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Original Research Paper

Simpson-Type Inequalities for Conformable Fractional Operators Concerning Twice-Differentiable Functions

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Abstract. The authors of the paper propose a new method of investigation of an an equality for the case of twice-differentiable convex functions with respect to the conformable fractional integrals. With the help of this equality, we establish several Simpson-type inequalities for twice-differentiable convex functions by using conformable fractional integrals. Sundry significant inequalities are obtained by taking advantage of the convexity, the Hölder inequality, and the power mean inequality. By using the specific selection of our results, we give several new and well-known results in the literature.

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1 Introduction and Preliminaries

Investigating inequality theory is crucial due to its widespread applications in various fields of science. It is widely acknowledged that the Hermite–Hadamard-type inequalities are a prominent inequalities for the case of convex functions. Convexity theory is an interesting and coercive methodology for the case of considering the great issues that arise in several different areas of the pure and applied sciences. Sundry structures have been presented and explored, including convex sets and related functions. This theory has a rich history and has been the focus and motivation of special mathematical research. Moreover, convexity theory has a critical place in the advancement of the idea of inequality. There are many types of convexity in the literature.

Definition 1.1. [23] Suppose that I is an interval of real numbers. Then, a function $f : I \to \mathbb{R}$ is said to be convex, if

$$f\left(tx+(1-t)\,y\right) \leq tf\left(x\right)+(1-t)\,f\left(y\right)$$

is satisfied $\forall x, y \in I$ and $\forall t \in [0,1]$.

Inequalities have an interesting mathematical model because of their important applications in traditional calculus, fractional calculus, quantum calculus, interval-valued, fractal sets, etc. Recently, many researchers, including mathematicians and engineers, have dedicated themselves to considering the inequalities and properties associated with convexity. Many integral inequalities have been developed so far by different mathematicians. In the literature, we have many types of inequalities that include convex functions, such as Hermite–Hadamard-type inequalities, Simpson-type inequalities, Bullen-type inequalities, etc. Hence, there are a lot of well-known integral inequalities but the most notable one is the Simpson-type inequalities.

Theorem 1.2. *[9] Let $f : [a,b] \to \mathbb{R}$ be a four times continuously differentiable function on (a,b) and $\left\|f^{(4)}\right\|_{\infty} = \sup_{x\in(a,b)} |f^{(4)}(x)| < \infty$. Then, the following inequality holds:*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq \frac{1}{2880}\left\|f^{(4)}\right\|_{\infty}(b-a)^{4}.$$

Simpson-type inequality via fractional calculus have been considered widely by many interested researchers. Fractional calculus is a field of mathematics that expands the traditional derivative and integral ideas to non-integer orders. The popularity of this topic among mathematicians continues to increase very strongly in recent years (see [8, 11, 25, 5, 3]). Riemann-Liouville fractional integrals, conformable fractional integrals, and many types of fractional integrals have been investigated with Simpson-type inequalities. Fractional derivatives are also used to model a wide range of mathematical biology, as well as chemical processes, physics and engineering problems [15, 4, 10]. By using the derivative's fundamental limit formulation, a newly well-behaved fundamental fractional derivative known as the conformable derivative is improved in the paper [20]. Several major requirements that cannot be implemented by the Riemann-Liouville and Caputo definitions are implemented by the conformable derivative. On the other hand, in paper [2] the author established that the conformable approach in [20] cannot yield good results when compared to the Caputo definition for specific functions. This flaw in the conformable definition was avoided by several extensions of the conformable approach [26, 14, 21].

Let us put forth some preliminaries which will be utilized in the sequel. The fundamental definitions of Riemann-Liouville integrals and conformable integrals, which are used throughout the paper, are given as follows:

Definition 1.3. The gamma function, beta function, and incomplete beta function are represented

$$\Gamma(x) := \int_{0}^{\infty} t^{x-1}e^{-t}dt,$$

$$\mathfrak{B}(x,y) := \int_{0}^{1} t^{x-1}(1-t)^{y-1}dt,$$

and

$$\mathscr{B}(x,y,r) := \int_0^r t^{x-1}(1-t)^{y-1}\,dt,$$

respectively for $0 < x, y < \infty$ and $x, y \in \mathbb{R}$.

Kilbas et al. [19] defined fractional integrals, also called Riemann-Liouville integrals as follows:

Definition 1.4. [19] The Riemann-Liouville integrals $J_{a+}^{\beta} f(x)$ and $J_{b-}^{\beta} f(x)$ of order $\beta > 0$ are given by

$$J_{a+}^{\beta} f(x) = \frac{1}{\Gamma(\beta)} \int_a^x (x-t)^{\beta-1} f(t)dt, \quad x > a \tag{1}$$

and

$$J_{b-}^{\beta} f(x) = \frac{1}{\Gamma(\beta)} \int_x^b (t-x)^{\beta-1} f(t)dt, \quad x < b, \tag{2}$$

respectively for $f \in L_1[a,b]$. Note that the Riemann-Liouville integrals coincides with the classical integrals for the case of $\beta = 1$.

Many mathematicians have considered the twice-differentiable convex functions in order to get sundry significant inequalities. For example, several fractional Simpson-type inequalities [12] established for functions whose second derivatives in absolute value are convex. Moreover, several generalized trapezoid-type and midpoint-type fractional integral inequalities for the case of twice-differentiable convex functions are obtained in paper [22]. Furthermore, Sarikaya and Aktan [24] proved several new inequalities of the Simpson-type and the Hermite–Hadamard-type for functions whose absolute values of derivatives are convex. For results connected with several properties of Riemann-Liouville fractional integrals and twice-differentiable convex functions one can see Refs. [7, 13, 12, 18] and the references therein.

Jarad et al. [17] established the fractional conformable integral operators. They also derived certain characteristics and relationships between these operators and several other fractional operators in the literature. The fractional conformable integral operators are described as follows:

Definition 1.5. [17] The conformable fractional operators ${}^{\beta}\mathcal{J}_{a+}^{\alpha}f(x)$ and ${}^{\beta}\mathcal{J}_{b-}^{\alpha}f(x)$ of order $\beta \in C$, $Re(\beta) > 0$ and $\alpha \in (0,1]$ are given by

$$ {}^{\beta}\mathcal{J}_{a+}^{\alpha}f(x) = \frac{1}{\Gamma(\beta)}\int_a^x \left(\frac{(x-a)^{\alpha}-(t-a)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{f(t)}{(t-a)^{1-\alpha}}dt, \quad t > a \tag{3} $$

and

$$ {}^{\beta}\mathcal{J}_{b-}^{\alpha}f(x) = \frac{1}{\Gamma(\beta)}\int_x^b \left(\frac{(b-x)^{\alpha}-(b-t)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{f(t)}{(b-t)^{1-\alpha}}dt, \quad t < b, \tag{4} $$

respectively for $f \in L_1[a,b]$.

If we consider $\alpha = 1$ in (3), then the fractional integral in (3) equals to the Riemann-Liouville fractional integral in (1). Moreover, the fractional integral in (4) is equal to the Riemann-Liouville fractional integral in (2) if $\alpha = 1$. There have been a great number of research papers written on these subjects, [1, 16] and the references therein.

The purpose of this paper is to prove Simpson-type inequalities for the case of twice-differentiable convex functions with respect to conformable fractional integrals. The entire form of the study takes the form of three sections including introduction. Here, we also presented the fundamental definitions of convex functions, Riemann-Liouville integrals and conformable integrals in order to build our principal outcomes. In Section 2, an equality will be established for the case of twice-differentiable convex functions according to the conformable fractional integrals. With the help of this equality, we will present several Simpson-type inequalities for twice-differentiable convex functions related to conformable fractional integrals. More precisely, several important inequalities are obtained by taking advantage of the convexity, the Hölder inequality, and the power mean inequality. Furthermore, we also give several corollaries and remarks in this section. Finally, concluding remarks are given in Section 3.

2 Statement of the Problem

In this section, Simpson-type inequalities are created for the case of twice-differentiable convex functions with respect to the conformable fractional integrals. Let us first set up the following identity in order to obtain conformable fractional versions of Simpson-type inequalities.

Lemma 2.1. *Let us consider that $f:[a,b]\to\mathbb{R}$ is a twice-differentiable function on (a,b) such that $f''\in L_1[a,b]$. Then, the following equality holds:*

$$\begin{aligned}
&\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(b-a)^{\alpha\beta}}\left[\ {}^{\beta}\mathcal{J}_{b-}^{\alpha}f(a)+\ {}^{\beta}\mathcal{J}_{a+}^{\alpha}f(b)\right]-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right] \qquad (5)\\
&\qquad=\frac{(b-a)^{2}\alpha^{\beta}}{2}\sum_{i=1}^{4}\Omega_{i},
\end{aligned}$$

where ${}^{\beta}\mathcal{J}_{b-}^{\alpha}$ and ${}^{\beta}\mathcal{J}_{a+}^{\alpha}$ denote conformable fractional operators defined in Definition 1.5 and

$$\begin{cases}
\Omega_1=\int\limits_0^{\frac12}\left(\int\limits_0^{t}\left[\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{3\alpha^{\beta}}\left(1-\left(\frac12\right)^{\alpha}\right)^{\beta}\right]ds\right)f''(tb+(1-t)a)\,dt,\\
\Omega_2=\int\limits_0^{\frac12}\left(\int\limits_0^{t}\left[\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{3\alpha^{\beta}}\left(1-\left(\frac12\right)^{\alpha}\right)^{\beta}\right]ds\right)f''(ta+(1-t)b)\,dt,\\
\Omega_3=\int\limits_{\frac12}^{1}\left(\int\limits_t^{1}\left[\frac{2}{3\alpha^{\beta}}+\frac{1}{3\alpha^{\beta}}\left(1-\left(\frac12\right)^{\alpha}\right)^{\beta}-\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}\right]ds\right)\\
\qquad\times f''(tb+(1-t)a)\,dt,\\
\Omega_4=\int\limits_{\frac12}^{1}\left(\int\limits_t^{1}\left[\frac{2}{3\alpha^{\beta}}+\frac{1}{3\alpha^{\beta}}\left(1-\left(\frac12\right)^{\alpha}\right)^{\beta}-\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}\right]ds\right)\\
\qquad\times f''(ta+(1-t)b)\,dt.
\end{cases}$$

Proof. From the fact of the integrating by parts, it follows

$$\begin{aligned}
\Omega_1&=\int\limits_0^{\frac12}\left(\int\limits_0^{t}\left[\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{3\alpha^{\beta}}\left(1-\left(\frac12\right)^{\alpha}\right)^{\beta}\right]ds\right) \qquad (6)\\
&\quad\times f''(tb+(1-t)a)\,dt\\
&=\frac{1}{b-a}\left(\int\limits_0^{t}\left[\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{3\alpha^{\beta}}\left(1-\left(\frac12\right)^{\alpha}\right)^{\beta}\right]ds\right)\\
&\quad\times f'(tb+(1-t)a)\Big|_0^{\frac12}
\end{aligned}$$

$$
\begin{aligned}
& -\frac{1}{b-a}\int\limits_{0}^{\frac{1}{2}}\left[\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{3\alpha^{\beta}}\left(1-\left(\frac{1}{2}\right)^{\alpha}\right)^{\beta}\right] \\
& \times f'\left(tb+(1-t)a\right)dt \\
= & \frac{1}{b-a}\left(\int\limits_{0}^{\frac{1}{2}}\left[\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{3\alpha^{\beta}}\left(1-\left(\frac{1}{2}\right)^{\alpha}\right)^{\beta}\right]ds\right) \\
& \times f'\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\left\{\frac{1}{b-a}\left[\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}\right.\right. \\
& \left.-\frac{1}{3\alpha^{\beta}}\left(1-\left(\frac{1}{2}\right)^{\alpha}\right)^{\beta}\right]f\left(tb+(1-t)a\right)\Big|_{0}^{\frac{1}{2}} \\
& \left.-\frac{\beta}{b-a}\int\limits_{0}^{\frac{1}{2}}\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta-1}(1-t)^{\alpha-1}f\left(tb+(1-t)a\right)dt\right\} \\
= & \frac{1}{b-a}\left(\int\limits_{0}^{\frac{1}{2}}\left[\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{3\alpha^{\beta}}\left(1-\left(\frac{1}{2}\right)^{\alpha}\right)^{\beta}\right]ds\right) \\
& \times f'\left(\frac{a+b}{2}\right)-\frac{1}{3\left(b-a\right)^{2}}\left(\frac{1-\left(\frac{1}{2}\right)^{\alpha}}{\alpha}\right)^{\beta}\left(2f\left(\frac{a+b}{2}\right)+f\left(a\right)\right) \\
& +\frac{\beta}{\left(b-a\right)^{2}}\int\limits_{0}^{\frac{1}{2}}\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta-1}(1-t)^{\alpha-1}f\left(tb+(1-t)a\right)dt.
\end{aligned}
$$

Similar to foregoing process, we get

$$
\begin{aligned}
\Omega_{2}= & -\frac{1}{b-a}\left(\int\limits_{0}^{\frac{1}{2}}\left[\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{3\alpha^{\beta}}\left(1-\left(\frac{1}{2}\right)^{\alpha}\right)^{\beta}\right]ds\right) \quad (7) \\
& \times f'\left(\frac{a+b}{2}\right)-\frac{1}{3\left(b-a\right)^{2}}\left(\frac{1-\left(\frac{1}{2}\right)^{\alpha}}{\alpha}\right)^{\beta}\left(2f\left(\frac{a+b}{2}\right)+f\left(b\right)\right)
\end{aligned}
$$

$$+\frac{\beta}{(b-a)^2}\int_0^{\frac{1}{2}}\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\beta-1}(1-t)^{\alpha-1}f(ta+(1-t)b)\,dt,$$

$$\begin{aligned}\Omega_3 = & -\frac{1}{b-a}\left(\int_{\frac{1}{2}}^{1}\left[\frac{2}{3\alpha^\beta}+\frac{1}{3\alpha^\beta}\left(1-\left(\frac{1}{2}\right)^\alpha\right)^\beta-\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta\right]ds\right) \\ & \times f'\left(\frac{a+b}{2}\right)-\frac{1}{3(b-a)^2} \\ & \times\left[\frac{1}{\alpha^\beta}-\left(\frac{1-\left(\frac{1}{2}\right)^\alpha}{\alpha}\right)^\beta\right]\left(2f\left(\frac{a+b}{2}\right)+f(b)\right) \\ & +\frac{\beta}{(b-a)^2}\int_{\frac{1}{2}}^{1}\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\beta-1}(1-t)^{\alpha-1}f(tb+(1-t)a)\,dt,\end{aligned} \tag{8}$$

and

$$\begin{aligned}\Omega_4 = & \frac{1}{b-a}\left(\int_{\frac{1}{2}}^{1}\left[\frac{2}{3\alpha^\beta}+\frac{1}{3\alpha^\beta}\left(1-\left(\frac{1}{2}\right)^\alpha\right)^\beta-\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta\right]ds\right) \\ & \times f'\left(\frac{a+b}{2}\right)-\frac{1}{3(b-a)^2} \\ & \times\left[\frac{1}{\alpha^\beta}-\left(\frac{1-\left(\frac{1}{2}\right)^\alpha}{\alpha}\right)^\beta\right]\left(2f\left(\frac{a+b}{2}\right)+f(a)\right) \\ & +\frac{\beta}{(b-a)^2}\int_{\frac{1}{2}}^{1}\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\beta-1}(1-t)^{\alpha-1}f(ta+(1-t)b)\,dt.\end{aligned} \tag{9}$$

If we collect from equality (6) to equality (9), then we can obtain

$$\sum_{i=1}^{4}\Omega_i = \frac{\beta}{(b-a)^2}\int_0^1 \left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta-1}(1-t)^{\alpha-1} f(tb+(1-t)a)\,dt \tag{10}$$
$$+\frac{\beta}{(b-a)^2}\int_0^1 \left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta-1}(1-t)^{\alpha-1} f(ta+(1-t)b)\,dt$$
$$-\frac{1}{3(b-a)^2\alpha^{\beta}}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right].$$

If we use change of variables in (10), then equality (10) is converted as follows:

$$\begin{aligned}\sum_{i=1}^{4}\Omega_i &= \left(\frac{1}{b-a}\right)^{\alpha\beta+2}\frac{\Gamma(\beta+1)}{\Gamma(\beta)} \\ &\quad\times\int_a^b \left(\frac{(b-a)^{\alpha}-(b-x)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{f(x)}{(b-x)^{1-\alpha}}dx \\ &\quad+\left(\frac{1}{b-a}\right)^{\alpha\beta+2}\frac{\Gamma(\beta+1)}{\Gamma(\beta)} \\ &\quad\times\int_a^b \left(\frac{(b-a)^{\alpha}-(x-a)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{f(x)}{(x-a)^{1-\alpha}}dx \\ &\quad-\frac{1}{3(b-a)^2\alpha^{\beta}}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right] \\ &= \frac{\Gamma(\beta+1)}{(b-a)^{\alpha\beta+2}}\left[{}^{\beta}\mathcal{J}_{b-}^{\alpha}f(a)+{}^{\beta}\mathcal{J}_{a+}^{\alpha}f(b)\right] \\ &\quad-\frac{1}{3(b-a)^2\alpha^{\beta}}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right].\end{aligned} \tag{11}$$

If equality (11) is multiplied by $\frac{(b-a)^2\alpha^{\beta}}{2}$, then the proof of Lemma 2.1 is completed. □

Theorem 2.2. *Suppose that $f:[a,b]\to\mathbb{R}$ is a twice-differentiable function on (a,b) such that $f''\in L_1[a,b]$ and $|f''|$ is convex on $[a,b]$. Then, the following inequality*

$$\left|\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(b-a)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}_{b-}^{\alpha}f(a)+{}^{\beta}\mathcal{J}_{a+}^{\alpha}f(b)\right]-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right|$$
$$\leq\frac{(b-a)^2\alpha^{\beta}}{2}\left\{\varphi_1(\alpha,\beta)+\varphi_2(\alpha,\beta)\right\}\left[\left|f''(a)\right|+\left|f''(b)\right|\right]$$

is valid. Here, ${}^{\beta}\mathcal{J}_{b-}^{\alpha}$ and ${}^{\beta}\mathcal{J}_{a+}^{\alpha}$ denote conformable fractional operators defined in Definition 1.5 and

$$\begin{cases}\varphi_1(\alpha,\beta)=\int\limits_0^{\frac{1}{2}}\left|\int\limits_0^{t}\left[\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{3\alpha^{\beta}}\left(1-\left(\frac{1}{2}\right)^{\alpha}\right)^{\beta}\right]ds\right|dt\\ =\int\limits_0^{\frac{1}{2}}\left|\frac{1}{\alpha^{\beta+1}}\left\{\mathfrak{B}\left(\frac{1}{\alpha},\beta+1\right)-\mathscr{B}\left(\frac{1}{\alpha},\beta+1,(1-t)^{\alpha}\right)\right\}\right.\\ \qquad\left.-\frac{1}{3\alpha^{\beta}}\left(1-\left(\frac{1}{2}\right)^{\alpha}\right)^{\beta}t\right|dt,\\ \varphi_2(\alpha,\beta)=\int\limits_{\frac{1}{2}}^{1}\left|\int\limits_t^{1}\left[\frac{2}{3\alpha^{\beta}}+\frac{1}{3\alpha^{\beta}}\left(1-\left(\frac{1}{2}\right)^{\alpha}\right)^{\beta}-\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}\right]ds\right|dt\\ =\frac{1}{\alpha^{\beta}}\int\limits_{\frac{1}{2}}^{1}\left|\left\{\frac{2}{3}+\frac{1}{3}\left(1-\left(\frac{1}{2}\right)^{\alpha}\right)^{\beta}\right\}(1-t)-\frac{1}{\alpha}\mathscr{B}\left(\frac{1}{\alpha},\beta+1,(1-t)^{\alpha}\right)\right|dt,\end{cases}\tag{12}$$

where $\mathfrak{B}$ and $\mathscr{B}$ denote the beta function and incomplete beta function, respectively.

Proof. Let us take the absolute value of both sides of (5). Then, we have the following inequality

$$\left|\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(b-a)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}_{b-}^{\alpha}f(a)+{}^{\beta}\mathcal{J}_{a+}^{\alpha}f(b)\right]-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right|\tag{13}$$

$$\leq\frac{(b-a)^2\alpha^{\beta}}{2}\left\{\int\limits_0^{\frac{1}{2}}\left|\int\limits_0^{t}\left[\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{3\alpha^{\beta}}\left(1-\left(\frac{1}{2}\right)^{\alpha}\right)^{\beta}\right]ds\right|\right.$$

$$
\begin{aligned}
& \times\left|f''(tb+(1-t)a)\right|dt \\
& +\int_0^{\frac{1}{2}}\left|\int_0^t\left[\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta-\frac{1}{3\alpha^\beta}\left(1-\left(\frac{1}{2}\right)^\alpha\right)^\beta\right]ds\right| \\
& \times\left|f''(ta+(1-t)b)\right|dt \\
& +\int_{\frac{1}{2}}^1\left|\int_t^1\left[\frac{2}{3\alpha^\beta}+\frac{1}{3\alpha^\beta}\left(1-\left(\frac{1}{2}\right)^\alpha\right)^\beta-\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta\right]ds\right| \\
& \times\left|f''(tb+(1-t)a)\right|dt \\
& +\int_{\frac{1}{2}}^1\left|\int_t^1\left[\frac{2}{3\alpha^\beta}+\frac{1}{3\alpha^\beta}\left(1-\left(\frac{1}{2}\right)^\alpha\right)^\beta-\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta\right]ds\right| \\
& \times\left|f''(ta+(1-t)b)\right|dt\Big\}.
\end{aligned}
$$

Since $|f''|$ is convex on $[a,b]$, we can easily obtain

$$
\begin{aligned}
& \left|\frac{\alpha^\beta\Gamma(\beta+1)}{2(b-a)^{\alpha\beta}}\left[\,{}^\beta\mathcal{J}^\alpha_{b-}f(a)+\,{}^\beta\mathcal{J}^\alpha_{a+}f(b)\right]-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right| \\
& \leq\frac{(b-a)^2\alpha^\beta}{2}\left\{\int_0^{\frac{1}{2}}\left|\int_0^t\left[\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta-\frac{1}{3\alpha^\beta}\left(1-\left(\frac{1}{2}\right)^\alpha\right)^\beta\right]ds\right|\right. \\
& \quad\times\left[t\left|f''(b)\right|+(1-t)\left|f''(a)\right|+t\left|f''(a)\right|+(1-t)\left|f''(b)\right|\right]dt \\
& \quad+\int_{\frac{1}{2}}^1\left|\int_t^1\left[\frac{2}{3\alpha^\beta}+\frac{1}{3\alpha^\beta}\left(1-\left(\frac{1}{2}\right)^\alpha\right)^\beta-\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta\right]ds\right| \\
& \quad\left.\times\left[t\left|f''(b)\right|+(1-t)\left|f''(a)\right|+t\left|f''(a)\right|+(1-t)\left|f''(b)\right|\right]dt\right\} \\
& =\frac{(b-a)^2\alpha^\beta}{2}\left\{\int_0^{\frac{1}{2}}\left|\int_0^t\left[\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta-\frac{1}{3\alpha^\beta}\left(1-\left(\frac{1}{2}\right)^\alpha\right)^\beta\right]ds\right|dt\right.
\end{aligned}
$$

$$+\int_{\frac{1}{2}}^{1}\left|\int_{t}^{1}\left[\frac{2}{3\alpha^{\beta}}+\frac{1}{3\alpha^{\beta}}\left(1-\left(\frac{1}{2}\right)^{\alpha}\right)^{\beta}-\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}\right]ds\right|dt\Bigg\}$$
$$\times\left[\left|f''(a)\right|+\left|f''(b)\right|\right].$$

This finishes the proof of Theorem 2.2. □

Remark 2.3. Let us consider $\alpha=1$ and $\beta=1$ in Theorem 2.2. Then, Theorem 2.2 reduces to

$$\left|\frac{1}{b-a}\int_{a}^{b}f(x)\,dx-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right|$$
$$\leq\frac{(b-a)^{2}}{162}\left[\left|f''(a)\right|+\left|f''(b)\right|\right],$$

which is given in [24, Proposition 1].

Theorem 2.4. *Assume that $f:[a,b]\to\mathbb{R}$ is a twice-differentiable function on (a,b) such that $f''\in L_1[a,b]$. Assume also that $|f''|^q$ is convex on $[a,b]$ with $q>1$. Then, the following inequalities hold:*

$$\left|\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(b-a)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}_{b-}^{\alpha}f(a)+{}^{\beta}\mathcal{J}_{a+}^{\alpha}f(b)\right]-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right| \tag{14}$$
$$\leq\frac{(b-a)^{2}\alpha^{\beta}}{2^{1+\frac{1}{q}}}\left(\left(\psi_1^{\alpha,\beta}(p)\right)^{\frac{1}{p}}+\left(\psi_2^{\alpha,\beta}(p)\right)^{\frac{1}{p}}\right)$$
$$\times\left[\left(\frac{|f''(b)|^{q}+3|f''(a)|^{q}}{4}\right)^{\frac{1}{q}}+\left(\frac{|f''(a)|^{q}+3|f''(b)|^{q}}{4}\right)^{\frac{1}{q}}\right]$$
$$\leq\frac{(b-a)\alpha^{\beta}}{2^{1+\frac{1}{q}}}\left(\left(4\psi_1^{\alpha,\beta}(p)\right)^{\frac{1}{p}}+\left(4\psi_2^{\alpha,\beta}(p)\right)^{\frac{1}{p}}\right)\left[\left|f''(a)\right|+\left|f''(b)\right|\right].$$

Here, $\frac{1}{p}+\frac{1}{q}=1$ and

$$\begin{cases}\psi_1^{\alpha,\beta}(p)=\int_{0}^{\frac{1}{2}}\left|\int_{0}^{t}\left[\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{3\alpha^{\beta}}\left(1-\left(\frac{1}{2}\right)^{\alpha}\right)^{\beta}\right]ds\right|^{p}dt,\\ \psi_2^{\alpha,\beta}(p)=\int_{\frac{1}{2}}^{1}\left|\int_{t}^{1}\left[\frac{2}{3\alpha^{\beta}}+\frac{1}{3\alpha^{\beta}}\left(1-\left(\frac{1}{2}\right)^{\alpha}\right)^{\beta}-\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}\right]ds\right|^{p}dt.\end{cases}$$

Proof. If we use Hölder inequality in (13), then we obtain

$$\left|\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(b-a)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}_{b-}^{\alpha}f(a)+{}^{\beta}\mathcal{J}_{a+}^{\alpha}f(b)\right]-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right|$$
$$\leq\frac{(b-a)^{2}\alpha^{\beta}}{2}$$
$$\times\left\{\left(\int_{0}^{\frac{1}{2}}\left|\int_{0}^{t}\left[\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{3\alpha^{\beta}}\left(1-\left(\frac{1}{2}\right)^{\alpha}\right)^{\beta}\right]ds\right|^{p}dt\right)^{\frac{1}{p}}\right.$$
$$\times\left[\left(\int_{0}^{\frac{1}{2}}\left|f''(tb+(1-t)a)\right|^{q}dt\right)^{\frac{1}{q}}+\left(\int_{0}^{\frac{1}{2}}\left|f''(ta+(1-t)b)\right|^{q}dt\right)^{\frac{1}{q}}\right]$$
$$+\left(\int_{\frac{1}{2}}^{1}\left|\int_{t}^{1}\left[\frac{2}{3\alpha^{\beta}}+\frac{1}{3\alpha^{\beta}}\left(1-\left(\frac{1}{2}\right)^{\alpha}\right)^{\beta}-\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}\right]ds\right|^{p}dt\right)^{\frac{1}{p}}$$
$$\left.\times\left[\left(\int_{\frac{1}{2}}^{1}\left|f''(tb+(1-t)a)\right|^{q}dt\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^{1}\left|f''(ta+(1-t)b)\right|^{q}dt\right)^{\frac{1}{q}}\right]\right\}.$$

Note that $|f''|^{q}$ is convex on $[a,b]$. Then, we get

$$\left|\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(b-a)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}_{b-}^{\alpha}f(a)+{}^{\beta}\mathcal{J}_{a+}^{\alpha}f(b)\right]-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right|$$
$$\leq\frac{(b-a)^{2}\alpha^{\beta}}{2}$$
$$\times\left\{\left(\int_{0}^{\frac{1}{2}}\left(\int_{0}^{t}\left[\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{3\alpha^{\beta}}\left(1-\left(\frac{1}{2}\right)^{\alpha}\right)^{\beta}\right]ds\right)^{p}dt\right)^{\frac{1}{p}}\right.$$

$$
\begin{aligned}
&\times\left[\left(\int_0^{\frac{1}{2}}\left(t\left|f''(b)\right|^q+(1-t)\left|f''(a)\right|^q\right)dt\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_0^{\frac{1}{2}}\left(t\left|f''(a)\right|^q+(1-t)\left|f''(b)\right|^q\right)dt\right)^{\frac{1}{q}}\right]\\
&+\left(\int_{\frac{1}{2}}^{1}\left(\int_t^1\left[\frac{2}{3\alpha^\beta}+\frac{1}{3\alpha^\beta}\left(1-\left(\frac{1}{2}\right)^\alpha\right)^\beta-\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta\right]ds\right)^p dt\right)^{\frac{1}{p}}\\
&\times\left[\left(\int_{\frac{1}{2}}^{1}\left(t\left|f''(b)\right|^q+(1-t)\left|f''(a)\right|^q\right)dt\right)^{\frac{1}{q}}\right.\\
&\left.\left.+\left(\int_{\frac{1}{2}}^{1}\left(t\left|f''(a)\right|^q+(1-t)\left|f''(b)\right|^q\right)dt\right)^{\frac{1}{q}}\right]\right\}\\
&=\frac{(b-a)^2\alpha^\beta}{2^{1+\frac{1}{q}}}\left(\left(\psi_1^{\alpha,\beta}(p)\right)^{\frac{1}{p}}+\left(\psi_2^{\alpha,\beta}(p)\right)^{\frac{1}{p}}\right)\\
&\times\left[\left(\frac{\left|f''(b)\right|^q+3\left|f''(a)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{\left|f''(a)\right|^q+3\left|f''(b)\right|^q}{4}\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

The second inequality of Theorem 2.4 can be obtained simultaneously by letting $\phi_1 = 3\left|f''(a)\right|^q$, $\varrho_1 = \left|f''(b)\right|^q$, $\phi_2 = \left|f''(a)\right|^q$ and $\varrho_2 = 3\left|f''(b)\right|^q$ and applying the following inequality:

$$\sum_{k=1}^{n}(\phi_k+\varrho_k)^s \leq \sum_{k=1}^{n}\phi_k^s+\sum_{k=1}^{n}\varrho_k^s, \qquad 0 \leq s < 1,$$

which ends the proof of Theorem 2.4. □

Corollary 2.5. *Let us consider $\alpha = 1$ and $\beta = 1$ in Theorem 2.4. Then,* *we have*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right| \\
&\leq \frac{(b-a)^2}{3\cdot 2^{2+\frac{1}{q}}}\left(\left(\int_0^{\frac{1}{2}}\left|3t^2-t\right|^p dt\right)^{\frac{1}{p}}+\left(\int_{\frac{1}{2}}^1\left|2-5t+3t^2\right|^p dt\right)^{\frac{1}{p}}\right) \\
&\quad\times\left[\left(\frac{|f'(b)|^q+3|f'(a)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^q+3|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right] \\
&\leq \frac{(b-a)^2}{3\cdot 2^{2+\frac{1}{q}}}\left[\left(4\int_0^{\frac{1}{2}}\left|3t^2-t\right|^p dt\right)^{\frac{1}{p}}\right. \\
&\quad\left.+\left(4\int_{\frac{1}{2}}^1\left|2-5t+3t^2\right|^p dt\right)^{\frac{1}{p}}\right]\left[\left|f''(a)\right|+\left|f''(b)\right|\right].
\end{aligned}$$

Theorem 2.6. *If $f:[a,b]\to\mathbb{R}$ is a twice-differentiable function on (a,b) such that $f''\in L_1[a,b]$ and $|f''|^q$ is convex on $[a,b]$ with $q\geq 1$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(b-a)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}_{b-}^{\alpha}f(a)+{}^{\beta}\mathcal{J}_{a+}^{\alpha}f(b)\right]-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right| \\
&\leq \frac{(b-a)^2\alpha^{\beta}}{2}\Big\{(\varphi_1(\alpha,\beta))^{1-\frac{1}{q}} \\
&\quad\times\Big[\left(\varphi_3(\alpha,\beta)\left|f''(b)\right|^q+(\varphi_1(\alpha,\beta)-\varphi_3(\alpha,\beta))\left|f''(a)\right|^q\right)^{\frac{1}{q}} \\
&\quad+\left(\varphi_3(\alpha,\beta)\left|f''(a)\right|^q+(\varphi_1(\alpha,\beta)-\varphi_3(\alpha,\beta))\left|f''(b)\right|^q\right)^{\frac{1}{q}}\Big] \\
&\quad+(\varphi_2(\alpha,\beta))^{1-\frac{1}{q}}\Big[\left(\varphi_4(\alpha,\beta)\left|f''(b)\right|^q+(\varphi_2(\alpha,\beta)-\varphi_4(\alpha,\beta))\left|f''(a)\right|^q\right)^{\frac{1}{q}} \\
&\quad+\left(\varphi_4(\alpha,\beta)\left|f''(a)\right|^q+(\varphi_2(\alpha,\beta)-\varphi_4(\alpha,\beta))\left|f''(b)\right|^q\right)^{\frac{1}{q}}\Big]\Big\}.
\end{aligned}$$

Here, ${}^{\beta}\mathcal{J}_{b-}^{\alpha}$ and ${}^{\beta}\mathcal{J}_{a+}^{\alpha}$ denote conformable fractional operators defined in Definition 1.5 and $\varphi_1(\alpha,\beta)$, $\varphi_2(\alpha,\beta)$ are described in (12) and

$$\left\{\begin{array}{l}\varphi_3(\alpha,\beta)=\int\limits_0^{\frac{1}{2}} t\left|\int\limits_0^t\left[\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{3\alpha^{\beta}}\left(1-\left(\frac{1}{2}\right)^{\alpha}\right)^{\beta}\right]ds\right|dt\\ =\int\limits_0^{\frac{1}{2}} t\left|\frac{1}{\alpha^{\beta+1}}\left\{\mathfrak{B}\left(\frac{1}{\alpha},\beta+1\right)-\mathscr{B}\left(\frac{1}{\alpha},\beta+1,(1-t)^{\alpha}\right)\right\}\right.\\ \qquad\left.-\frac{1}{3\alpha^{\beta}}\left(1-\left(\frac{1}{2}\right)^{\alpha}\right)^{\beta}t\right|dt,\\ \varphi_4(\alpha,\beta)=\int\limits_{\frac{1}{2}}^{1} t\left|\int\limits_t^1\left[\frac{2}{3\alpha^{\beta}}+\frac{1}{3\alpha^{\beta}}\left(1-\left(\frac{1}{2}\right)^{\alpha}\right)^{\beta}-\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}\right]ds\right|dt\\ =\frac{1}{\alpha^{\beta}}\int\limits_{\frac{1}{2}}^{1} t\left|\left\{\frac{2}{3}+\frac{1}{3}\left(1-\left(\frac{1}{2}\right)^{\alpha}\right)^{\beta}\right\}(1-t)-\frac{1}{\alpha}\mathscr{B}\left(\frac{1}{\alpha},\beta+1,(1-t)^{\alpha}\right)\right|dt,\end{array}\right.$$

where $\mathfrak{B}$ and $\mathscr{B}$ denote the beta function and incomplete beta function, respectively.

Proof. Let us apply power-mean inequality in (13). Then, we have

$$\left|\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(b-a)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}_{b-}^{\alpha}f(a)+{}^{\beta}\mathcal{J}_{a+}^{\alpha}f(b)\right]-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right|$$

$$\leq\frac{(b-a)^2\alpha^{\beta}}{2}$$

$$\times\left\{\left(\int\limits_0^{\frac{1}{2}}\left|\int\limits_0^t\left[\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{3\alpha^{\beta}}\left(1-\left(\frac{1}{2}\right)^{\alpha}\right)^{\beta}\right]ds\right|dt\right)^{1-\frac{1}{q}}\right.$$

$$\times\left[\left(\int\limits_0^{\frac{1}{2}}\left|\int\limits_0^t\left[\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{3\alpha^{\beta}}\left(1-\left(\frac{1}{2}\right)^{\alpha}\right)^{\beta}\right]ds\right|\right.\right.$$

$$\left.\times\left|f''(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\left(\int\limits_0^{\frac{1}{2}}\left|\int\limits_0^t\left[\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{3\alpha^{\beta}}\left(1-\left(\frac{1}{2}\right)^{\alpha}\right)^{\beta}\right]ds\right|\right.$$

$$\times \left|f''\left(ta+(1-t)b\right)\right|^{q} dt\Big)^{\frac{1}{q}}\Bigg]$$

$$+\left(\int_{\frac{1}{2}}^{1}\left|\int_{t}^{1}\left[\frac{2}{3\alpha^{\beta}}+\frac{1}{3\alpha^{\beta}}\left(1-\left(\frac{1}{2}\right)^{\alpha}\right)^{\beta}-\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}\right]ds\right|dt\right)^{1-\frac{1}{q}}$$

$$\times\Bigg[\Bigg(\int_{\frac{1}{2}}^{1}\left|\int_{t}^{1}\left[\frac{2}{3\alpha^{\beta}}+\frac{1}{3\alpha^{\beta}}\left(1-\left(\frac{1}{2}\right)^{\alpha}\right)^{\beta}-\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}\right]ds\right|$$

$$\times \left|f''\left(tb+(1-t)a\right)\right|^{q} dt\Big)^{\frac{1}{q}}$$

$$+\Bigg(\int_{\frac{1}{2}}^{1}\left|\int_{t}^{1}\left[\frac{2}{3\alpha^{\beta}}+\frac{1}{3\alpha^{\beta}}\left(1-\left(\frac{1}{2}\right)^{\alpha}\right)^{\beta}-\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}\right]ds\right|$$

$$+\left|f''\left(ta+(1-t)b\right)\right|^{q} dt\Big)^{\frac{1}{q}}\Bigg]\Bigg\}.$$

Note that $|f''|^q$ is convex on $[a,b]$. Then, we have

$$\left|\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(b-a)^{\alpha\beta}}\left[\,{}^{\beta}\mathcal{J}_{b-}^{\alpha}f(a)+\,{}^{\beta}\mathcal{J}_{a+}^{\alpha}f(b)\right]-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right|$$

$$\leq\frac{(b-a)^{2}\alpha^{\beta}}{2}$$

$$\times\left\{\left(\int_{0}^{\frac{1}{2}}\left|\int_{0}^{t}\left[\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{3\alpha^{\beta}}\left(1-\left(\frac{1}{2}\right)^{\alpha}\right)^{\beta}\right]ds\right|dt\right)^{1-\frac{1}{q}}\right.$$

$$\times\Bigg[\Bigg(\int_{0}^{\frac{1}{2}}\left|\int_{0}^{t}\left[\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{3\alpha^{\beta}}\left(1-\left(\frac{1}{2}\right)^{\alpha}\right)^{\beta}\right]ds\right|$$

$$\times\left(t\left|f''(b)\right|^{q}+(1-t)\left|f''(a)\right|^{q}\right)dt\Big)^{\frac{1}{q}}$$

$$+\Bigg(\int_{0}^{\frac{1}{2}}\left|\int_{0}^{t}\left[\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{3\alpha^{\beta}}\left(1-\left(\frac{1}{2}\right)^{\alpha}\right)^{\beta}\right]ds\right|$$

$$+\left(t\left|f''(a)\right|^{q}+(1-t)\left|f''(b)\right|^{q}\right)dt\right)^{\frac{1}{q}}\Bigg]$$

$$+\left(\int_{\frac{1}{2}}^{1}\left|\int_{t}^{1}\left[\frac{2}{3\alpha^{\beta}}+\frac{1}{3\alpha^{\beta}}\left(1-\left(\frac{1}{2}\right)^{\alpha}\right)^{\beta}-\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}\right]ds\right|dt\right)^{1-\frac{1}{q}}$$

$$\times\Bigg[\left(\int_{\frac{1}{2}}^{1}\left|\int_{t}^{1}\left[\frac{2}{3\alpha^{\beta}}+\frac{1}{3\alpha^{\beta}}\left(1-\left(\frac{1}{2}\right)^{\alpha}\right)^{\beta}-\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}\right]ds\right|\right.$$

$$\times\left(t\left|f''(b)\right|^{q}+(1-t)\left|f''(a)\right|^{q}\right)dt\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{2}}^{1}\left|\int_{t}^{1}\left[\frac{2}{3\alpha^{\beta}}+\frac{1}{3\alpha^{\beta}}\left(1-\left(\frac{1}{2}\right)^{\alpha}\right)^{\beta}-\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}\right]ds\right|\right.$$

$$+\left(t\left|f''(a)\right|^{q}+(1-t)\left|f''(b)\right|^{q}\right)dt\right)^{\frac{1}{q}}\Bigg]\Bigg\},$$

which finishes the proof of Theorem 2.6. □

Remark 2.7. Consider $\alpha = 1$ and $\beta = 1$ in Theorem 2.6. Then, we obtain

$$\left|\frac{1}{b-a}\int_{a}^{b}f(x)\,dx-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right|$$

$$\leq\frac{(b-a)^{2}}{48}\left[\left(\frac{5\left|f''(b)\right|^{q}+3\left|f''(a)\right|^{q}}{8}\right)^{\frac{1}{q}}+\left(\frac{5\left|f''(a)\right|^{q}+3\left|f''(b)\right|^{q}}{8}\right)^{\frac{1}{q}}\right],$$

$$\left|\frac{\Gamma(\beta+1)}{2(b-a)^{\beta}}\left[J_{b-}^{\beta}f(a)+\ J_{a+}^{\beta}f(b)\right]-f\left(\frac{a+b}{2}\right)\right|\leq\frac{(b-a)^{2}}{162}$$

$$\times\left[\left(\frac{59\left|f''(b)\right|^{q}+133\left|f''(a)\right|^{q}}{192}\right)^{\frac{1}{q}}+\left(\frac{59\left|f''(a)\right|^{q}+133\left|f''(b)\right|^{q}}{192}\right)^{\frac{1}{q}}\right],$$

which is given in [24, Proposition 7].

3 Concluding Remarks

This paper establishes an equality for the case of convex differentiable functions. By using this identity, we proved Simpson-type inequalities related to the conformable fractional integrals. To be more precise, some significant inequalities are established by using advantage of the convexity, the Hölder inequality, and the power mean inequality. Moreover, we also give several corollaries and remarks in this section. Furthermore, our results generalized known results in the literature.

In future studies, the ideas for our results about Simpson-type inequalities with respect to conformable fractional integrals may open new avenues for further research in this area. In addition, one can obtain likewise inequalities of Simpson-type by conformable fractional integrals for twice-differentiable convex functions with the aid of the quantum calculus. Furthermore, one can apply these resulting inequalities to different types of fractional integrals.

Author contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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