and $d, D : \mathcal{A} \to \mathcal{B}$ be two non-zero σ-derivations such that $\mathbf{1} - \sigma(\mathbf{1})$ is invertible. If $d(\mathbf{1}) = D(\mathbf{1})$ then D=d.*

Proof. Suppose that $a \in \mathcal{A}$ is a non-zero arbitrary element. We have $d(a) = d(\mathbf{1})\sigma(a) + \sigma(\mathbf{1})d(a)$ and $D(a) = D(\mathbf{1})\sigma(a) + \sigma(\mathbf{1})D(a) = d(\mathbf{1})\sigma(a) +$

$\sigma(\mathbf{1})D(a)$. Hence $d(\mathbf{1})\sigma(a) = d(a) - \sigma(\mathbf{1})d(a) = D(a) - \sigma(\mathbf{1})D(a)$. It implies that $(\mathbf{1} - \sigma(\mathbf{1}))(d(a) - D(a)) = 0$. Using the fact that $\mathbf{1} - \sigma(\mathbf{1})$ is invertible, we arrive at $d(a) = D(a)$ for all $a \in \mathcal{A}$ and it means that $D = d$. □

Theorem 2.9. *Suppose that $\mathcal{A}$ is unital and $\phi : \mathcal{A} \times \mathcal{A} \to \mathcal{M}$ is a two variable σ-derivation such that $\phi(a, b) = \phi(\mathbf{1}, ab)$ for all $a, b \in \mathcal{A}$. Then ϕ is symmetric, i.e. $\phi(a, b) = \phi(b, a)$ $(a, b \in \mathcal{A})$.*

Proof. First of all, we show that $\phi(ab, c) = \phi(a, bc)$ if and only if $\phi(\mathbf{1}, ab) = \phi(a, b)$ for all $a, b, c \in \mathcal{A}$. If $\phi(ab, c) = \phi(a, bc)$ then clearly, $\phi(\mathbf{1}, ab) = \phi(a, b)$. Conversely, assume that $\phi(\mathbf{1}, ab) = \phi(a, b)$. Then $\phi(ab, c) = \phi(\mathbf{1}ab, c) = \phi(\mathbf{1}, abc)$ and $\phi(a, bc) = \phi(\mathbf{1}a, bc) = \phi(\mathbf{1}, abc)$. Hence $\phi(ab, c) = \phi(a, bc)$ for all $a, b, c \in \mathcal{A}$. By hypothesis and using Remark 2.4, for every $a, b \in \mathcal{A}$ we have

$$\begin{aligned}
\phi(a, b) &= \phi(\mathbf{1}, ab) \\
&= \phi(\mathbf{1}, a)\sigma(b) + \sigma(a)\phi(\mathbf{1}, b) \\
&= \phi(b, \mathbf{1})\sigma(a) + \sigma(b)\phi(a, \mathbf{1}) \\
&= \phi(ba, \mathbf{1}) \\
&= \phi(b, a).
\end{aligned}$$

It means that ϕ is symmetric. □

Remark 2.10. *If $\mathcal{A}$ has an approximate identity such as $\{e_i\}_{i \in I}$ and ϕ is a continuous two variable σ-derivation such that $\phi(a, b) = \lim_{i \in I} \phi(e_i, ab)$, then ϕ is symmetric.*

Definition 2.11. *Let $\phi : \mathcal{A} \times \mathcal{A} \to \mathcal{X}$, where $\mathcal{X}$ is a Banach space, be a bilinear map. We say that ϕ preserves zero product if*

$$a, b \in \mathcal{A}, ab = 0 \Rightarrow \phi(a, b) = 0 \quad (B_1).$$

Definition 2.12. *A Banach algebra $\mathcal{A}$ has the property $(\mathbb{B})$ if every continuous bilinear map $\phi : \mathcal{A} \times \mathcal{A} \to \mathcal{X}$, where $\mathcal{X}$ is an arbitrary Banach*

space, (B_1) implies that $\phi(ab, c) = \phi(a, bc) \quad (a, b, c \in \mathcal{A})$
(for more details see [2]).

Corollary 2.13. *Let* $\mathcal{A}$ *has the property* ($\mathbb{B}$) *and having a bounded approximate identity. Then every continuous two variable* σ*-derivation* $\phi : \mathcal{A} \times \mathcal{A} \to \mathcal{M}$ *satisfying* (B_1) *is symmetric.*

Theorem 2.14. *Suppose that* $\mathcal{A}$ *has the property* ($\mathbb{B}$) *and having a bounded approximate identity. Then* $\mathcal{A}$ *is commutative if and only if for every* $a, b \in \mathcal{A}$, $ab = 0$ *implies that* $ba = 0$.

Proof. Suppose for every $a, b \in \mathcal{A}$, $ab = 0$ implies that $ba = 0$. We define $\phi : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ by $\phi(a, b) = ab - ba$. Hence, ϕ is continuous bi-linear mapping which preserves zero product. Let $\{e_i\}$ be a bounded approximate identity. According to Remark 2.6, ϕ is identically zero. So, $ab - ba = 0$ for all $a, b \in \mathcal{A}$. It means that $\mathcal{A}$ is commutative. The converse is clear. $\square$

Note that if $\mathcal{A}$ is a domain with the property ($\mathbb{B}$) and furthermore, $\mathcal{A}$ has a bounded approximate identity, then $\mathcal{A}$ is commutative.
Note: Let $\mathcal{A}$ be a unital Banach algebra with the property ($\mathbb{B}$) and let $\phi : \mathcal{A} \times \mathcal{A} \to \mathcal{M}$ be a continuous two variable $\sigma-$*derivation* which preserves zero product. It follows from Theorem 2.9, that ϕ is symmetric.

Theorem 2.15. *Suppose that* $\mathcal{M}$ *is symmetric and* $\mathcal{A}$ *has a bounded approximate identity. Assume that* $\phi : \mathcal{A} \times \mathcal{A} \to \mathcal{M}$ *is a continuous two variable* σ*-derivation. If* $\sigma(a)\phi(b, c) = \phi(a, b)\sigma(c)$ *for all* $a, b, c \in \mathcal{A}$ *then* ϕ *is symmetric and there exists a continuous* σ*-derivation* $d : \mathcal{A} \to \mathcal{M}$ *such that* $\phi(a, b) = d(ab)$ *for all* $a, b \in \mathcal{A}$.

Proof. First of all, we show that the following are equivalent:
(i) $\phi(ab, c) = \phi(a, bc)$,
(ii) $\sigma(a)\phi(b, c) = \phi(a, b)\sigma(c)$,
(iii) $\phi(ab, c) = \phi(ac, b)$,
for all $a, b, c \in \mathcal{A}$. Clearly, (i) $\Leftrightarrow$ (ii). We are going to prove that (ii) $\Leftrightarrow$

(iii). Suppose (ii) is true. We have therefore the relations

$$\begin{aligned}\phi(ab,c)-\phi(ac,b)&=\phi(a,c)\sigma(b)+\sigma(a)\phi(b,c)-\phi(a,b)\sigma(c)-\sigma(a)\phi(c,b)\\&=\sigma(a)\phi(c,b)+\sigma(a)\phi(b,c)-\sigma(a)\phi(b,c)-\sigma(a)\phi(c,b)\\&=0\end{aligned}$$

for all $a,b,c\in\mathcal{A}$. Conversely, suppose (iii) is true, then for every $a,b\in\mathcal{A}$ and every bounded approximate identity $\{e_i\}_{i\in I}$ we have

$$\begin{aligned}\phi(a,b)&=\lim_{i\in I}\phi(e_ia,b)\\&=\lim_{i\in I}\phi(e_ib,a)\\&=\phi(b,a),\end{aligned}$$

which means that ϕ is symmetric. Reusing (iii) together with the fact that ϕ and $\mathcal{M}$ are symmetric, we obtain $\phi(b,a)\sigma(c)=\phi(a,b)\sigma(c)=\phi(a,c)\sigma(b)=\sigma(b)\phi(a,c)$ for all $a,b,c\in\mathcal{A}$. Consequently, we can find that $\phi(a,b)\sigma(c)=\sigma(a)\phi(b,c)$ and (ii) is achieved. So (i), (ii) and (iii) are equivalent. It follows from (i) that

$$\phi(a,b)=\lim_{i\in I}\phi(e_ia,b)=\lim_{i\in I}\phi(e_i,ab) \tag{1}$$

for all $a,b\in\mathcal{A}$. Hence the net $\{\phi(e_i,a)\}_{i\in I}$ is convergent for each $a\in\mathcal{A}^2$ and so we may define a linear operator $d:\mathcal{A}^2\to\mathcal{M}$ by $d(a)=\lim_{i\in I}\phi(e_i,a)$ $(a\in\mathcal{A}^2)$. According to the definition of d and in view of (1), we have $\phi(a,b)=d(ab)$ for all $a,b\in\mathcal{A}$. Since $\mathcal{A}$ has a bounded approximate identity, $\mathcal{A}^2=\mathcal{A}$. Hence d is a σ-derivation from $\mathcal{A}$ into $\mathcal{M}$. Moreover, for every $a\in\mathcal{A}$,

$$\|\phi(e_i,a)\|\leqslant\|\phi\|\|e_i\|\|a\|\leqslant\|\phi\|\sup_{i\in I}\|e_i\|\|a\|$$

and so d is continuous with $\|d\|\leqslant\|\phi\|\sup_{i\in I}\|e_i\|$. □

Theorem 2.16. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are unital and $\phi:\mathcal{A}\times\mathcal{A}\to\mathcal{B}$ is a two variable σ-derivation. If $\phi(\mathbf{1},a_0)=\mathbf{1}$ for some $a_0\in\mathcal{A}$, then ϕ is symmetric and there exists a unital homomorphism $\theta:\mathcal{A}\to\mathcal{B}$*

such that $\phi(a,b) = \theta(ab)(\theta(a_0))^{-1}$ *for all* $a, b \in \mathcal{A}$. *Moreover,* $\frac{\mathcal{A}}{ker(\theta)}$ *is a commutative algebra.*

Proof. First note that $\sigma(\mathbf{1}) = \frac{\mathbf{1}}{2}$. Because,

$$\begin{aligned}\mathbf{1} &= \phi(\mathbf{1}, a_0) = \phi(\mathbf{1}\ \mathbf{1}, a_0)\\ &= \sigma(\mathbf{1})\phi(\mathbf{1}, a_0) + \phi(\mathbf{1}, a_0)\sigma(\mathbf{1})\\ &= 2\sigma(\mathbf{1}).\end{aligned}$$

We know that $\phi(a, \mathbf{1}) = \phi(\mathbf{1}, a)$ for all $a \in \mathcal{A}$. Assume that a and b are two arbitrary elements of $\mathcal{A}$. Then we have

$$\begin{aligned}\phi(a,b) &= \phi(\mathbf{1}a, b)\\ &= \phi(\mathbf{1}, b)\sigma(a) + \sigma(\mathbf{1})\phi(a,b)\\ &= \phi(\mathbf{1}, b)\sigma(a) + \frac{\phi(a,b)}{2}.\end{aligned}$$

So $\phi(a,b) = 2\phi(\mathbf{1}, b)\sigma(a)$. By Remark 2.4 and reusing the fact that $\phi(a, \mathbf{1}) = \phi(\mathbf{1}, a)$ for all $a \in \mathcal{A}$, we can get

$$\begin{aligned}\phi(b,a) &= 2\phi(\mathbf{1}, a)\sigma(b)\\ &= 2\phi(b, \mathbf{1})\sigma(a)\\ &= 2\phi(\mathbf{1}, b)\sigma(a)\\ &= \phi(a,b).\end{aligned}$$

It means that ϕ is symmetric. Let a be an arbitrary element of $\mathcal{A}$. Then

$$\begin{aligned}\phi(a, a_0) &= \phi(a\mathbf{1}, a_0)\\ &= \phi(a, a_0)\sigma(\mathbf{1}) + \sigma(a)\phi(\mathbf{1}, a_0)\\ &= \frac{\phi(a, a_0)}{2} + \sigma(a).\end{aligned}$$

Hence $\sigma(a) = \frac{\phi(a,a_0)}{2}$. We define a linear operator $\theta : \mathcal{A} \to \mathcal{B}$ by $\theta(a) = \phi(a, a_0)$. Obviously, $\theta(\mathbf{1}) = \mathbf{1}$ and it means that θ is unital. Furthermore, we have

$$\begin{aligned}\theta(ab) &= \phi(ab, a_0)\\ &= \phi(a, a_0)\sigma(b) + \sigma(a)\phi(b, a_0)\\ &= \phi(a, a_0)\frac{\phi(b, a_0)}{2} + \frac{\phi(a, a_0)}{2}\phi(b, a_0)\\ &= \phi(a, a_0)\phi(b, a_0)\\ &= \theta(a)\theta(b).\end{aligned}$$

So θ is a unital homomorphism. Furthermore,

$$\begin{aligned}\phi(a, a_0) &= \phi(a, \mathbf{1}a_0)\\ &= \phi(a, \mathbf{1})\sigma(a_0) + \sigma(\mathbf{1})\phi(a, a_0)\\ &= \phi(a, \mathbf{1})\frac{\phi(a_0, a_0)}{2} + \frac{\phi(a, a_0)}{2}.\end{aligned}$$

So $\phi(a, a_0) = \phi(a, \mathbf{1})\phi(a_0, a_0)$. This results that $\mathbf{1} = \phi(\mathbf{1}, a_0) = \phi(\mathbf{1}, \mathbf{1})\phi(a_0, a_0)$. Similarly, by using the equation $\phi(a, a_0) = \phi(a, a_0\mathbf{1})$, we can obtain that $\mathbf{1} = \phi(a_0, a_0)\phi(\mathbf{1}, \mathbf{1})$. Hence $\phi(a_0, a_0)^{-1} = \phi(\mathbf{1}, \mathbf{1})$. Now, we define a σ-derivation $d : \mathcal{A} \to \mathcal{B}$ by $d(a) = \phi(a, \mathbf{1})$. Since $\phi(a, a_0) = \phi(a, \mathbf{1})\phi(a_0, a_0)$, $\theta(a) = d(a)\theta(a_0)$ for all $a \in \mathcal{A}$. So $d(a) = \theta(a)\theta(a_0)^{-1}$. Moreover, we have

$$\begin{aligned}d(a) &= d(\mathbf{1})\sigma(a) + \sigma(\mathbf{1})d(a)\\ &= d(\mathbf{1})\sigma(a) + \frac{d(a)}{2},\end{aligned}$$

and so $d(a) = 2d(\mathbf{1})\sigma(a) = 2\theta(a_0)^{-1}\frac{\theta(a)}{2} = \theta(a_0)^{-1}\theta(a)$. Similarly, $d(a) = d(a)\sigma(\mathbf{1}) + \sigma(a)d(\mathbf{1}) = \frac{d(a)}{2} + \sigma(a)d(\mathbf{1})$ and therefore, $d(a) = 2\sigma(a)d(\mathbf{1}) = \theta(a)\theta(a_0)^{-1}$. Hence $\theta(a)\theta(a_0)^{-1} = \theta(a_0)^{-1}\theta(a)$ for all

$a \in \mathcal{A}$. Assume that a and b are two arbitrary elements of $\mathcal{A}$. Then

$$\begin{aligned}\phi(a,b) &= 2\phi(a,\mathbf{1})\sigma(b)\\ &= 2d(a)\frac{\theta(b)}{2}\\ &= d(a)\theta(b)\\ &= \theta(a)\theta(a_0)^{-1}\theta(b)\\ &= \theta(a)\theta(b)\theta(a_0)^{-1}\\ &= \theta(ab)\theta(a_0)^{-1}. \quad \square\end{aligned}$$

Our next task is to prove that $\frac{\mathcal{A}}{ker\theta}$ is a commutative algebra. Since ϕ is symmetric, i.e. $\phi(a,b) = \phi(b,a)$ for all $a,b \in \mathcal{A}$, we have $\theta(ab)\theta(a_0)^{-1} = \theta(ba)\theta(a_0)^{-1}$. This implies that $\theta(ab) = \theta(ba)$, i.e. $ab - ba \in ker(\theta)$ for all $a,b \in \mathcal{A}$. Consequently, $ab + ker(\theta) = ba + ker(\theta)$ and it results that $\frac{\mathcal{A}}{ker(\theta)}$ is a commutative algebra.

Corollary 2.17. *Suppose that $\mathcal{A}$ is unital and $\mathcal{B}$ is a unital, commutative Banach algebra. Let $\phi : \mathcal{A} \times \mathcal{A} \to \mathcal{B}$ be a two variable σ-derivation such that $\phi(\mathbf{1}, a_0) = \mathbf{1}$ for some $a_0 \in \mathcal{A}$. If $\mathcal{B}$ is semisimple then ϕ is continuous.*

Proof. It is an immediate conclusion from the previous theorem and Proposition 5.1.1 of [5]. $\square$

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