Journal of Mathematical Extension
Vol. 17, No. 4, (2023) (3)1-14
URL: https://doi.org/10.30495/JME.2023.2593
ISSN: 1735-8299
Original Research Paper

# Probability of Normality of Chains in Finite Groups

**M. Sajedi**
Mashhad Branch, Islamic Azad University

**M. R. R. Moghaddam** *
Ferdowsi University of Mashhad

**H. Darabi**
Esfarayen University of Technology

**Abstract.** In this paper we introduce the concept of probability of normality of chains in finite groups. For any normal subgroup $N$ of a finite group $G$, the relation between the probability of normality of chains of $G$ and of its factor group $G/N$ are obtained. Finally, we give explicit formulas for such probability of dihedral groups $D_{2n}$, quasi-dihedral groups $QD_{2^n}$, generalized quaternion groups $Q_{2^n}$, and the modular $p$-groups $M_{p^n}$.

**AMS Subject Classification:** Primary: 20E36, 20F28; Secondary: 20E45.

**Keywords and Phrases:** Probability of normality, Dihedral groups, Generalized quaternion groups, Modular $p$-groups.

Received: January 2022; Accepted: February 2023
* Corresponding author

# 1 Introduction

The notion of probability in finite groups has been studied by many authors. Saeedi et al. [4] and Tarnauceanu [9] studied the probability of randomly chosen a subgroup of a given finite group $G$ can be normal. Tarnauceanu and Toth [7] studied the probability of chosen a subgroup of a given finite group to be cyclic.

In this paper, we study the probability of a randomly chosen chain of subgroups of a given finite group $G$, that can be a normal chain. Note that such a chain of subgroups corresponds to an equivalence class of fuzzy subgroups. More precisely, let

$$H = H_1 \subsetneq H_2 \subsetneq \cdots \subsetneq H_n = G$$

be a chain of subgroups of a given group $G$, which starts from any subgroup $H$ and ends in $G$. In this article, we consider and study the chains of subgroups which started from any subgroup $H$ (including the identity), and ending in $G$. The collection of all such chains of the group $G$ is denoted by $Ch(G)$.

A chain of subgroups of $G$ which ends in $G$ is called *normal*, whenever all its components are normal subgroups of $G$ (see also [5] for calculating the probability cyclicity chain of a finite group). We denote the set of all normal chains of a group $G$ by $N^*(G)$ and introduce the *probability of normality(or normality degree) of chains of* $G$, denoted by $pnc(G)$, as follows:

$$pnc(G) = \frac{|N^*(G)|}{|Ch(G)|}.$$

Throughout this paper, we consider all groups to be finite and by a *chain* we mean a chain of subgroups of a group $G$, which starts from any subgroup and ends in $G$. The properties and counting the number of chains in a given finite group have been investigated by many authors. For example, the number of chains of finite Dedekind groups and finite elementary abelian $p$-groups is obtained in [8]. Also, the number of chains of nonabelian groups $D_{2n}$, $Q_{4n}$, $QD_{2^n}$, and $M_{p^n}$ is computed in [1]. Our aim is to give some inequalities for the normality degree of chains in finite groups. We also obtain the probability of normality (or normality degree) of chains of the groups $D_{2n}$, $Q_{2^n}$, $QD_{2^n}$, and $M_{p^n}$.

## 2 Preliminaries

It is obvious that the probability of normality of chains of any finite group $G$ satisfies $0 < pnc(G) \leq 1$, and $pnc(G) = 1$ if and only if $G$ is an abelian group or it is a Dedekind group. Clearly, the group $G$ itself is a chain, and so as in [2] using the subgroup lattice of a group $G$, we define the number of chains start from any non-trivial subgroup $H$ of $G$ and end in $G$ in the following way

$$n(H) = \sum_{1 \leq i \leq r} n(H_i),$$

in which $H_i$'s are the subgroups of $G$ containing $H$ properly. Clearly $n(G) = 1$ and if $H = \langle e \rangle$ is the trivial subgroup, then the number of all chains of subgroups of $G$ is $|Ch(G)| = 2n(\langle e \rangle)$. Using the above technique, one can easily conclude that

$$|Ch(\mathbb{Z}_{p^n})| = 2^n.$$

The following example shows the normal subgroup lattice of the direct product of the symmetric group of degree $n$ by itself, i.e. $S_n \times S_n$.

**Example 2.1.** The normal subgroups of the symmetric group $S_n$ ($n \geq 5$) are only $S_n, A_n, \{1\}$, and hence the lattice of normal subgroups of $S_n \times S_n$ is as follows:

$S_n \times S_n, A_n \times A_n, 1 \times A_n, A_n \times 1, S_n \times A_n, A_n \times S_n, S_n \times 1, 1 \times S_n, A, \{e\}$,

where $A_n$ is the alternating subgroup of $S_n$ and $A = \{(x_1, x_2) : x_1 x_2 \in A_n\}$. The normal subgroup lattice of the group $S_n \times S_n$ is given by the diagram in Figure 1.

Using the above method, the number of normal chains of the group $S_n \times S_n$ can be obtained in the following way:

$$\begin{aligned}
n(S_n \times S_n) = n(S_n \times A_n) = n(A_n \times S_n) = n(A) = 1,\\
n(A_n \times A_n) = 3,\\
n(S_n \times 1) = n(1 \times S_n) = 2,\\
n(A_n \times 1) = n(1 \times A_n) = 8,\\
n(\{e\}) = 1 + 1 + 1 + 1 + 3 + 2.2 + 2.8 = 27.
\end{aligned}$$

Therefore $|N^*(S_n \times S_n)| = 2.27 = 54$, in which "." is used for the product of two numbers. So the probability of normality of chains in the group $S_n \times S_n$ $(n \geq 5)$ is $pnc = \frac{54}{|Ch(S_n \times S_n)|}$.

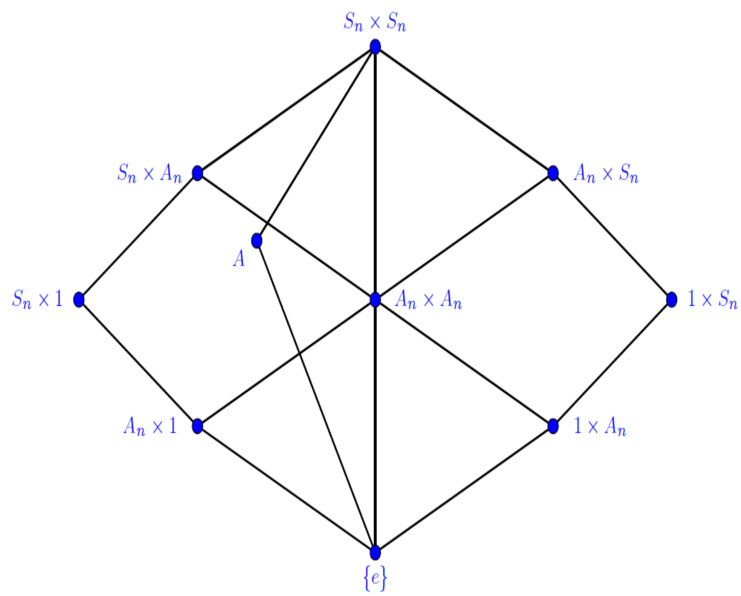


**Figure 1:** Normal subgroup lattice of $S_n \times S_n$.

Assume $L(G)$ denotes the subgroup lattice of a given group $G$. Then the number of chains of a group $G$ with maximum length 2 that ends in $G$ is equal to $|L(G)|$. Similarly, the number of the set of all normal chains of a group $G$ with maximum length 2 is the same as the number of the set of all normal subgroups of the group $G$. Hence, the probability of normality of chains with maximum length 2 is equal to the normality degree of subgroups of $G$ defined in [9] and

$$pnc(G) = \frac{|N(G)|}{|L(G)|},$$

where $N(G)$ denotes the normal subgroup lattice of $G$.

We investigate finite $p$-groups having a cyclic maximal subgroup. One observes that, the following finite nonabelian $p$-groups all have a cyclic maximal subgroup. So we calculate the probability of their normality degrees of their subgroups chains.

$$
\begin{aligned}
D_{2n} &= \langle a, b : a^n = b^2 = 1, a^b = a^{-1} \rangle, \\
Q_{2^n} &= \langle x, y : x^{2^{n-1}} = y^4 = 1, x^y = x^{2^{n-1}-1} \rangle, \\
QD_{2^n} &= \langle x, y : x^{2^{n-1}} = y^2 = 1, x^y = x^{2^{n-1}-1} \rangle, \\
M_{p^n} &= \langle x, y : x^{p^{n-1}} = y^p = 1, x^y = x^{p^{n-2}+1} \rangle.
\end{aligned}
$$

The following Theorem of [1] is needed in proving our main results.

**Theorem 2.2.** *[1].*
*(i) The number of all subgroup chains of dihedral group $D_{2n}$ is*

$$|Ch(D_{2n})| = \sum_{k|n} \frac{n}{k} |Ch(\mathbb{Z}_{\frac{n}{k}})|(k + |Ch(\mathbb{Z}_k)|) - (2n-1)|Ch(\mathbb{Z}_n)| + n.$$

*(ii) The number of all subgroup chains of generalized quaternion group $Q_{4n}$ is*

$$|Ch(Q_{4n})| = |Ch(D_{2n})| + \sum_{d|m} |ch(\mathbb{Z}_d)| \times |ch(D_{\frac{2n}{d}})|,$$

*where $n = 2^k m$, for some $k$ and odd integer $m$.*
*(iii) The number of all subgroup chains of quasi-dihedral group $QD_{2^n}(n \geq 4)$ is*

$$Ch(QD_{2^n})| = 3.2^{2n-3}.$$

*(iv) The number of all subgroup chains of Modular $p$-group $M_{p^n}(n \geq 3, p^n \neq 8)$ is*

$$2^{n-1}(n-1)p + 2^n.$$

**Lemma 2.3.** *The proper normal subgroups of the dihedral group $D_{2n}$ are of the form $\langle a^d \rangle$, where $d|n$, if $n$ is odd, and $\langle a^d \rangle$, $\langle a^2, b \rangle$, $\langle a^2, ab \rangle$, when $n$ is even.*

**Example 2.4.** The normal subgroup lattice of the group $D_{30}$ is as follows:

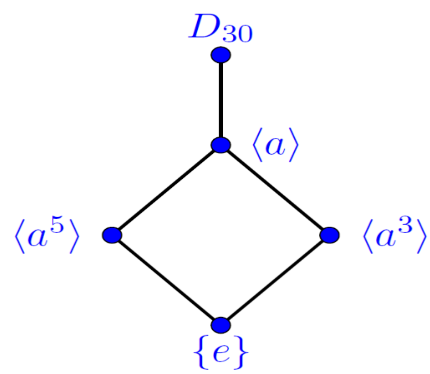


**Figure 2:** Normal subgroup lattice of $D_{30}$.

By Theorem 2.2(i) and Proposition 2.2 of [1], the number of all chains of the group $D_{30}$ is $|Ch(D_{30})| = 2 \times 67 = 134$. Also, using Lemma 2.3, the normal subgroups of $D_{30}$ reads as follows: $\langle 1 \rangle$, $\langle a \rangle$, $\langle a^5 \rangle$ and $\langle a^3 \rangle$, and $D_{30}$. Therefore, as in Example 2.1, we have $|N^*(D_{30})| = 12$, and so $pnc(D_{30}) = \frac{12}{134}$.

Clearly, $pnc(G_1) = pnc(G_2)$ for any two isomorphic groups $G_1$ and $G_2$, but the converse is not true, in general. For example, $pnc(\mathbb{Z}_6) = pnc(\mathbb{Z}_{15}) = 1$, while $\mathbb{Z}_6 \ncong \mathbb{Z}_{15}$.

Usually, if $G$ and $H$ are two lattice isomorphic groups then $pnc(G) = pnc(H)$, as lattice isomorphisms preserve normal subgroups (for more details see [6], Theorem 1.2.10).

Also, it is obvious that $pnc(G_1 \times G_2) = pnc(G_1)pnc(G_2)$, for any two groups $G_1$ and $G_2$ with coprime orders. For

$$pnc(G_1 \times G_2) = \frac{|N^*(G_1 \times G_2)|}{|Ch(G_1 \times G_2)|} = \frac{|N^*(G_1)||N^*(G_2)|}{|Ch(G_1)||Ch(G_2)|} = pnc(G_1)pnc(G_2).$$

However, the above equality does not hold in general, for instance

$$pnc(S_3 \times \mathbb{Z}_2) = \frac{20}{68} \neq \frac{4}{10} = pnc(S_3).pnc(\mathbb{Z}_2).$$

On the other hand, if the groups in the direct factors are pairwise coprime orders, then one can extend the above equality to arbitrary direct products of finite groups.

**Proposition 2.5.** *Assume $G_1, G_2, \ldots, G_r$ are finite groups with mutually coprime orders. Then*

$$pnc(\Pi_{i=1}^r G_i) = \Pi_{i=1}^r pnc(G_i).$$

The following corollary explains that the probability of normality of chains of a finite nilpotent group can be obtained from its Sylow $p_i$-subgroups.

**Corollary 2.6.** *If $G$ is a finite nilpotent group and $P_1, P_2, \ldots, P_r$ are its Sylow $p_i$ subgroups, then*

$$pnc(G) = \Pi_{i=1}^r pnc(P_i).$$

# 3 Main Results

In this section, we give some inequalities which connects $pnc(G)$ with $pnc(N)$ and $pnc(\frac{G}{N})$, for any normal subgroup $N$ of $G$. It is obvious that if $N$ is the unique smallest normal subgroup of $G$, then

$$|N(G)| = |N(\frac{G}{N})| + 1.$$

**Theorem 3.1.** *Let $N$ be a normal subgroup of a finite group $G$, then the probability of normality of chains of $G$ has the following upper bound.*

$$pnc(G) \geq \frac{1}{|Ch(G)|}\left[pnc(N)|Ch(N)|pnc(\frac{G}{N})|Ch(\frac{G}{N})|\right].$$

**Proof.** We define the map

$$f : Ch(G) \longrightarrow \{0, 1\},$$

given by

$$f(C) = \begin{cases} 1, & \text{if all components of the chain } C \text{ are normal,} \\ 0, & \text{if the chain } C \text{ has a non-normal component.} \end{cases}$$

It is obvious that

$$pnc(G) = \frac{1}{|Ch(G)|}\sum_{C\in Ch(G)} f(C).$$

Let $N$ be a fixed normal subgroup of $G$ and consider the following sets:

$$\Gamma_1 = \{C \in Ch(G) : \text{all components of } C \text{ contains } N\},$$

and

$$\Gamma_2 = \{C \in Ch(G) : \text{all components of } C \text{ are proper subgroups of } N, \text{ except } G\}.$$

We infer that

$$\begin{aligned} pnc(G) &= \frac{1}{|Ch(G)|} \sum_{C \in Ch(G)} f(C) \\ &\geq \frac{1}{|Ch(G)|} \left( \sum_{C \in \Gamma_1} f(C) \sum_{C \in \Gamma_2} f(C) \right). \end{aligned} \tag{1}$$

Clearly, there exists a one-to-one corresponding between the subgroups (or normal subgroups) of $G$ containing $N$ and the ones of $\frac{G}{N}$. Hence one can calculate all three terms on the right hand side of (3.1), as follows:

$$\sum_{C \in \Gamma_1} f(C) = pnc(\frac{G}{N})|Ch(\frac{G}{N})|,$$

$$\sum_{C \in \Gamma_2} f(C) = pnc(N)|Ch(N)|,$$

and

$$\sum_{C \in \Gamma_1} f(C) \sum_{C \in \Gamma_2} f(C) = pnc(\frac{G}{N})|Ch(\frac{G}{N})|pnc(N)|Ch(N)|.$$

By replacing the above quantities in (3.1), it follows that

$$pnc(G) \geq \frac{1}{|Ch(G)|} \left[ pnc(N)|Ch(N)|pnc(\frac{G}{N})|Ch(\frac{G}{N})| \right],$$

which gives the result. $\square$

**Corollary 3.2.** *If $N$ is a normal subgroup of a finite group $G$ such that $N$ and $G/N$ are abelian, then*

$$pnc(G) \geq \frac{|Ch(N)||Ch(\frac{G}{N})|}{|Ch(G)|}.$$

**Corollary 3.3.** *If $G$ is a finite group with a normal subgroup $N$ of prime index, then*

$$pnc(G) \geq \frac{2pnc(N)|Ch(N)|}{|Ch(G)|}.$$

**Proof.** As the order of $G/N$ is a prime number, the only subgroups of $G$ containing $N$ are $G$ and $N$. Therefore the group $G/N$ has only two subgroups. Thus $pnc(\frac{G}{N}) = 1$, and so

$$\begin{aligned} pnc(G) &\geq \frac{1}{|Ch(G)|}\left[pnc(N)|Ch(N)|pnc(\frac{G}{N})|Ch(\frac{G}{N})|\right] \\ &= \frac{2pnc(N)|Ch(N)|}{|Ch(G)|}, \end{aligned}$$

as required. □

**Corollary 3.4.** *If $G$ is a finite soluble group with the series $1 = G_0 \subseteq G_1 \subseteq G_2 \subseteq \ldots \subseteq G_k = G$, in which $G_i/G_{i-1}$ is cyclic of prime order, for $i = 1, 2, \ldots, k$. Then*

$$pnc(G) \geq \frac{2^k}{|Ch(G)|}.$$

**Proof.** Using the previous corollary, we have $pnc(G_i)|Ch(G_i)| \geq 2pnc(G_{i-1})|Ch(G_{i-1})|$, for all $1 \leq i \leq k$. Summing up all these inequalities, they yield

$$pnc(G)|Ch(G)| \geq 2^k,$$

and so $pnc(G) \geq \frac{2^k}{|Ch(G)|}$. □

Now, we are able to calculate the probability of normality of chains of the groups $D_{2n}$, $Q_{2^n}$, $QD_{2^n}$, and $M_{p^n}$.

**Theorem 3.5.** *The probability of normality of chains of dihedral group $D_{2n}$ is*

$$pnc(D_{2n}) = \frac{1 + \sum_{k|n} |Ch(\mathbb{Z}_k)|}{\sum_{k|n} \frac{n}{k}|Ch(\mathbb{Z}_{\frac{n}{k}})|(k + |Ch(\mathbb{Z}_k)|) - (2n-1)|Ch(\mathbb{Z}_n)| + n},$$

*or*

$$pnc(D_{2n}) = \frac{1 + 4|Ch(\mathbb{Z}_{\frac{n}{2}})| + \sum_{k|n} |Ch(\mathbb{Z}_k)|}{\sum_{k|n} \frac{n}{k}|Ch(\mathbb{Z}_{\frac{n}{k}})|(k + |Ch(\mathbb{Z}_k)|) - (2n-1)|Ch(\mathbb{Z}_n)| + n},$$

*when $n$ is odd or even, respectively.*

**Proof.** By Lemma 2.3, if $n$ is odd then the normal subgroups of $D_{2n}$ are $\langle a^{\frac{n}{k}} \rangle$, for all $k|n$. Hence the chains with all normal components in $D_{2n}$ are of the form $\ldots \subset \langle a^{\frac{n}{k}} \rangle \subset D_{2n}$. So

$$N^*(D_{2n}) = 1 + \sum_{k|n} Ch(\mathbb{Z}_k),$$

which gives the result.

Now, assume $n$ is even, then all normal subgroups of $D_{2n}$ are $\langle a^{\frac{n}{k}} \rangle$, $k|n$, $\langle a^2, b \rangle$ and $\langle a^2, ab \rangle$. In this case, the chains with all normal components are as follows:

(1) Chains of the form $D_{2n}$, $\langle a^2, b \rangle \subset D_{2n}$ and $\langle a^2, ab \rangle \subset D_{2n}$,

(2) Chains of the form $\ldots \subset \langle a^{\frac{n}{k}} \rangle \subset D_{2n}$,

(3) Chains of the form $\ldots \subset \langle a^2, b \rangle \subset D_{2n}$,

(4) Chains of the form $\ldots \subset \langle a^2, ab \rangle \subset D_{2n}$.

In case (2), the number of chains is equal to $\sum_{k|n} |Ch(\mathbb{Z}_k)|$. In case (3), the chains are of the form $\ldots \subset \langle a^2 \rangle \subset \langle a^2, b \rangle \subset D_{2n}$ or $\ldots \subset X \subset \langle a^2, b \rangle \subset D_{2n}$, where $X$ is a proper subgroup of $\langle a^2 \rangle$. Hence the number of chains are equal to $|Ch(\mathbb{Z}_{\frac{n}{2}})|$ and $|Ch(\mathbb{Z}_{\frac{n}{2}})| - 1$, respectively. Case (4) is similar to (3), so the chains are of the following form

$$\cdots \subset \langle a^2 \rangle \subset \langle a^2, ab \rangle \subset D_{2n} \quad \text{or} \quad \cdots \subset X \subset \langle a^2, ab \rangle \subset D_{2n},$$

where $X$ is a proper subgroup of $\langle a^2\rangle$. Hence, the number of chains are equal to $|Ch(\mathbb{Z}_{\frac{n}{2}})|$ and $|Ch(\mathbb{Z}_{\frac{n}{2}})|-1$, respectively. Therefore

$$|N^*(D_{2n})| = 1 + 4|Ch(\mathbb{Z}_{\frac{n}{k}})| + \sum_{k|n} |Ch(\mathbb{Z}_k)|.$$

Now, the proof is obtained by using the definition of probability of normality of chains and applying Theorem 2.2 (i). $\square$

**Corollary 3.6.** *If $p$ is a prime number, then the probability of normality of chains of dihedral group $D_{2p^m}$ is equal to*

$$pnc(D_{2p^m}) = \frac{2p-2}{(p^{m+1}+p-2)}.$$

*In particular,*

$$pnc(D_{2^m}) = 2^{2-m}.$$

**Proof.** By Theorem 2.2(i), the number of chains of subgroups of the group $D_{2p^m}$ ending in $D_{2p^m}$ is equal to $|Ch(D_{2p^m})| = \frac{2^m}{p-1}(p^{m+1}+p-2)$. On the other hand, we have

$$\begin{aligned}|N^*(D_{2p^m})| &= \sum_{k|p^m} |Ch(\mathbb{Z}_k)| + 1 = \sum_{0\leq i\leq m} |Ch(\mathbb{Z}_{p^i})| + 1\\ &= 1 + 2 + \cdots + 2^m + 1 = 2^{m+1}.\end{aligned}$$

So the definition gives our claim. $\square$

Now, in the following result we obtain the probability of normality of chains of the generalized quaternion group, $Q_{2^n} = \langle x, y : x^{2^{n-1}} = y^4 = 1, x^y = x^{2^{n-1}-1}\rangle$ of order $2^n$ and the quasi-dihedral group $QD_{2^n} = \langle x, y : x^{2^{n-1}} = y^2 = 1, x^y = x^{2^{n-2}-1}\rangle$. Clearly $Z(QD_{2^n}) = \langle x^{2^{n-2}}\rangle$ and $QD_{2^n}/Z(QD_{2^n}) \cong D_{2^{n-1}}$. Furthermore, if $H$ is a subgroup of $QD_{2^n}$ such that $H \cap Z(QD_{2^n}) = \langle 1\rangle$, then either $H = \langle 1\rangle$ or $H = \langle x^{2i}y\rangle$, for some $0 \leq i \leq 2^{n-2}$. Hence the normal subgroups of $QD_{2^n}$ are $\langle x^{2^i}\rangle$ for $0 \leq i \leq n-1$, $\langle x^i y\rangle$ with $0 \leq i \leq 2^{n-2}$, $\langle x^2, y\rangle$ and $\langle x^2, xy\rangle$. It is easy to check that $Z(Q_{2^n}) = \langle x^{2^{n-2}}\rangle$ and $Q_{2^n}/Z(Q_{2^n}) \cong D_{2^{n-1}}$. Now, if $H$ is a subgroup of $Q_{2^n}$ such that $H \cap Z(Q_{2^n}) = \langle 1\rangle$, then either $H = \langle 1\rangle$ or $H = \langle x^{2i}y\rangle$, for some $0 \leq i \leq 2^{n-2}$. Hence the normal subgroups of $Q_{2^n}$ are $\langle x^{2^i}\rangle$, for $0 \leq i \leq n-1$. Therefore the number of normal subgroups of $Q_{2^n}$ is equal to $n + 3$. This argument leads to the following result.

**Theorem 3.7.** *The probability of normality of chains of the groups $Q_{2^n}$ and $QD_{2^n}$ are, respectively, equal to*

$$pnc(Q_{2^n}) = \frac{1}{2^{n-3}}, \quad \text{and} \quad pnc(QD_{2^n}) = \frac{1}{3.2^{n-4}}.$$

**Proof.** Using the normal subgroup lattices of $Q_{2^n}$ and $QD_{2^n}$ are calculated as follows:

$$\begin{aligned}|N^*(Q_{2^n})| &= 2(1+1+1+1+4+8+\cdots+2^{n-1})\\ &= 2^{n+1},\end{aligned}$$

and

$$\begin{aligned}|N^*(QD_{2^n})| &= 2(1+1+1+1+4+8+\cdots+2^{n-1})\\ &= 2^{n+1}.\end{aligned}$$

Now, by Theorem 2.2 (iii), (ii) and using "." for the product of two numbers, we obtain $|Ch(QD_{2^n})| = 3.2^{2n-3}$, and

$$|Ch(Q_{2^n})| = |Ch(Q_{4.2^{n-2}})| = |Ch(D_{2^{n-1}})| + 1 \times |Ch(D_{2^{n-1}})|$$

$$= 2|Ch(D_{2^{n-1}})| = 2.2^{2n-3} = 2^{2n-2},$$

from which the both results follow. $\square$

Finally, we calculate the probability of normality of chains of the modular $p$-groups of order $p^n$ with the following presentation

$$M_{p^n} = \langle x, y : x^{p^{n-1}} = y^p = 1, x^y = x^{p^{n-2}+1} \rangle.$$

One observes that $Z(M_{p^n}) = \langle x^{p^{n-2}} \rangle$ and $M_{p^n}/Z(M_{p^n}) \cong C_{p^{n-2}} \times C_p$. Furthermore, If $H$ is a subgroup of $M_{p^n}$ such that $H \cap Z(M_{p^n}) = \langle 1 \rangle$, then either $H = \langle 1 \rangle$ or $H = \langle x^{ip^{n-2}} y \rangle$, for $0 \leq i \leq p$. Hence the group $M_{p^n}$ has the following normal subgroups: $\langle x^{p^i} \rangle$, with $0 \leq i \leq n-1$, and $\langle x^{p^i}, x^j y \rangle$, with $1 \leq i \leq n-2$ and $1 \leq j \leq p-1$, in which $p$ is a prime number and $p^n \neq 8$.

**Theorem 3.8.** *The probability of normality of chains of the modular $p$-group $M_{p^n}$ is*

$$pnc(M_{p^n}) = \frac{2^{n-2}(2pn - 3p + 2) + 8(n-3)}{2^{n-1}(n-1)p + 2^n},$$

*where $n \geq 3$ and $p^n \neq 8$. In particular,*

$$pnc(M_{2^n}) = \frac{n2^n - 2^n + 8n - 24}{n2^n}.$$

**Proof.** One can easily check that

$$\begin{aligned}|N^*(M_{p^n})| &= 4p[2^{n-2} - 1 + 2^{n-3} - 1 + 2(2^{n-4} - 1) + 2^2(2^{n-5} - 1) \\ &\quad + 2^3(2^{n-6} - 1) + \cdots + 2^{n-5}(2^2 - 1)] + 2^{n-2}(p+2) + 8(n-3) + p.2^{n-2} \\ &= 2^{n-2}(2pn - 3p + 2) + 8(n-3).\end{aligned}$$

Now, Theorem 2.2($iv$) implies that the number of chains of modular $p$-group $M_{p^n}$ ($n \geq 3$ and $p^n \neq 8$) is equal to $2^{n-1}(n-1)p + 2^n$, which gives the result. $\square$

## References

[1] H. Darabi, F. Saeedi, and M. Farrokhi, The number of fuzzy subgroups of some non-abelian groups, *Iran. J. Fuzzy Syst*, 10:6 (2013), 101–107.

[2] H. Darabi and M. Imanparast, Counting number of fuzzy subgroups of some of dihedral groups, *Int. J. Pure Appl .Math.*, 85 (2013) 563–575.

[3] M. D. Millier, On the lattice of normal subgroups of a direct product, *Pacific Journal of Mathematics.* **60** (1975) 153–158.

[4] F. Saeedi, M. Farrokhi and S. H. Jafari, Subgroup normality degrees of finite groups I, *Arch. Math.*, 96 (2011) 215–224.

[5] M. Sajedi and M. R. R. Moghaddam, Probability of cyclicity of chains in finite groups, To appear in Southeast Asian Bull. of Mathematics.

[6] R. Schmidt, *Subgroup Lattices of Groups*, de Gruyter Expositions in Mathematics **14**, *de Gruyter, Berlin, 1994*.

[7] M. Tarnauceanu and L. Toth, Cyclicity degrees of finite groups, *Acta Math. Hungar.*, 145:2 (2015), 489–504.

[8] M. Tarnauceanu, Subgroup commutativity degrees of finite groups, *J. Algebra.*, 321 (2009) 2508–2520.

[9] M. Tarnauceanu, Normality degrees of finite groups, *Carpathian J. Math.*, 33:1 (2017), 115-126.

**Mostafa Sajedi**
Department of Mathematics,
Mashhad Branch, Islamic Azad University,
Mashhad, Iran
E-mail: m-sajedi1979@yahoo.com

**Mohammad Reza Rajabzadeh Moghaddam**
Department of Mathematics, Khayyam University, Mashhad, Iran
and Department of Pure Mathematics, Centre of Excellence in Analysis on Algebraic Structures (CEAAS), Ferdowsi University of Mashhad, P.O. Box 1159, Mashhad, Iran
E-mails: m.r.moghaddam@khayyam.ac.ir; rezam@ferdowsi.um.ac.ir

**Hamid Darabi**
Department of Mathematics,
Esfarayen University of Technology,
Esfarayen, Iran
E-mail: darabi@esfarayen.ac.ir