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On domatic number of graphs

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Abstract. The domatic number $d(G)$ of a graph G is the maximum k for which $V(G)$ can be partitioned into k pairwise disjoint dominating sets. In this paper, we determine the domatic numbers of complete graphs, complete k-partite graphs, Johnson graphs $J(n, 2)$, unicyclic graphs, bicyclic graphs and generalized Θ-graphs.

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1 Introduction

The vertices set and edges set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. An adjacent vertex of a vertex v in a graph is a vertex that is connected to v by an edge. The neighbourhood of a vertex v in a graph G is the subgraph of G induced by all vertices adjacent to v. Also, the closed-neighbourhood of a vertex v in a graph G is the subgraph of G induced by v and all vertices adjacent to v. If X is a

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subset of vertices of a graph G, the induced subgraph resulting of the deletion of X is denoted by $G - X$. Also, the subgraph of G induced by X is denoted by $G[X]$. A vertex of degree one is called a pendant. The distance of two vertices v and u of a graph G is denoted by $d_G(v, u)$. The maximum degree and minimum degree of a graph G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. A unicyclic graph is a connected graph containing exactly one cycle. Bicyclic graphs are connected graphs in which the number of edges equals the number of vertices plus one. The Johnson graphs are a special class of undirected graphs defined from systems of sets. The vertices of the Johnson graph $J(n, k)$ are the k-element subsets of an n-element set, two vertices are adjacent when the intersection of the two vertices (subsets) contains $(k-1)$-elements.

A dominating set for a graph G is a subset D of $V(G)$ such that every vertex not in D is adjacent to at least one member of D. Also, a total dominating set for a graph G is a subset D of $V(G)$ such that every vertex in $V(G)$ is adjacent to at least one member of D. A total domatic partition of a graph G is a partition of $V(G)$ into disjoint sets $V_1, V_2, ..., V_k$ such that each V_i is a dominating set for G. In addition, a total domatic partition of a graph G is a partition of $V(G)$ into disjoint sets $V_1, V_2, ..., V_k$ such that each V_i is a total dominating set for G. Chen et al. [5] defined a k-coupon coloring of G is an assignment of colors from $[k] := \{1, 2, ..., k\}$ to the vertices of G such that the neighborhood of every vertex of G contains vertices of all colors from $[k]$. The maximum k for which a k-coupon coloring exists is called the coupon coloring number of G, and is denoted by $\chi_c(G)$. For more results on Coupon coloring of graphs refer to [12, 13]. There are applications in network science which finding coupon coloring number helps to have a better performance. One application is to large multi-robot networks [3]. An example described in [1] is about a group of robots is deployed to monitor an environ-ment. Also, another similar example about allocating resources to a network is described in [1] which each vertex of a graph may only use resources available at the vertex or its neighbors. The concept of coupon coloring is equivalent to the concept of total domatic partition.

A k-coloring c of G is an injective coloring if for every vertex $v \in V(G)$, all the neighbours of v are assigned distinct colors. The smallest integer k such that G has an injective k-coloring is the injective chromatic number of G, denoted by $\chi_i(G)$. It is easy to see that $\chi_i(G) \leq \Delta(G)$. Also, some works has been done for finding the injective chromatic number of graphs. For example, Doyon et al. [9], Cranston et al. [7, 8], Bu and Lu [4], and Chen et al. [6] have some results on the injective chromatic number of graphs.

We directly have $d(G) \leq \delta+1$ for any graph G with minimum degree δ. Therefore, we have $\chi_c(G) \leq d(G) \leq \delta + 1$ for any graph G. Also, every non-regular graph G satisfies the inequalities $\chi_c(G) \leq d(G) \leq \delta + 1 \leq \Delta \leq \chi_i(G)$.

In addition, recently, in [10], Ghanbari and Alikhani, defined the notions of strong dominating set and strong domination number and then they initiate the studied of the strong domatic number, and presented different sharp bounds on $d_{st}(G)$. In addition, they determined this parameter for some classes of graphs, such as cubic graphs of order at most 10. Also, in [11], improved the results for the middle and central graphs of a cycle, respectively. Furthermore, they discussed the domatic number for some other cycle-related graphs and graphs of convex polytopes.

In Section 2, we determine the domatic numbers of some special graphs, namely complete graphs, complete k-partite graphs, cycles, unicyclic graphs, bicyclic graphs and generalized Θ-graphs.

2 Main Results

In this section, a domatic coloring of a graph G is a domatic partition of G.

Theorem 2.1. *Let G be a complete graph with n vertices. Then $d(G) = n$.*

Proof. Consider a coloring $c : V(G) = \{v_1, v_2, ...v_n\} \rightarrow \{1, 2, ..., n\}$ of G such that $c(v_i) = i$. It is easy to see that c is a domatic coloring of G. So, $d(G) = n$, since $d(G) \leq n$. □

Theorem 2.2. *Let G be a connected graph with $n \geq 2$ vertices. Then $d(G) \geq 2$.*

Proof. It suffices to give a domatic coloring of G with 2 colors. Consider a search tree T of G with root v. First, assign to vertex v, color 1. Now, color a vertex u of T with color 2, if $d_T(v,u)$ is odd and color with color 1, if $d_T(v,u)$ is even. It is easy to check that this coloring is a domatic coloring of G. $\square$

Theorem 2.3. *Let G be a graph obtained from two graphs H and K_n by joining every vertex of H to every vertex of K_n. Then $d(G) = d(H)+n$.*

Proof.
Set $k = d(H)$. Consider a domatic coloring c of H with k colors. First, color all of vertices of H using c. Next, color all of vertices of K_n with n fresh distinct colors. It is easy to check that this coloring is a domatic coloring of G with $k+n$ colors. So, $d(G) \geq k+n$. Now, by way of contradiction suppose that there is a domatic coloring c' of G with more than $k+n$ colors. One can easily prove that there is more than k colors which are used only in H. Donote by $A = \{a_1, a_2, ...a_{k'}\}$ the set of all of the colors which are used only in H where $k' > k$. Let X be the set of all vertices of H which are colored with a color of A. One can easily prove that there is a domatic coloring of the induced subgraph $H[x]$ with k' colors, a contradiction.
$\square$

A k-partite graph is one whose vertex set can be partitioned into k subsets, or parts, in such a way that no edge has both ends in the same part.

In a k-coloring c, we call a vertex v a bad vertex if its closed-neighborhood does not contain vertices of all colors from $[k]$. Also, in a k-coloring c, we call a vertex v a good vertex if it is not bad. Obviously, there is no bad vertices in a domatic coloring. In other words, all vertices in a domatic coloring are good.

Theorem 2.4. *Let $G = K_{n_1,n_2,...,n_k}$ be a complete k-partite graph on n vertices and let $k \geq 2$. Suppose that $2 \leq n_1 \leq n_2 \leq ... \leq n_k$. Then*

$$d(G) = \begin{cases} \lfloor \frac{n}{2} \rfloor & \text{if } n_k \leq \frac{n}{2}, \\ n - n_k & \text{if } n_k > \frac{n}{2}. \end{cases}$$

Proof.

Let $G = [V_1, V_2, ..., V_k]$. Set $V' = V(G) - V_k$. We have two cases:

Suppose that $n_k \leq \frac{n}{2}$. It is proved in [13, Theorem 2.2] that $\chi_c(G') = \lfloor \frac{n}{2} \rfloor$. So, $d(G) \geq \lfloor \frac{n}{2} \rfloor$. Now, it suffices to show $d(G) \leq \lfloor \frac{n}{2} \rfloor$. Suppose, by way of contradiction that there is a domatic coloring c' of G with more than $\lfloor \frac{n}{2} \rfloor$ colors. So, there must exist a color a appears in G exactly once. Assume that a is assigned only to v in G. It follows that $d(v) = n - 1$. But, since for all V_i, $1 \leq i \leq k$, $|V_i| \geq 2$, we have $d(v) \leq n - 2$ which is a contradiction.

Suppose that $n_k > \frac{n}{2}$. Now, we color G as follows. We have $|V'| = n - n_k$ and assign $n - n_k$ distinct colors to vertices in V' such that every color appear in V' exactly once. Next, we color the vertices in V_k with the same $n - n_k$ colors such that all these colors appear in V_k. The overall number of colors used is $n - n_k$. It is easy to verify that the above coloring is a domatic coloring of G. Hence, $d(G) \geq n - n_k$. Now, it suffices to show that $d(G) \leq n - n_k$. Suppose, by way of contradiction that there is a domatic coloring c' of G with more than $n - n_k$ colors. So, there is a color a in coloring c' which does not appear in V'. In fact, the color a appear only in V_k. Now, if there exists in V_k a vertex u such that $c'(u) \neq a$, then u is a bad vertex, since a does not appear in its closed-neighbourhood, a contradiction. And if for every $u \in V_k$, $c'(u) = a$, then since, for any V_i, $1 \leq i \leq k - 1$, we have $|V_i| \geq 2$, we can conclude that every vertex in V' is a bad vertex, since for every $v, u \in V_k$, we have $c'(u)$ does not appear in neighbourhood of v, a contradiction. Therefore, $d(G) = n - n_k$.

□

In the following theorem, we determine the domatic coloring number of complete k-partite graphs.

Theorem 2.5. *Let $G = K_{n_1, n_2, ..., n_k}$ be a complete k-partite graph where $k \geq 2$ and let $1 \leq n_1 \leq n_2 \leq ... \leq n_k$ such that $n = \sum_{i=1}^{k} n_i$. Suppose that s is the number of parts of G which are isomorphic to K_1 and set $n' = n - s$. Then*

$$d(G) = \begin{cases} s + 1 & \text{if } k - s = 1, \\ \lfloor \frac{n'}{2} \rfloor + s & \text{if } k - s \geq 2 \text{ and } n_k \leq \frac{n'}{2}, \\ n' - n_k + s & \text{if } k - s \geq 2 \text{ and } n_k > \frac{n'}{2}. \end{cases}$$

Proof. Let $V_1, V_2, ..., V_k$ be the partite sets of G. We distinguish the following two cases:

If $k - s = 1$, then for all V_i, $1 \leq i \leq k-1$, we have $|V_i| = 1$. Also, we have $|V_k| \geq 2$. It is easy to see that $d(G[V_k]) = 1$. Now, Theorem 2.3 implies that $d(G) = s + 1$.

If $k - s \geq 2$, then set $V' = V_1 \cup V_2 ... \cup V_s$. The induced subgraph $G - V'$ is a k'-partite graph where $k' \geq 2$ in which every part contains at least 2 vertices. This induced subgraph is on n' vertices. So, Theorems 2.3 and 2.4 imply that

$$d(G) = \begin{cases} \lfloor \frac{n'}{2} \rfloor + s & \text{if } n_k \leq \frac{n'}{2}, \\ n' - n_k + s & \text{if } n_k > \frac{n'}{2}. \end{cases}$$

□

Theorem 2.6. *For the cycle C_n of order n, we have*

$$d(C_n) = \begin{cases} 3 & n \overset{3}{\equiv} 0, \\ 2 & \textit{otherwise}. \end{cases}$$

Proof. Let $C_n = v_0 v_1 v_2 ... v_{n-1} v_0$. We know that $d(C_n) \leq 3$, since $\delta(C_n) = 2$. Now, we distinguish the following two cases:

Suppose, first, that $n \overset{3}{\equiv} 0$. We give a 3-coloring c to C_n as follows.

$$c(v_i) = \begin{cases} 0 & i \overset{3}{\equiv} 0, \\ 1 & i \overset{3}{\equiv} 1, \\ 2 & i \overset{3}{\equiv} 2. \end{cases}$$

It is easy to check that c is a domatic coloring of C_n (see C_6 in Figure 1). Therefore, $d(C_n) = 3$.

Suppose, now, that $n \overset{3}{\not\equiv} 0$. Suppose by way of contradiction that there is a domatic coloring c of C_n with 3 colors. It is easy to check that the vertices v_0, v_1 and v_2 must be colored by three distinct colors in coloring c. Denote the colors by 0, 1 and 2, respectively. Now, one can easily prove that since c is a domatic coloring, colors of vertices of C_n are as follows.

$$c(v_i) = \begin{cases} 0 & i \overset{3}{\equiv} 0, \\ 1 & i \overset{3}{\equiv} 1, \\ 2 & i \overset{3}{\equiv} 2. \end{cases}$$

If $n \overset{3}{\equiv} 1$, then the vertices v_{n-1} and v_0 are colored by the same color 0 which contradicts to that c is a domatic coloring with 3 colors (see C_4 in Figure 1). Also, if $n \overset{3}{\equiv} 2$, then three vertices v_{n-2}, v_{n-1} and v_0 are colored by two colors 0 and 1 which contradicts to that c is a domatic coloring with 3 colors (see C_5 in Figure 1). Therefore, we have $d(C_n) \leq 2$. Also, Theorem 2.2 implies that $d(C_n) \geq 2$. So, $d(C_n) = 2$.
□

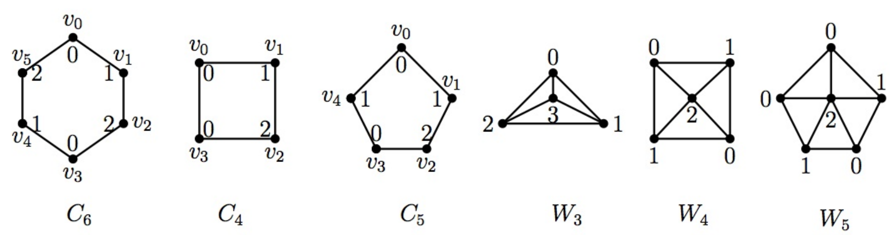

Figure 1: Some graphs in proof of Theorems 2.6.

Theorem 2.7. *Let G be a unicyclic graph on $n \geq 2$ vertices. Then $d(G) = 3$ if and only if G is a cycle C_n such that $n \overset{3}{\equiv} 0$, otherwise $d(G) = 2$.*

Proof. If G is a cycle, then Theorem 2.6 implies that $d(G) = 3$ when $n \overset{3}{\equiv} 0$ and $d(G) = 2$ when $n \overset{3}{\not\equiv} 0$.

If G is not a cycle, then $\delta(G) = 1$. So, $d(G) \leq 2$. Now, Theorem 2.2 implies that $d(G) \geq 2$. Therefore, $d(G) = 2$.
□

Now, we study the domatic numbers of bicyclic graphs. Because the domatic number of any graph containing pendant vertices is 2, we only consider bicyclic graphs without pendant vertices in the following.

Yongtang Shi et al. explained in [13] that bicyclic graphs without pendant vertex divided into two classes. Class I contains all of the bicyclic graphs with no cycles sharing common edges. Class II contains all of bicyclic graphs whose cycles share common edges. In other words, any graph in Class II, consists of three internally disjoint paths with common end vertices. The graphs in Class II are known as the Θ-graphs (see Figure 2 in [13]).

All of the graphs in Figure 2 belong to Class I and all of the graphs in Figure 3 belong to Class II.

The length of a path P_n or a cycle C_n is the number of its edges which is denoted by $l(P_n)$ or $l(C_n)$, respectively

In Theorems 2.8 and 2.9, we determine the domatic number of bicyclic graphs which are of Class I.

Theorem 2.8. *Let G be a bicyclic graph of Class I with n vertices and let C and C' be two the cycles of G. Then $d(G) = 3$ if and only if $l(C) \overset{3}{\equiv} 0$ and $l(C') \overset{3}{\equiv} 0$.*

Proof. Let P be the path connecting C and C' such that v and v' are the two ends of P. Suppose $l(C) \overset{3}{\equiv} 0$ and $l(C') \overset{3}{\equiv} 0$. Now, we give a domatic coloring c of G with 3 colors $1, 2$ and 3 as follows.

First, we color all of the vertices of C such that the coloring is domatic. Next, we continue coloring of all vertices on P such that the internal vertices of P are good. Finally, we give a domatic coloring of C' by considering the vertex v' has color $c(v')$. It is easy to check that c is a 3-domatic coloring of G (see G in Figure 2).

If $l(C) \overset{3}{\not\equiv} 0$, then suppose by way of contradiction that there is a domatic coloring c of G with 3 colors. Set $C = vv_1v_2...v_{n-1}v$. Consider the three vertices v, v_1 and v_2. It is easy to check that these three vertices must be colored by three distinct colors in coloring c (see H in Figure 2). Denote these colors by $0, 1$ and 2, respectively. Now, if vertices $v_3, v_4, ...v_{n-1}$ are colored, respectively, then it is easy to see that vertices must be colored as follows.

$$c(v_i) = \begin{cases} 0 & i \overset{3}{\equiv} 1, \\ 1 & i \overset{3}{\equiv} 2, \\ 2 & i \overset{3}{\equiv} 0. \end{cases}$$

If $n \overset{3}{\equiv} 1$, then the vertices v_{n-1} and v are colored by the same color 0 which contradicts to that c is a domatic coloring with 3 colors (see H in Figure 2). Also, if $n \overset{3}{\equiv} 2$, then three vertices v_{n-2}, v_{n-1} and v are colored by two colors 0 and 1 which contradicts to that c is a domatic coloring with 3 colors, since in closed-neighborhood of v_{n-1} does not appear color 2 (see F in Figure 2).
□

Theorem 2.9. *Let G be a bicyclic graph of Class I with n vertices and let C and C' be the two cycles of G. If $l(C) \overset{3}{\not\equiv} 0$, then $d(G) = 2$.*

Proof. Theorem 2.8 implies that $d(G) \leq 2$. Also, Theorem 2.4 implies that $d(G) \geq 2$. So, $d(G) = 2$.
□

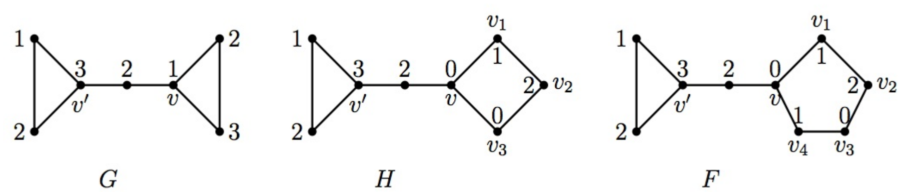

Figure 2: Some graphs in proof of Theorem 2.8.

Lemma 2.10. *Let P be a path of length k such that $k \overset{3}{\not\equiv} 0$ and let v and u be the two ends of P. Then there is no coloring c with 3 colors of P such that $c(v) = c(u)$ and all of the internal vertices of P are good.*

Proof. The proof is straightforward. □

Lemma 2.11. *Let P be a path of length k such that $k \overset{3}{\equiv} 0$ and let v and u be the two ends of P. Then there is no coloring c with 3 colors of P such that $c(v) \neq c(u)$ and all of the internal vertices of P are good.*

Proof. The proof is straightforward. □

Lemma 2.12. *Let $P = vv_1...v_{k-1}u$ such that $k \overset{3}{\equiv} 1$ and $k \geq 2$ and let v and u be the two ends of P such that v and u are colored with two distinct colors 1 and 2, respectively. In order for all of internal vertices of P to be good in a 3-coloring, the internal vertices must be colored as follows.*

$$c(v_i) = \begin{cases} 2 & i \overset{3}{\equiv} 1, \\ 3 & i \overset{3}{\equiv} 2, \\ 1 & i \overset{3}{\equiv} 0. \end{cases}$$

Proof. The proof is straightforward (see P_5 in Figure 3). □

The generalized $\ominus$-graph $\ominus_{q_1,q_2,...,q_t}$ is the union of $t \geq 2$ paths P_1, P_2,...,P_t with length $q_1 \geq ...q_{t-1} \geq q_t \geq 1$ where $q_{t-1} \geq 2$. and the paths are pairwise internally vertex-disjoint with the same two end vertices. Using the following theorem, we can determine the domatic number of a bicyclic graph of Class II.

Theorem 2.13. *Let $G = \ominus_{q_1,q_2,...,q_t}$, then $d(G) = 3$, if and only if one the following holds, otherwise $d(G) = 2$.*

(1) $l(P_i) \overset{3}{\equiv} 0$ for any path P_i, $1 \leq i \leq t$.

(2) $l(P_1) = 1$ and $l(P_i) \overset{3}{\equiv} 0$ for any path P_i, $2 \leq i \leq t$,

(3) $l(P_i) \overset{3}{\equiv} 1$ or $l(P_i) \overset{3}{\equiv} 2$ for any path P_i, $1 \leq i \leq t$, and $\{1, 2\} \subseteq \{l(P_i) : 1 \leq i \leq t\}$ $(mod 3)$.

Proof. Denote by v and u the two common ends of all paths P_i, $1 \leq i \leq t$. Also, set $P_i = vv_{i,1}v_{i,2}...v_{i,q_i-1}u$ for $1 \leq i \leq t$.

Case 1. If condition (1) holds, then we give a 3-domatic coloring c of G. First, color v and u with color 1. Next, color vertices $v_{i,1}$, $v_{i,2}$,..., v_{i,q_i-1} on the path P_i, $1 \leq i \leq t$, as follows.

$$c(v_{i,j}) = \begin{cases} 1 & j \overset{3}{\equiv} 0, i = 1 \\ 2 & j \overset{3}{\equiv} 1, i = 1 \\ 3 & j \overset{3}{\equiv} 2, i = 1 \\ 1 & j \overset{3}{\equiv} 0, i \neq 1 \\ 3 & j \overset{3}{\equiv} 1, i \neq 1 \\ 2 & j \overset{3}{\equiv} 2. i \neq 1 \end{cases}$$

It is easy to check that c is a 3-domatic coloring of G (see $\Theta_{3,3,3}$ in Figure 3).

Case 2. If condition (2) holds, then proof is similar to Case 1 (see $\Theta_{1,3,3}$ in Figure 3).

Case 3. If condition (3) holds, then we give a 3-domatic coloring c of G. First, color v and u with colors 1 and 2, respectively. Next, color vertices $v_{i,1}$, $v_{i,2}$,..., v_{i,q_i-1} on the path P_i, $1 \leq i \leq t$, as follows (see $\Theta_{1,4,5}$ in Figure 3):

$$c(v_{i,j}) = \begin{cases} 1 & j \overset{3}{\equiv} 0, l(P_i) \overset{3}{\equiv} 1, \\ 2 & j \overset{3}{\equiv} 1, l(P_i) \overset{3}{\equiv} 1, \\ 3 & j \overset{3}{\equiv} 2, l(P_i) \overset{3}{\equiv} 1, \\ 1 & j \overset{3}{\equiv} 0, l(P_i) \overset{3}{\equiv} 2, \\ 3 & j \overset{3}{\equiv} 1, l(P_i) \overset{3}{\equiv} 2, \\ 2 & j \overset{3}{\equiv} 2, l(P_i) \overset{3}{\equiv} 2. \end{cases}$$

Now, suppose that none of the three above conditions hold. Theorem 2.4 implies that $d(G) \geq 2$. So, it suffices to show $d(G) \leq 2$. Now, we distinguish the following three cases:

Case 1. Suppose there is two paths P_i and P_j such that $l(P_i) \overset{3}{\equiv} 0$ and $l(P_j) \overset{3}{\not\equiv} 0$ and $l(P_j) \geq 2$. Suppose by way of contradiction that there is a domatic coloring c of G with 3 colors.

If $c(v) = c(u)$, then Lemma 2.10 implies that there is no coloring of internal vertices P_j with 3 colors such that all of the internal vertices are good, a contradiction.

If $c(v) \neq c(u)$, then Lemma 2.11 implies that there is no coloring of internal vertices P_i with 3 colors such that all of the internal vertices are good, a contradiction.

Case 2. Suppose that for all of paths P_i, $1 \leq i \leq t$, we have $l(P_i) \overset{3}{\equiv} 1$. Suppose by way of contradiction that there is a domatic coloring c of G with 3 colors $1, 2$ and 3.

If $c(v) = c(u)$, then Lemma 2.10 implies that there is no coloring of internal vertices P_j with 3 colors such that all of the internal vertices are good, a contradiction.

If $c(v) \neq c(u)$, then without loose of generality assume that $c(v) = 1$ and $c(u) = 2$. Now, Lemma 2.12 implies that all of adjacent vertices to v are colored with 2 in coloring c. So, the closed-neighbourhood of v does not contain color 3, a contradiction.

Case 3. Suppose that for all of paths P_i, $1 \leq i \leq t$, we have $l(P_i) \overset{3}{\equiv} 2$. In this case, we can obtain the needed contradiction by a similar way to Case 2.

□

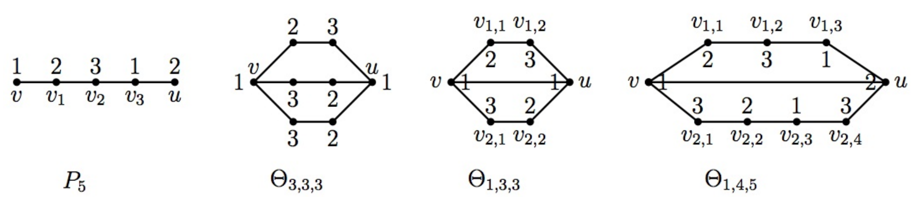

Figure 3: Some graphs in proof of Theorem 2.13.

Lemma 2.14. [2] *Any complete graph K_n has a decomposition into maximal matchings.*

Lemma 2.15. *Let X be a class of all the 2-element subsets of a set $A = \{1, 2, ..., n\}$.*

1) If n is even, then X is partitioned to $n-1$ subclasses of size $\frac{n}{2}$ such that all members of any subclass are disjoint

2) If n is odd, then X is partitioned to n subclasses of size $\frac{n-1}{2}$ such that all members of any subclass are disjoint.

Proof. It is proved by Lemma 2.14. $\square$

Theorem 2.16. *For Johnson graph $J(n, 2)$, we have*

$$d(J(n,2)) = \begin{cases} n-1 & n \text{ is even}, \\ n & n \text{ is odd}. \end{cases}$$

Proof.

Let $d(J(n, 2)) = k$. Denote by C_i, the set of vertices with color i for $1 \leq i \leq k$. Suppose, first, that n is even. It is easy to see that if $|C_i| < \frac{n}{2}$, then there is vertex of graph $J(n, 2)$ which is adjacent to none of vertices of C_i, a contradiction. So, $|C_i| \geq \frac{n}{2}$, for all $1 \leq i \leq k$. Suppose, now, that n is odd. It is easy to see that if $|C_i| < \frac{n-1}{2}$, then there is vertex of graph $J(n, 2)$ which is adjacent to none of vertices of C_i, a contradiction. So, $|C_i| \geq \frac{n-1}{2}$, for all $1 \leq i \leq k$.

Hence, we have $d(J(n, 2)) \leq n-1$ when n is even, and $d(J(n, 2)) \leq n$ when n is odd.

Now, using Lemma 2.15, consider subclasses $A_1, A_2, ..., A_{n-1}$ of size $\frac{n}{2}$ when n is even, and consider subclasses $B_1, B_2, ..., B_n$ of size $\frac{n-1}{2}$ when n is odd.

Now, we give a domatic coloring c for $J(n, 2)$. If n is even, color all vertices of A_i by color i, where $1 \leq i \leq n-1$. Also if n is odd, color all vertices of B_i by color i, where $1 \leq i \leq n$.

$\square$

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