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GN-operators

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Abstract. In this article, first we are going to review the concept of ordinary frames , in more general case in measure spaces, namely, gc-frames. We try to develop the use of measure space in describing frames. Then by means of the gc-frames, we shall introduce gn-operators, which we shall show that each trace class operator has a vector-valued integral representation and vice-versa

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1 Introduction

Frame were first introduced in 1952 by Duffin and Schaeffer [10], reintroduced in 1986 by Daubechies, Grossman, and Meyer [9]. In the last twenty years, the theory of frames have been employed in numerous applications such as filter bank theory [5], sigma-data quantization [4],

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signal and image processing [6], sensor networks [14], and wireless communications [12]. However, a large number of new applications have emerged.

One of the main virtues of frames is that, given a frame we can get properties of a function and reconstruct it only from the frame coefficients, a sequence of complex numbers.

Various generalizations of frame theory have been proposed recently. For example,frames of subspaces [3, 7],g-frames [23],continuous g-frames [1],Continuous k-frames[17, 16],Continuous weaving G-frames[2], Continuous weaving fusion frames[20], Continuous k-fusion frames[22] and K-g-fusion frames[21].

The point of view of the g-frames is based on operator theory. The point of view of the gc-frames is completely different and it is based on reconstruction a quantity by means of a repeated sequence of members of a Hilbert space, which is worthy in applications. However, we shall consider it in more general case, generalization of the ordinary frames according to repeated Lebesgue integrals.

Throughout this paper H will be a Hilbert space. Also, (X, μ) will be a measure space and $\{(Y_x, \mu_x)\}_{x\in X}$ will be a class of measure spaces. We shall denote by $\mathcal{M}_H$ the class of all mappings of a measure space to H. Also H_1 will be denote the unit ball of H.

Definition 1.1. Let $L^2(X, H)$ be the class of all measurable mappings $f : X \to H$ such that

$$||f||_2^2 = \int_X ||f(x)||^2 d\mu < \infty.$$

By the polar identity we conclude that for each $f, g \in L^2(X, H)$, the mapping $x \mapsto \langle f(x), g(x)\rangle$ of X to $\mathbb{C}$ is measurable, and it can be proved that $L^2(X, H)$ is a Hilbert space with the inner product defined by

$$\langle f, g\rangle = \int_X \langle f(x), g(x)\rangle d\mu.$$

We shall write $L^2(X)$ when $H = \mathbb{C}$.

Definition 1.2. Let $\{f_n\}_{n\in\mathbb{N}}$ be sequence in H. We say that $\{f_n\}_{n\in\mathbb{N}}$ is a frame for H if there exist constants $0 < A \leq B < \infty$ such that

$$A||h||^2 \leq \sum_{n\in\mathbb{N}} |\langle f_n, h\rangle|^2 \leq B||h||^2, \quad h \in H.$$

The next definition is the continuous version of frames.

Definition 1.3. Let $f : X \to H$ be weakly measurable (i.e. for each $h \in H$, the mapping $x \mapsto \langle f(x), h\rangle$ is measurable). We say that f is a c-frame (continuous frame) for H if there exist constants $0 < A_f \leq B_f < \infty$ such that

$$A_f ||h||^2 \leq \int_X |\langle f(x), h\rangle|^2 d\mu \leq B_f ||h||^2, \quad h \in H.$$

The following Lemmas can be found in operator theory text books [15, 18], which we shall use them.

Lemma 1.4. *Let $u : K \to H$ be a bounded operator with closed range $\mathcal{R}_u$.Then there exists a bounded operator $u^\dagger : H \to K$ for which*

$$uu^\dagger f = f, \quad f \in \mathcal{R}_u.$$

Also, $u^* : H \to K$ has closed range and $(u^*)^\dagger = (u^\dagger)^*$.

Lemma 1.5. *Let $u : K \to H$ be a bounded surjective operator. Given $y \in H$, the equation $ux = y$ has a unique solution of minimal norm, namely, $x = u^\dagger y$.*

Lemma 1.6. *Let u be a self-adjoint bounded operator on H. Let*

$$m_u = \inf_{||h||=1} \langle uh, h\rangle \quad \textit{and} \quad M_u = \sup_{||h||=1} \langle uh, h\rangle.$$

Then, $m_u, M_u \in \sigma(u)$.

2 GC-frames

In this section we shall introduce gc-frames which is the generalization of the ordinary frames.

Definition 2.1. Let $f : X \to \mathcal{M}_H$, $x \mapsto f_x$ and let for each $x \in X$, $f_x : Y_x \to H$. We say that f is weakly measurable if
(i) for each $x \in X$, $f_x : (Y_x, \mu_x) \to H$ is weakly measurable, and
(ii) for each $h \in H$, the mapping

$$X \to \mathbb{C}, \quad x \mapsto \int_{Y_x} \langle f_x(y), h\rangle d\mu_x$$

is measurable.

We are now ready to give the central definition.

Definition 2.2. We say that $f : X \to \mathcal{M}_H$ is a gc-frame (generalized continuous frame) for H if f is weakly measurable and there exist constants $0 < A_f \leq B_f < \infty$ such that

$$A_f||h||^2 \leq \int_X |\int_{Y_x} \langle f_x(y), h\rangle d\mu_x|^2 d\mu \leq B_f||h||^2, \quad h \in H.$$

We call A_f and B_f the gc-frame bounds. The gc-frame f is called a tight gc-frame if A_f and B_f can be chosen so that $A_f = B_f$, and a Parseval gc-frame provided that $A_f = B_f = 1$. If just the right hand inequality satisfies then we say that f is a gc-Bessel mapping for H with gc-Bessel bound B_f.

By the following example, each c-frame for H is indeed a gc-frame for H.

Example 2.3. Let $f : X \to H$ be weakly measurable. Let $g : X \to \mathcal{M}_H$ be a gc-frame for H where (Y, λ) is a finite measure space with $\lambda(Y) = 1$. Let for each $x \in X$ and $y \in Y$, $g_x(y) = f(x)$. Then we get

$$A_g||h||^2 \leq \int_X |\langle f(x), h\rangle|^2 d\mu \leq B_g||h||^2, \quad h \in H,$$

which is the usual c-frame.

Let $f : X \to \mathcal{M}_H$ be a gc-Bessel mapping for H with gc-Bessel bound B_f. We define the gc-pre-frame mapping $T_f : L^2(X) \to H$ as a vector-valued mapping by

$$\langle T_f(g), h\rangle = \int_X \int_{Y_x} \langle f_x, h\rangle g d\mu_x d\mu, \quad h \in H.$$

Since

$$\begin{aligned} |\int_X \int_{Y_x} \langle f_x, h\rangle g d\mu_x d\mu| &\leq \int_X |\int_{Y_x} \langle f_x, h\rangle d\mu_x||g| d\mu \\ &\leq (\int_X |\int_{Y_x} \langle f_x, h\rangle d\mu_x|^2 d\mu)^{1/2} (\int_X |g|^2 d\mu)^{1/2} \\ &\leq B_f^{1/2}||g||_2||h||, \end{aligned}$$

T_f is well-defined. Since

$$||T_f(g)|| = \sup_{h \in H_1} |\langle T(g), h\rangle| \leq B_f^{1/2}||g||,$$

T_f is a bounded linear mapping. Let $T_f^* : H \to L^2(X)$ be its adjoint. We have

$$\langle T_f^*(h), g\rangle = \langle h, T_f(g)\rangle = \int_X \int_{Y_x} \langle h, f_x\rangle \overline{g} d\mu_x d\mu.$$

Since for each $h \in H$, the mapping

$$\int_{Y_\cdot} \langle h, f_\cdot\rangle d\mu_\cdot : X \to \mathbb{C}, \quad x \mapsto \int_{Y_x} \langle h, f_x\rangle d\mu_x$$

belongs to $L^2(X)$,

$$\langle T_f^*(h), g\rangle = \langle \int_{Y_\cdot} \langle h, f_\cdot\rangle d\mu_\cdot, g\rangle.$$

Thus the gc-analysis operator is defined by

$$T_f^* : H \to L^2(X), \quad T_f^*(h) = \int_{Y_\cdot} \langle h, f_\cdot\rangle d\mu_\cdot.$$

By composing T_f and T_f^*, we obtain the gc-frame operator

$$S_f : H \to H, \quad S_f(h) = TT^*(h).$$

Let $f : X \to \mathcal{M}_H$ be a gc-frame with frame bounds A_f and B_f. Then

$$A_f I \leq S_f \leq B_f I.$$

Hence S is a positive invertible operator.

Let $f : X \to \mathcal{M}_H$ be a gc-Bessel mapping. Then the gc-pre-frame and gc-frame operators are vector-valued mappings which are defined by

$$T_f(g) = \int_X g(x) \int_{Y_x} f_x d\mu_x d\mu, \quad g \in L^2(X)$$

where

$$\langle T_f(g), h\rangle = \int_X g(x) \int_{Y_x} \langle f_x, h\rangle d\mu_x d\mu, \quad h \in H$$

and

$$S_f(h) = T_f(\int_{Y_.} \langle h, f_.\rangle d\mu_.) = \int_X \int_{Y_x} \langle h, f_x\rangle d\mu_x \int_{Y_x} f_x d\mu_x d\mu, \quad h \in H$$

where

$$\begin{aligned}\langle S(h), h\rangle &= \int_X \int_{Y_x} \langle h, f_x\rangle d\mu_x \int_{Y_x} \langle f_x, h\rangle d\mu_x d\mu \\ &= \int_X |\int_{Y_x} \langle f_x, h\rangle d\mu_x|^2 d\mu \\ &= ||\int_{Y_.} \langle h, f_.\rangle d\mu_.||^2.\end{aligned}$$

We now give a characterization of gc-frames in term of the gc-pre-frame operator. It is not involve any knowledge of the frame bounds.

Lemma 2.4. *Let $f : X \to \mathcal{M}_H$ be a gc-Bessel mapping. Then the following assertions are equivalent:*
(i) The frame operator S_f is invertible .
(ii) The mapping f is a gc-frame.
(iii) The gc-pre-frame operator T_f is surjective.

Proof. $(i) \Rightarrow (ii)$ Let S_f be invertible. Since

$$\inf_{h\in H_1} \langle T_f T_f^*(h), h\rangle \in \sigma(S),$$

$\inf_{h\in H_1} \langle T_f T_f^*(h), h\rangle > 0$. Hence f is a gc-frame for H.
$(ii) \Rightarrow (iii)$ Let f be a gc-frame for H with gc-frame lower bound A_f. Since

$$A_f ||h||^2 \leq ||T_f^*(h)||^2, \quad h \in H$$

T_f is surjective.
$(iii) \Rightarrow (i)$ Let T_f be surjective. Then there exists $A > 0$ such that

$$A||h|| \leq ||T_f^*(h)||, \quad h \in H.$$

Thus f is a gc-frame for H. So S_f is invertible. □

The following Theorem indicates a relation between operators and compositions with gc-frames.

Lemma 2.5. *Let K be a Hilbert space and $f : X \to \mathcal{M}_H$ be a gc-Bessel mapping for H with gc-pre-frame operator T_f and upper bound B_f. Let $u : H \to K$ be a bounded linear mapping. Then*
(i) $uf : X \to \mathcal{M}_K$, $x \mapsto uf_x$ is a gc-Bessel mapping for K with gc-pre-frame operator $T_{uf} = uT_f$ and gc-frame operator $S_{uf} = uS_f u^*$.
(ii) If f is a gc-frame for H then uf is a gc-frame for K if and only if u is surjective.

Proof. (i) It is clear that $uf : X \to \mathcal{M}_K$ is weakly measurable. Since

$$\int_X |\int_{Y_x} \langle uf_x, k\rangle d\mu_x|^2 d\mu \le ||u||^2 B_f ||k||^2, \quad k \in K,$$

uf is a gc-Bessel mapping. Let $g \in L^2(X)$. For each $k \in H$

$$\begin{aligned} \langle T_{uf}(g), k\rangle &= \int_X \int_{Y_x} g(x)\langle uf_x, k\rangle d\mu_x d\mu \\ &= \int_X \int_{Y_x} g(x)\langle f_x, u^*(k)\rangle d\mu_x d\mu \\ &= \langle uT_f(g), k\rangle. \end{aligned}$$

So $T_{uf} = uT_f$. Also, we have

$$S_{uf} = T_{uf}T_{uf}^* = T_{uf}(T_f^* u^*) = uTT_f^* u^* = uS_f u^*.$$

(ii) It is clear by the Lemma 2.4.
□

Let f be a gc-frame for H with gc-frame operator S_f. Then $S_f^{-1} f$ is a gc-frame for H with gc-frame operator

$$S_{s^{-1}f} = S_f^{-1} S_f S_f^{-1} = S_f^{-1}.$$

So for any $h \in H$ we have the following retrieval formulas

$$h = S_f S_f^{-1}(h) = \int_X \int_{Y_x} \langle h, S_f^{-1} f_x\rangle d\mu_x \int_{Y_x} f_x d\mu_x d\mu,$$

and

$$\begin{aligned} h = S_f^{-1} S_f(h) &= S_{S^{-1}f}(S_f(h)) \\ &= \int_X \int_{Y_x} \langle S_f(h), S_f^{-1} f_x \rangle d\mu_x \int_{Y_x} S_f^{-1} f_x d\mu_x d\mu \\ &= \int_X \int_{Y_x} \langle h, f_x \rangle d\mu_x \int_{Y_x} S_f^{-1} f_x d\mu_x d\mu. \end{aligned}$$

In the next Theorem, $T_f^\dagger$ will be the pseudo-inverse of T_f .

Theorem 2.6. *Let $f : X \to \mathcal{M}_H$ be a gc-frame for H. Then we have:*
(i) Let $g \in L^2(X)$ and $h = T_f(g)$. Then

$$||g||_2^2 = \int_X |\int_{Y_x} \langle h, S_f^{-1} f_x \rangle d\mu_x|^2 d\mu + \int_X |g(x) - \int_{Y_x} \langle h, S_f^{-1} f_x \rangle d\mu_x|^2 d\mu.$$

(ii) For each $h \in H$, $T_f^\dagger(h) = \int_{Y_\cdot} \langle h, S_f^{-1} f_\cdot \rangle d\mu_\cdot$.
(iii) $||T_f^\dagger||^{-2} = ||S_f^{-1}||$.

Proof. (i) Since

$$\begin{aligned} T_f(g - \int_{Y_\cdot} \langle h, S_f^{-1} f_\cdot \rangle d\mu_\cdot) &= h - T_f(T^*_{S_f^{-1}f})(h) \\ &= h - T_f(T_f^* S_f^{-1})(h) = 0, \end{aligned}$$

$$g - \int_{Y_\cdot} \langle h, S_f^{-1} f_\cdot \rangle d\mu_\cdot \in \ker(T_f) = (\mathcal{R}(T_f^*))^\perp.$$

Since $\int_{Y_\cdot} \langle h, S_f^{-1} f_\cdot \rangle d\mu_\cdot \in \mathcal{R}(T_f^*)$,

$$||g||_2^2 = ||g - \int_{Y_\cdot} \langle h, S_f^{-1} f_\cdot \rangle d\mu_\cdot||_2^2 + ||\int_{Y_\cdot} \langle h, S_f^{-1} f_\cdot \rangle d\mu_\cdot||_2^2.$$

(ii) Since by the Lemma 1.5, $T_f^\dagger(h)$ is the unique solution of minimal norm of $T_f(g) = h$, so

$$\int_X |g(x) - \int_{Y_x} \langle h, S_f^{-1} f_x \rangle d\mu_x|^2 d\mu = 0.$$

Hence, $g = \int_{Y_\cdot} \langle h, S_f^{-1} f_\cdot \rangle d\mu_\cdot = T_f^\dagger(h)$.
(iii) Since f is a gc-frame for H, by the Lemma 2.5, $S_f^{-1} f$ is also a gc-frame. Therefore

$$\begin{aligned} ||T_f^\dagger||^2 &= \sup_{h \in H_1} \int_X |\int_{Y_x} \langle h, S_f^{-1} f_x \rangle d\mu_x|^2 d\mu \\ &= \sup_{h \in H_1} ||T^*_{S_f^{-1} f}(h)||^2 \\ &= ||T^*_{S_f^{-1} f}||^2 = ||S_f^{-1}||. \end{aligned}$$

□

Christensen [8] proved that every frame in a complex Hilbert space is a multiple of a sum of three orthonormal bases. Now, we shall show that a derived vector-valued integral of a gc-Bessel mapping is a multiple of a sum of three orthonormal bases.

Theorem 2.7. *Let $f : X \to \mathcal{M}_H$ be a gc-Bessel mapping with gc-pre-frame operator T_f and $e = \{e_\alpha\}_{\alpha \in X}$ be an orthonormal basis for H. Let $\{\delta_\alpha\}_{\delta \in X}$ be the canonical orthonormal basis for $l^2(X)$. Let $u : H \to l^2(X)$ be the isomorphism which maps e_α to δ_α. Then:*
(i) If $0 < \epsilon < 1$ then there exist oryhonormal bases $e^i = \{e^i_\alpha\}_{\alpha \in X}, i = 1, 2, 3$ for H such that

$$\int_{Y_\cdot} f_\cdot d\mu_\cdot = \frac{||T_f||}{1-\epsilon}(e^1 + e^2 + e^3).$$

(ii) Let $0 < \epsilon < 1$ and $T_{uf} : l^2(X) \to l^2(X)$ be positive. Then there exist orthonormal bases $e^i = \{e^i_\alpha\}_{\alpha \in X}, i = 1, 2$ for H such that

$$\int_{Y_\cdot} f_\cdot d\mu_\cdot = \frac{||T_f||}{2\epsilon}(e^1 + e^2).$$

Proof. (i) If $||T_f|| = 0$ then $T_f^* = 0$. Therefore, for each $h \in H$, $\int_{Y_\cdot} \langle f_\cdot, h \rangle d\mu_\cdot = 0$, so (i) is satisfied. Now, let $||T_f|| > 0$. Let $w : H \to H$ be defined by

$$w = \frac{1}{2} I + \frac{1-\epsilon}{2} \frac{T_f u}{||T_f||}.$$

Since $||I - w|| < 1$, w is invertible. So by using the polar decomposition we can write $w = vp$, where v is an unitary and p is a positive operator.

But, $||p|| < 1$, so we can write $p = \frac{1}{2}(z + z^*)$, where z, z^* are unitary operators. Thus

$$T_f u = \frac{||T_f||}{1-\epsilon}(vz + vz^* - I).$$

For each $\alpha \in X$, we have

$$\begin{aligned} T_f u(e_\alpha) &= T_f(\delta_\alpha) \\ &= \int_X \int_{Y_x} \delta_\alpha(x) f_x d\mu_x d\mu \\ &= \int_{Y_\alpha} f_\alpha d\mu_\alpha. \end{aligned}$$

Therefore

$$\int_{Y_.} f_. d\mu_. = T_f ue = \frac{||T_f||}{1-\epsilon}(vze + vz^*e - e).$$

Since, vz and vz^* are unitary operators, vze and vz^*e are orthonormal bases for H. Thus

$$\int_{Y_.} f_. d\mu_. = \frac{||T_f||}{1-\epsilon}(e^1 + e^2 + e^3),$$

where, $e^i = \{e^i_\alpha\}_{\alpha \in X}, i = 1, 2, 3$ are orthonormal bases for H.
(ii) Since, $T_{uf} : l^2(X) \to l^2(X)$ is positive and u is a unitary

$$\begin{aligned} uT_f &= \frac{||T_{uf}||}{2\epsilon}(w + w^*) \\ &= \frac{||T_f||}{2\epsilon}(w + w^*), \end{aligned}$$

where w is a unitary operator. We have

$$\int_{Y_.} f_. d\mu_. = \frac{||T_f||}{2\epsilon}(u^{-1}w + u^{-1}w^*).$$

Thus

$$\int_{Y_.} f_. d\mu_. = \frac{||T_f||}{2\epsilon}(e^1 + e^2),$$

where, $e^i = \{e^i_\alpha\}_{\alpha \in X}, i = 1, 2$ are orthonormal bases for H. □

The following theorem shows that the role of two gc-Bessel mapping can be interchanged .

Theorem 2.8. *Let $f, g : X \to \mathcal{M}_H$ be a gc-Bessel mapping for H with gc-Bessel mapping bounds B_f and B_g such that for each $x \in X$, $f_x : (Y_x, \mu_x) \to H$ and $g_x : (Z_x, \lambda_x) \to H$. Then the following assertions are equivalent:*
(i) For each $h \in H$, $h = \int_X \int_{Y_x} \langle h, f_x\rangle d\mu_x \int_{Z_x} g_x d\lambda_x d\mu$.
(ii) For each $h \in H$, $h = \int_X \int_{Z_x} \langle h, g_x\rangle d\lambda_x \int_{Y_x} f_x d\mu_x d\mu$.
(iii) For each $h, k \in H$, $\langle h, k\rangle = \int_X \int_{Y_x} \langle h, f_x\rangle d\mu_x \int_{Z_x} \langle g_x, k\rangle d\lambda_x d\mu$.
(iv) For each $h \in H$, $||h||^2 = \int_X \int_{Y_x} \langle h, f_x\rangle d\mu_x \int_{Z_x} \langle g_x, h\rangle d\lambda_x d\mu$.

Proof. $(i) \Rightarrow (ii)$ Let $f, k \in H$. We have

$$\begin{aligned} \overline{\langle h, k\rangle} &= \int_X \int_{Y_x} \langle f_x, h\rangle d\mu_x \int_{Z_x} \langle k, g_x\rangle d\lambda_x d\mu \\ &= \langle \int_X \int_{Z_x} \langle k, g_x\rangle d\lambda_x \int_{Y_x} f_x d\mu_x d\mu, h\rangle. \end{aligned}$$

Hence, $k = \int_X \int_{Z_x} \langle k, g_x\rangle d\lambda_x \int_{Y_x} f_x d\mu_x d\mu$.
$(ii) \Rightarrow (iii)$ It is clear.
$(iv) \Rightarrow (i)$ Let

$$F(h) = \int_X \int_{Y_x} \langle h, f_x\rangle d\mu_x \int_{Z_x} g_x d\mu_x d\lambda.$$

It is clear that $F : H \to H$ is linear. Since

$$\begin{aligned} ||F(h)|| &= \sup_{k \in H_1} |\langle F(h), k\rangle| \\ &= \sup_{k \in H_1} |\int_X \int_{Y_x} \langle h, f_x\rangle d\mu_x \int_{Z_x} \langle g_x, k\rangle d\lambda_x d\mu| \\ &\leq \sup_{k \in H_1} (\int_X |\int_{Y_x} \langle k, f_x\rangle d\mu_x|^2 d\mu)^{1/2} (\int_X |\int_{Z_x} \langle h, g_x\rangle d\lambda_x|^2 d\mu)^{1/2} \\ &\leq B_f^{1/2} B_g^{1/2} ||h||, \end{aligned}$$

$F \in B(H)$. For each $h \in H$ we have

$$\begin{aligned} \langle h, h\rangle &= ||h||^2 \\ &= \int_X \int_{Y_x} \langle h, f_x\rangle d\mu_x \int_{Z_x} \langle g_x, h\rangle d\lambda_x d\mu \\ &= \langle \int_X \int_{Y_x} \langle h, f_x\rangle d\mu_x \int_{Z_x} g_x d\lambda_x d\mu, h\rangle. \end{aligned}$$

Hence, for each $h \in H$, $h = \int_X \int_{Y_x} \langle h, f_x\rangle d\mu_x \int_{Z_x} g_x d\lambda_x d\mu$.
$(iii) \to (iv)$ It is clear. □

Definition 2.9. Let $f, g : X \to \mathcal{M}_H$ be a gc-Bessel mapping for H. We say that f, g is a dual pair if one of the assertions of the Theorem 2.8 satisfies.

Now, we show that for a dual pair the lower gc-frame condition automatically is satisfied .

Theorem 2.10. *Let $f, g : X \to \mathcal{M}_H$ be a dual pair. Then f and g are gc-frames for H.*

Proof. For each $h \in H$ we have

$$\begin{aligned} ||h||^2 &\leq (\int_X |\int_{Y_x} \langle h, f_x\rangle d\mu_x|^2 d\mu)^{1/2} (\int_X |\int_{Z_x} \langle g_x, h\rangle d\lambda_x|^2 d\mu)^{1/2} \\ &\leq B_g(\int_X |\int_{Y_x} \langle h, f_x\rangle d\mu_x|^2 d\mu)^{1/2} ||h||. \end{aligned}$$

Thus

$$B_g^{-2}||h||^2 \leq \int_X |\int_{Y_x} \langle h, f_x\rangle d\mu_x|^2 d\mu.$$

So f is a gc-frame for H, and similarly, g is a gc-frame for H. □

The following theorem will indicate all of the dual pairs of a gc-frame.

Theorem 2.11. *Let $f, g : X \to \mathcal{M}_H$ be a gc-frame such that for each $x \in X$, $f_x, g_x : (Y_x, \mu_x) \to H$. Let $h \in H$. Then :*
(i) In the retrieval formula

$$h = \int_X \int_{Y_x} \langle S_f^{-1}(h), f_x\rangle d\mu_x \int_{Y_x} f_x d\mu_x d\mu,$$

$\int_{Y_.} \langle S_f^{-1}(h), f_.\rangle d\mu_.$ has least norm among all retrieval formulas.
(ii) For each $h \in H$, $h = \int_X \int_{Y_x} \langle h, g_x\rangle \int_{Y_x} f_x d\mu_x$ if and only if there exists a gc-Bessel mapping $l : X \to \mathcal{M}_H$ such that $g = S_f^{-1} f + l$ where for each $k \in H$, $\int_{Y_.} \langle k, l_.\rangle d\mu_. \in \ker(T_f)$.

Proof. (i) Let $q \in L^2(X)$ and $h = \int_X q(x) \int_{Y_x} f_x d\mu_x d\mu$. Then for each $k \in H$

$$\begin{aligned} \langle h, k \rangle &= \int_X \int_{Y_x} \langle S_f^{-1}(h), f_x \rangle d\mu_x \int_{Y_x} \langle f_x, k \rangle d\mu_x d\mu \\ &= \int_X q(x) \int_{Y_x} \langle f_x, k \rangle d\mu_x d\mu. \end{aligned}$$

Therefore

$$\langle \int_X \int_{Y_x} (\langle S_f^{-1}(h), f_x \rangle - q(x)) d\mu_x \int_{Y_x} f_x d\mu_x d\mu, k \rangle$$

$$= \int_X \int_{Y_x} (\langle S_f^{-1}(h), f_x \rangle - q(x)) d\mu_x \int_{Y_x} \langle f_x, k \rangle d\mu_x d\mu = 0.$$

So $T_f(\int_{Y_.} \langle S_f^{-1}(h), f_. \rangle d\mu_. - q) = 0$. Hence, $\int_{Y_.} \langle S_f^{-1}(h), f_. \rangle d\mu_. - q \in \ker(T_f)$. Since $L^2(X) = \ker(T_f) \oplus \mathcal{R}T_f^*$,

$$||q||_2^2 = || \int_{Y_.} \langle S_f^{-1}(h), f_. \rangle d\mu_. - q||^2 + || \int_{Y_.} \langle S_f^{-1}(h), f_. \rangle d\mu_. ||^2,$$

and (i) is proved.
(ii) Let for each $h \in H$, $h = \int_X \int_{Y_x} \langle h, g_x \rangle d\mu_x \int_{Y_x} f_x d\mu_x d\mu$. Let $g - S_f^{-1} f = l$. By the Theorem (2.8), for each $h, k \in H$

$$\begin{aligned} \langle \int_{Y_.} \langle k, l_. \rangle d\mu_., \int_{Y_.} \langle h, f_. \rangle d\mu_. \rangle &= \langle \int_{Y_.} \langle k, g_. \rangle d\mu_., \int_{Y_.} \langle h, f_. \rangle d\mu_. \rangle \\ &- \langle \int_{Y_.} \langle k, S_f^{-1} f_. \rangle d\mu_., \int_{Y_.} \langle h, f_. \rangle d\mu_. \rangle \\ &= \langle k, h \rangle - \langle k, h \rangle = 0. \end{aligned}$$

Hence, for each $k \in H$, $\int_{Y_.} \langle k, l_. \rangle d\mu_. \in (\mathcal{R}T_f^*)^{\perp} = kerT_f$. Now, let

$g = S_f^{-1} f + l$ where for each $k \in H$, $\int_{Y_.} \langle k, l_. \rangle d\mu_. \in ker(T_f)$. We have

$$\begin{aligned}
& \int_X \int_{Y_x} \langle h, f_x \rangle d\mu_x \int_{Y_x} \langle g_x, k \rangle d\mu_x d\mu \\
= & \int_X \int_{Y_x} \langle h, f_x \rangle d\mu_x \int_{Y_x} \langle S_f^{-1} f_x + l_x, k \rangle d\mu_x d\mu \\
= & \langle \int_{Y_.} \langle h, f_. \rangle d\mu_., \int_{Y_.} \langle k, S_f^{-1} f_. + l_. \rangle d\mu_. \rangle \\
= & \langle \int_{Y_.} \langle h, f_. \rangle d\mu_., \int_{Y_.} \langle k, S_f^{-1} f_. \rangle d\mu_. \rangle \\
+ & \langle \int_{Y_.} \langle h, f_. \rangle d\mu_., \int_{Y_.} \langle k, l_. \rangle d\mu_. \rangle \\
= & \langle \int_{Y_.} \langle h, f_. \rangle d\mu_., \int_{Y_.} \langle k, S_f^{-1} f_. \rangle d\mu_. \rangle \\
= & \int_X \int_{Y_x} \langle h, f_x \rangle d\mu_x \int_{Y_x} \langle S^{-1} f_x, k \rangle d\mu_x d\mu \\
= & \langle h, k \rangle.
\end{aligned}$$

Thus by the Theorem 2.8, for each $h \in H$,

$$h = \int_X \int_{Y_x} \langle h, g_x \rangle d\mu_x \int_{Y_x} f_x d\mu_x d\mu.$$

□

3 GN-operators

Let $\phi \in B(K, H)$. The operator ϕ is called an n-operator of K to H, if there exist families $\{x_\alpha\}_{\alpha \in I} \subseteq H, \{y_\alpha\}_{\alpha \in I} \subseteq K$ such that, $\sum_{\alpha \in I} ||x_\alpha|| ||y_\alpha|| < \infty$ and for each $k \in K$,

$$\phi(k) = \sum_{\alpha \in I} \langle k, y_\alpha \rangle x_\alpha.$$

Each n-operator on H is a trace class operator and vise versa, which in that case

$$tr(|\phi|) =$$

$$\inf\{\sum_{\alpha\in I}||x_\alpha||.||y_\alpha|| : \forall h \in H, \phi(h) = \sum_{\alpha\in I}\langle h, y_\alpha\rangle x_\alpha, \sum_{\alpha}||x_\alpha||||y_\alpha|| < \infty\}.$$

In this section, we shall generalized the concept of n-operators according to the gc-frames.

Definition 3.1. We shall denote by $L^2(X, \mathcal{M}_H)$ the class of all $f : X \to \mathcal{M}_H$ such that :
(i) For each $x \in X$, $||f_x|| : Y_x \to \mathbb{C}$ is measurable.
(ii) The mapping $\int_{Y_.} ||f_.||d\mu_. : X \to \mathbb{C}$, $x \mapsto \int_{Y_x} ||f_x||d\mu_x$ is measurable and

$$\int_X (\int_{Y_x} ||f_x||d\mu_x)^2 d\mu < \infty.$$

It is clear that each $f \in L^2(X, \mathcal{M}_H)$ is a gc-Bessel mapping for H.

Definition 3.2. Let $\phi \in B(K, H)$. We say that ϕ is a gn-operator of K into H if there exist $f \in L^2(X, \mathcal{M}_H)$ and $g \in L^2(X, \mathcal{M}_K)$ with $f_x : (Y_x, \mu_x) \to H$ and $g_x : (Z_x, \lambda_x) \to K$ such that

$$\phi(k) = \int_X \int_{Z_x} \langle k, g_x\rangle d\lambda_x \int_{Y_x} f_x d\mu_x d\mu, \quad k \in K.$$

We say that ϕ is σ-finite gn-operator if all of the measure spaces are σ-finite . Also, we say that ϕ is a gn-operator on H if $H = K$.
For each gn-operator ϕ we define its gn-norm by $||\phi||_{gn} = \inf M$, where M is the class of all $\int_X \int_{Y_x} ||f_x||d\mu_x \int_{Z_x} ||g_x||d\lambda_x d\mu$ such that $f \in L^2(X, \mathcal{M}_H)$ and $g \in L^2(X, \mathcal{M}_K)$ and

$$\phi(k) = \int_X \int_{Z_x} \langle k, g_x\rangle d\lambda_x \int_{Y_x} f_x d\mu_x d\mu, \quad k \in K.$$

The following Lemma indicates a relation between operator norms and gn-norms.

Lemma 3.3. *Let $\phi \in B(K, H)$ be a gn-operator. Then $||\phi|| \leq ||\phi||_{gn}$.*

Proof. Let $f \in L^2(X, \mathcal{M}_H)$, $g \in L^2(X, \mathcal{M}_K)$ and

$$\phi(k) = \int_X \int_{Z_x} \langle k, g_x\rangle d\lambda_x \int_{Y_x} f_x d\mu_x d\mu, \quad k \in K.$$

For each $k \in K$, we have

$$\begin{aligned} ||\phi(k)|| &= ||\int_X \int_{Z_x} \langle k, g_x\rangle d\lambda_x \int_{Y_x} f_x d\mu_x d\mu|| \\ &= \sup_{h\in H_1} |\int_X \int_{Z_x} \langle k, g_x\rangle \lambda_x \int_{Y_x} \langle f_x, h\rangle d\mu_x d\mu| \\ &\leq ||k|| \int_X \int_{Z_x} ||g_x|| d\lambda_x \int_{Y_x} ||f_x|| d\mu_x d\mu. \end{aligned}$$

So $||\phi|| \leq ||\phi||_{gn}$. □

The next Lemma shows that compositions of gn-operators and operators on Hilbert spaces are gn-operators, and it will indicates the representations of the compositions according to the representations of the gn-operators.

Lemma 3.4. *If $\phi \in B(K,H)$ is a gn-operator, $v \in B(K)$ and $u \in B(H)$ then $u\phi v$ is a gn-operator.*

Proof. Let

$$\phi(k) = \int_X \int_{Z_x} \langle k, g_x\rangle d\lambda_x \int_{Y_x} f_x d\mu_x d\mu, \quad k \in K.$$

By the Theorem (2.5), for each $k \in K$, we have

$$\begin{aligned} u\phi v(k) &= u\int_X \int_{Z_x} \langle v(k), g_x\rangle d\lambda_x \int_{Y_x} f_x d\mu_x d\mu \\ &= u\int_X \int_{Z_x} \langle k, v^* g_x\rangle d\lambda_x \int_{Y_x} f_x d\mu_x d\mu \\ &= \int_X \int_{Z_x} \langle k, v^* g_x\rangle d\lambda_x \int_{Y_x} u f_x d\mu_x d\mu. \end{aligned}$$

Thus $u\phi v \in B(K,H)$ is a gn-operator. □

Now, we shall show relations between trace class operators, n-operators and gn-operators.

Theorem 3.5. *The following assertions are satisfied:*
(i) If $\phi \in B(K,H)$ is an n-operator then ϕ is a σ-finite gn-operator.
(ii) Let ϕ be a σ-finite gn-operator on H. If

$$\phi(h) = \int_X \int_{Z_x} \langle h, g_x\rangle d\lambda_x \int_{Y_x} f_x d\mu_x d\mu, \quad h \in H$$

then

$$tr(\phi) = \int_X \int_{Z_x \times Y_x} \langle f_x, g_x \rangle d(\lambda_x \times \mu_x) d\mu.$$

(iii) If ϕ is a gn-operator on H then ϕ is a trace class operator and $||\phi||_{gn} = tr(|\phi|)$.

Proof. (i) Since ϕ is an n-operator, there exist $\{x_\alpha\}_{\alpha \in I} \subseteq H$, $\{y_\alpha\}_{\alpha \in I} \subseteq K$ such that $\sum_\alpha ||x_\alpha|| ||y_\alpha|| < \infty$ and

$$\phi(k) = \sum_\alpha \langle k, y_\alpha \rangle x_\alpha, \quad k \in K.$$

Without less of generality we can suppose that for each $\alpha \in I$, $x_\alpha \neq 0$ and $y_\alpha \neq 0$. Since $\{||x_\alpha|| ||y_\alpha||\}_{\alpha \in I} \in l^2(I)$, there exists $\{a_\alpha\}_{\alpha \in I}, \{b_\alpha\}_{\alpha \in I} \in l^2(I)$ such that $||x_\alpha|| ||y_\alpha|| = a_\alpha \overline{b_\alpha}$. Thus

$$\phi(k) = \sum_\alpha \langle k, y_\alpha \rangle x_\alpha = \sum_\alpha \langle k, \frac{a_\alpha y_\alpha}{||y_\alpha||} \rangle \frac{b_\alpha x_\alpha}{||x_\alpha||}.$$

Let $g = \{a_\alpha y_\alpha / ||y_\alpha||\}_{\alpha \in I}$ and $f = \{b_\alpha x_\alpha / ||x_\alpha||\}_{\alpha \in I}$. Let $f : I \to \mathcal{M}_H, \alpha \mapsto f_\alpha$ and $g : I \to \mathcal{M}_K, \alpha \mapsto g_\alpha$ be defined by

$$f_\alpha : Y \to H, \quad y \mapsto \frac{b_\alpha x_\alpha}{||x_\alpha||} \quad \text{and} \quad g_\alpha : Y \to K, \quad y \mapsto \frac{a_\alpha y_\alpha}{||y_\alpha||},$$

where (Y, λ) is any measure space with $\lambda(Y) = 1$. Let $X = I$ and μ be the counting measure. Then it is evident that f and g are weakly measurable and for each $h \in H$ and $k \in K$ we have

$$\begin{aligned}
\int_X \int_Y \langle k, g_\alpha \rangle d\lambda \int_Y \langle f_\alpha, h \rangle d\lambda d\mu &= \sum_\alpha \langle k, \frac{a_\alpha y_\alpha}{||y_\alpha||} \rangle \langle \frac{b_\alpha x_\alpha}{||x_\alpha||}, h \rangle \\
&= \langle \sum_\alpha \langle k, \frac{a_\alpha y_\alpha}{||y_\alpha||} \rangle \frac{b_\alpha x_\alpha}{||x_\alpha||}, h \rangle \\
&= \langle \phi(k), h \rangle.
\end{aligned}$$

So

$$\phi(k) = \int_X \int_Y \langle k, g_\alpha \rangle d\lambda \int_Y f_\alpha d\lambda d\mu.$$

Hence, ϕ is a σ-finite gn-operator.
(ii) Let

$$\phi(h)=\int_X\int_{Z_x}\langle h,g_x\rangle d\lambda_x\int_{Y_x}f_xd\mu_xd\mu,\quad h\in H$$

and let $\{e_\alpha\}$ be an orthonormal basis for H. We have

$$\begin{aligned} tr(\phi) &= \sum_\alpha\langle\int_X\int_{Z_x}\langle e_\alpha,g_x\rangle d\lambda_x\int_{Y_x}f_xd\mu_xd\mu,e_\alpha\rangle\\ &= \sum_\alpha\int_X\int_{Z_x}\langle e_\alpha,g_x\rangle d\lambda_x\int_{Y_x}\langle f_x,e_\alpha\rangle d\mu_xd\mu. \end{aligned}$$

Since

$$\begin{aligned} &\int_X|\int_{Z_x}\langle e_\alpha,g_x\rangle d\lambda_x\int_{Y_x}\langle f_x,e_\alpha\rangle d\mu_x|d\mu\\ \leq\ &\int_X\int_{Z_x}|\langle e_\alpha,g_x\rangle|d\lambda_x\int_{Y_x}|\langle f_x,e_\alpha\rangle|d\mu_xd\mu\\ \leq\ &(\int_X(\int_{Z_x}||g_x||d\lambda_x)^2d\mu)^{1/2}(\int_X(\int_{Y_x}||f_x||d\mu_x)^2d\mu)^{1/2}\\ <\ &\infty, \end{aligned}$$

and

$$\int_{Z_x}|\langle e_\alpha,g_x\rangle|d\lambda_x\int_{Y_x}|\langle f_x,e_\alpha\rangle|d\mu_x<\infty,\quad a.e[\mu].$$

Thus

$$\int_{Z_x}|\langle e_\alpha,g_x\rangle|d\lambda_x\int_{Y_x}|\langle f_x,e_\alpha\rangle|d\mu_x$$

$$=\int_{Z_x\times Y_x}|\langle e_\alpha,g_x\rangle||\langle f_x,e_\alpha|d(\lambda_x\times\mu_x).$$

Hence

$$\begin{aligned}
& \int_X \sum_\alpha |\int_{Z_x} \langle e_\alpha, g_x\rangle d\lambda_x \int_{Y_x} \langle f_x, e_\alpha\rangle d\mu_x| d\mu \\
\leq & \int_X \sum_\alpha \int_{Z_x \times Y_x} |\langle e_\alpha, g_x\rangle||\langle f_x, e_\alpha\rangle| d(\lambda_x \times \mu_x) d\mu \\
\leq & \int_X \int_{Z_x \times Y_x} \sum_\alpha |\langle e_\alpha, g_x\rangle||\langle f_x, e_\alpha\rangle| d(\lambda_x \times \mu_x) d\mu \\
\leq & \int_X \int_{Z_x \times Y_x} (\sum_\alpha |\langle e_\alpha, g_x\rangle|^2)^{1/2} (\sum_\alpha |\langle f_x, e_\alpha\rangle|^2)^{1/2} d(\lambda_x \times \mu_x) d\mu \\
= & \int_X \int_{Z_x} ||g_x|| d\lambda_x \int_{Y_x} ||f_x|| d\mu_x d\mu \\
\leq & (\int_X (\int_{Z_x} ||g_x|| d\lambda_x)^2 d\lambda)^{1/2} (\int_X (\int_{Y_x} ||f_x|| d\mu_x)^2 d\mu)^{1/2} \\
< & \infty.
\end{aligned}$$

Therefore

$$\begin{aligned}
tr(\phi) &= \int_X \sum_\alpha \int_{Z_x} \langle h, g_x\rangle d\lambda_x \int_{Y_x} \langle f_x, e_\alpha\rangle d\mu_x d\mu \\
&= \int_X \sum_\alpha \int_{Z_x \times Y_x} \langle e_\alpha, g_x\rangle \langle f_x, e_\alpha\rangle d(\lambda_x \times \mu_x) \\
&= \int_X \int_{Z_x \times Y_x} \sum_\alpha \langle e_\alpha, g_x\rangle \langle f_x, e_\alpha\rangle d(\lambda_x \times \mu_x) d\mu \\
&= \int_X \int_{Z_x \times Y_x} \langle f_x, g_x\rangle d(\lambda_x \times \mu_x) d\mu.
\end{aligned}$$

(iii) Let

$$\phi(h) = \int_X \int_{Z_x} \langle h, g_x\rangle d\lambda_x \int_{Y_x} f_x d\mu_x d\mu, \quad h \in H.$$

Let $\phi = u|\phi|$ be the polar decomposition of ϕ. So $|\phi| = u^*\phi$. By the Lemma 3.4, $|\phi|$ is a gn-operator and

$$|\phi|(h) = \int_X \int_{Z_x} \langle h, g_x\rangle d\lambda_x \int_{Y_x} u^* f_x d\mu_x d\mu, \quad h \in H.$$

Let $\{e_j\}_{\in J}$ be an orthonormal basis for H. We have

$$\begin{aligned}
tr(|\phi|) &= \sum_j \langle |\phi|(e_j), e_j\rangle \\
&\leq \sum_j \int_X \int_{Z_x} |\langle e_j, g_x\rangle| d\lambda_x \int_{Y_x} |\langle u^* f_x, e_j\rangle| d\mu_x d\mu \\
&\leq \int_X \int_{Z_x} \int_{Y_x} \sum_j |\langle e_j, g_x\rangle||\langle u^* f_x, e_j\rangle| d\lambda_x d\mu_x d\mu \\
&\leq \int_X \int_{Z_x} (\sum_j |\langle e_j, g_x\rangle|^2)^{1/2} d\lambda_x) \int_{Y_x} (\sum_j |\langle u^* f_x, e_j\rangle|^2)^{1/2} d\mu_x d\mu \\
&= \int_X \int_{Z_x} ||g_x|| d\lambda_x \int_{Y_x} ||f_x|| d\mu_x d\mu.
\end{aligned}$$

Hence

$$tr(|\phi|) \leq ||\phi||_{gn}.$$

Since $||\phi||_{gn} < \infty$, ϕ is a trace class operator. Since

$$tr(|\phi|)$$

$$= \inf\{\sum_{\alpha\in I} ||x_\alpha||.||y_\alpha|| : \forall h \in H, \phi(h) = \sum_{\alpha\in I}\langle h, y_\alpha\rangle x_\alpha, \sum_\alpha ||x_\alpha||||y_\alpha|| < \infty\},$$

thus $||\phi||_{gn} \leq tr(|\phi|)$, so

$$||\phi||_{gn} = tr(|\phi|),$$

and the Theorem is proved. □

The following result can be dedicated by the Theorem 3.5.

Corollary 3.6. *Let $\phi \in B(H)$. Then ϕ is a trace class operator if and only if ϕ is an gn-operator on H.*

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