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# Vertex Betweenness Centrality of Corona Graphs and Unicyclic Graphs

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**Abstract.** The idea of centrality measurements is quite appropriate for determining the important vertices or edges in a network. A vertex in a network may be an important vertex depending on its angle of assumption. There are many centrality measurements to find the characteristics of a vertex in a network. Betweenness centrality is an important variant of centrality measurement for analyzing complex networks based on shortest paths. The betweenness centrality of a node point $u$ is the sum of the fraction which has the number of shortest paths between any two node points $v$ and $w$ as denominator and the number of the shortest paths passing through the vertex $u$ between them as numerator. This paper describes some theoretical results relating to the betweenness centrality and relative betweenness centrality of different types of corona graphs ($P_n \odot P_m$, $P_n \odot K_m$, $C_n \odot K_m$, $C_n \odot P_m$, $C_n \odot C_m$ and $C_n \odot K_{l,m}$) and unicyclic graphs ($A(n, k, l)$, $B(n, k, l)$, $D(n, k, l)$ and $E(n, k, l)$).

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# 1 Introduction

Determination of an important vertex or edge in a complex network is one of the fundamental problems for analyzing a network. For this, a lots of centrality measurements were introduced and developed for years to years. In present years, betweenness centrality is widely used to analyze social-interaction networks [4, 9, 17, 42, 43], urban networks [16, 39, 26], biological networks [18, 30, 40, 44, 29], sexual networks and AIDS [36], transportation networks [6, 27, 43], food web networks [47, 1, 19], computer networks [34], supply chain networks [13], organizational behaviour [12], identify drug targets [40, 46], finding key person in terrorist networks [14, 33], etc. It is also used to make primary routine in some popular algorithms in the networks, such as Girvan-Newman iteratively partitioned a network in his algorithm by removing the high betweenness scores edges and again computing the centrality scores.

*Betweenness centrality* (in short, B-centrality) of a node point $u$ is the sum of the ratio whose consequent is the number of shortest paths between node points $v$ and $w$, and the antecedent is the number of shortest paths (passing through the vertex $u$) between them. We use the symbol $B_C(u)$ to represent the B-centrality of the node $u$. In mathematically, we can write

$$B_C(u) = \sum_{u\neq w\neq v} \frac{\delta_{vw}(u)}{\delta_{vw}},$$

where $\delta_{vw}$ indicates the number of shortest paths between the nodes $v$ and $w$ and $\delta_{vw}(u)$ indicates the number of shortest paths between $v$ and $w$ passing through $u$. If a node has the highest betweenness centrality in a network, then that node can pass more information than other nodes throughout the network. If there is only one shortest path between every pair of nodes in a network, then it can be determined easily. If many paths exist between a pair of nodes, then determining the betweenness centrality of a node in the general graph is complicated. If betweenness centrality increases with the number of vertices of the networks, then it is not easy to handle. So, we divide the value by the maximum value of $B_C(u)$, which lies between 0 and 1. This value is known as the relative betweenness centrality of $u$. Freeman [22] proved that the betweenness centrality of central vertex of the star graph with $n$ vertices is maximum and value is $\binom{n-1}{2} = \dfrac{(n-1)(n-2)}{2}$. The mathematical expression of relative betweenness centrality of the vertex $u$ is

$$B_C'(u) = \frac{B_C(u)}{MaxB_C(u)} = \frac{2B_C(u)}{(n-1)(n-2)},$$

where $MaxB_C(u) = \frac{(n-1)(n-2)}{2}$.

### 1.1 Review of the related works

Bavelas [5] first introduced the idea of centrality measurements and described its applications in a communication network in 1948. In 1954, Shaw [48] first gave only the concept of betweenness centrality. An improved index of centrality is found in [7]. In 1977, Freeman [22] introduced the formulae of B-centrality of node, relative B-centrality of node point, and graph betweenness centrality. He also proved that the B-centrality of the central node point of the star graph is maximum among all graphs with the same vertex cardinality. In the next year, Freeman [23] described the graph centrality for social networks. Few years later, Brandes [10] presented a faster algorithm which take $O(mn)$ time for $n$ nodes and $m$ edges unweighted graphs and $O(mn + n^2 \log n)$ time for $n$ nodes and $m$ edges weighted graphs to determine betweenness centrality. Subgraph centrality in complex networks was studied by Estrada et al. [20]. In 2007, Bader et al. [3] presented a novel approximation algorithm to find the betweenness centrality of a node of unweighted and weighted graphs. In 2007, Leydesdorff [35] showed that the betweenness centrality is an indicator of the interdisciplinary scientific journals. In 2008, Brandes [11] invented essential software to analyze the network using betweenness centrality. In 2008, Kintali [31] designed a randomized parallel algorithm and gave an algebraic method to measure the betweenness centrality of all nodes in the network. In the next year, Estrada et al. [21] introduced communicability centrality using the exponential of the adjacency matrix and Frechet derivative. Puzis et al. [45] defined heuristics speed up to betweenness centrality. In 2013, Gago et al. [24] determined the betweenness centrality for uniform graphs. In the same year, Ausiello et al. [2] studied the betweenness centrality of critical nodes and network cores. Zaoli et al. [52] proposed the definition of betweenness centrality for temporal multiplexes in the next year. In 2014, Unnithan et al. [51] determined the betweenness centrality of some class graphs. Suppa et al. [50] defined the betweenness centrality by clustered approach, and it applied in social networks in 2015. In 2017, Costa et al. [15] determined betweenness centrality in marine connectivity studies using transfer probabilities. In 2018, Bergamini et al. [8] proposed a dynamic algorithm to measure the betweenness centrality of a node by adding some edges. Again, Kirkley et al. [32] studied from the betweenness centrality in street networks to structural invariants in random planar graphs in 2018. In 2019, Matta et al. [37] worked on the speed and accuracy of approaches to betweenness centrality approximation. Recently, in 2020, Sunil et al. [49] worked on the betweenness centrality in Cartesian product of graphs.

### 1.2 Result

This paper studies the theoretical development of vertex betweenness centrality and relative betweenness centrality for different types of corona graphs (obtained by the corona product of different graphs) and uniclyclic graphs.

### 1.3 Arrangement of the paper

In the next section, we give some notations used throughout our paper. In Section 3, we describe the betweenness centrality and the relative betweenness centrality of each node point of different types of corona graphs. We present the betweenness centrality and relative betweenness centrality of each node point of the unicyclic graph in section 4. In Section 5, we give the conclusion of the paper.

## 2 Some Notations

| | | |
|---|---|---|
| $B_C(u)$ | : | betweenness centrality of the node $u$. |
| $B_C^{'}(u)$ | : | relative betweenness centrality of the node $u$. |
| $P_k$ | : | path graph with $k$ vertices. |
| $C_n$ | : | cycle graph with $n$ vertices. |
| $S_k$ | : | star graph with $k$ vertices. |

# 3 Betweenness Centrality Of Corona Graph

Let $G_1, G_2$ are two graphs with $n_1$ nodes, $m_1$ links/edges and $n_2$ nodes, $m_2$ edges, respectively. Now a corona graph $G_1 \odot G_2$ is formed by taking corona product of the graphs $G_1$ and $G_2$ by drawing one copy of $G_1$ and $n_1$ copies of $G_2$ and joining the $i$th node point of $G_1$ by an edge to each node point of the corresponding copy of $G_2$. The number of vertices and edges of corona graphs are, respectively, $n_1+n_1n_2$ and $m_1+n_1m_2+n_1n_2$.

## 3.1 Betweenness centrality of corona graph $P_n \odot P_m$

The corona graph $P_n \odot P_m$ having a path graph $P_n$ and $n$ copies of path graph $P_m$ is obtained by joining the $i$th node point of $P_n$ by an edge to each node point of the corresponding copy of $P_m$. The number of vertices of the corona graph $P_n \odot P_m$ is $n+nm$. Let the vertex set of $P_n$ be $\{u_i : i = 1, 2, \cdots, n\}$ and the vertex set of $P_m$ corresponding to the vertex $u_i$, $i = 1, 2, \cdots, n$ be $\{u_{i,j} : j = 1, 2, \cdots, m\}$. A Corona graph $P_3 \odot P_4$ is shown in Figure 1.

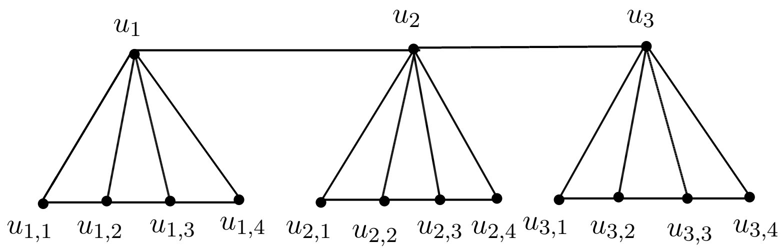


**Figure 1:** Corona graph $P_3 \odot P_4$

.

**Theorem 3.1.** *The betweenness centrality of any vertex $u$ of the corona graph $P_n \odot P_m$ is*

$$B_C(u) = \begin{cases} m(m+1)(n-1) + \frac{(m-2)^2}{2} + (n-i)(i-1)(m+1)^2, \text{if } u = u_i \in P_n, i = 1, 2, \cdots, n \\ 0, \quad \text{if } u = u_{i,j} \in P_m, i = 1, 2, \cdots, n \text{ and } j = 1, m \\ \frac{1}{2}, \text{ if } u = u_{i,j} \in P_m, i = 1, 2, \cdots, n \ \& \ j = 2, 3, \cdots, m-1. \end{cases}$$

**Proof.** Let us consider $\{u_i : i = 1, 2, \cdots, n\}$ be the set of vertices of $P_n$. Also, let $\{u_{i,j} : j = 1, 2, \cdots, m\}$ be the set of vertices of $P_m$ corresponding to the vertex $u_i, i = 1, 2, \cdots, n$. If $u = u_{i,1}$ or $u = u_{i,m}$, $i = 1, 2, \cdots, n$, then no shortest path between any pairs of vertices of $P_n \odot P_m$ (except $u$) pass through $u$. Therefore, $B_C(u) = 0$.

If $u = u_{i,j}, i = 1, 2, \cdots, n; j = 2, 3, \cdots, m-1$, then there exist two shortest path between the pairs of vertices $u_{i,j-1}$ and $u_{i,j+1}$. One of them passes through $u$, and no other shortest path between any pairs of vertices of $P_n \odot P_m$ (except $u$) passes through $u$. Therefore, $B_C(u) = \frac{1}{2}$.

If $u = u_1$, then there exist only one shortest path between a vertex of $\{u_{1,j} : j = 1, 2, \cdots, m\}$ and a vertex of $\{u_i : i = 2, 3, \cdots, n\} \cup \{u_{i,j} : j = 1, 2, \cdots, m; i = 2, 3, \cdots, n\}$ and that path pass through $u$. The total number of pairs of such vertices is $m(m+1)(n-1)$. So, these vertices of $P_n \odot P_m$ contribute the value $m(m+1)(n-1)$ for $B_C(u)$. Also, there exist two shortest paths between the vertices of $u_{1,1}$ and $u_{1,3}$ - one of them passes through $u$, and there is only one shortest path between $u_{1,1}$ and a vertex of $\{u_{1,j} : j = 4, 5, \cdots, m\}$ and the total number of pairs of such vertices is $m-3$. Therefore, the pairs of vertices between $u_{1,1}$ and a vertex of $\{u_{1,j} : j = 3, 4, \cdots, m\}$ contribute the centrality $\frac{1}{2} + (m-3)$ to $u$. Similarly, $u_{1,2}$ and a vertex of $\{u_{1,j} : j = 4, 5, \cdots, m\}$, $u_{1,3}$ and a vertex of $\{u_{1,j} : j = 5, 6, \cdots, m\}$ and the pair $(u_{1,m-2}, u_{1,m})$ contribute the centrality $\frac{1}{2} + (m-4)$, $\frac{1}{2} + (m-5)$ and $\frac{1}{2}$ respectively, to $u$. These pairs of vertices contributes the centrality

$$\begin{aligned}
&\{\tfrac{1}{2} + (m-3) + \tfrac{1}{2} + (m-4) + \tfrac{1}{2} + (m-5) + \cdots + \tfrac{1}{2} + 1 + \tfrac{1}{2}\} \\
&= (m-2) \cdot \tfrac{1}{2} + \{(m-3) + (m-4) + \cdots + 1\} \\
&= \tfrac{m-2}{2} + \tfrac{(m-3)(m-3+1)}{2} \\
&= \tfrac{m-2}{2} \cdot (m-3+1) \\
&= \tfrac{(m-2)^2}{2} \text{ to } u.
\end{aligned}$$

Therefore, $B_C(u) = m(m+1)(n-1) + \frac{(m-2)^2}{2}$. If $u = u_n$, then in similar way we can show that, $B_C(u) = m(m+1)(n-1) + \frac{(m-2)^2}{2}$.

If $u = u_p, p = 2, 3, \cdots, n-1$, then the from above result, the pair of vertices whose one vertex in $\{u_{p,j} : j = 1, 2, \cdots, m\}$ and other vertex in $\{u_i, u_{i,j} : i = 1, 2, \cdots, p-1, p+1, p+2 \cdots, n; j = 1, 2, \cdots, m\}$ contribute the centrality $m(m+1)(n-1) + \frac{(m-2)^2}{2}$ to $u$. The shortest path between a vertex of $\{u_i, u_{i,j} : i = 1, 2, \cdots, p-1; j = 1, 2, \cdots, m\}$ and a vertex of

$\{u_i, u_{i,j} : i = p+1, p+2, \cdots, n; j = 1, 2, \cdots, m\}$ passes through $u$. The total number of pairs of vertices is

$$\begin{aligned}(p-1)(n-p) + m(p-1)(n-p) &+ m(p-1)(n-p)\cdot m + m(p-1)(n-p)\\ &= (p-1)(n-p)(1+m+m^2+m)\\ &= (p-1)(n-p)(m+1)^2.\end{aligned}$$

These pairs of vertices contributes the value $(p-1)(n-p)(m+1)^2$ to $B_C(u)$.

Therefore,

$$B_C(u_p) = (p-1)(n-p)(m+1)^2 + m(m+1)(n-1) + \tfrac{(m-2)^2}{2}.$$

So, if $u = u_i, i = 1, 2, \cdots, n$, then

$$B_C(u_i) = (i-1)(n-i)(m+1)^2 + m(m+1)(n-1) + \tfrac{(m-2)^2}{2}. \qquad \square$$

**Relative betweenness centrality for the corona graph $P_n \odot P_m$.** We know

$$B'_C(u) = \frac{B_C(u)}{MaxB_C(u)}.$$

Now, using the result for the star graph with $nm + n$ vertices, we can write

$$MaxB_C(u) = \frac{(nm+n-1)(nm+n-2)}{2}.$$

Therefore,

$$B'_C(u) = \frac{2B_C(u)}{(nm+n-1)(nm+n-2)}.$$

**Corollary 3.2.** *The relative betweenness centrality $B'_C(u)$ of any vertex $u$ of $P_n \odot P_m$ is*

$$\begin{cases} \frac{2m(m+1)(n-1)}{(nm+n-1)(nm+n-2)} + \frac{(m-2)^2}{(nm+n-1)(nm+n-2)} + \frac{2(n-i)(i-1)(m+1)^2}{(nm+n-1)(nm+n-2)}, & \text{if } u = u_i \in P_n, i = 1, 2, \cdots, n \\ 0, & \text{if } u = u_{i,j} \in P_m, i = 1, 2, \cdots, n \ \& \ j = 1, m \\ \frac{1}{(nm+n-1)(nm+n-2)}, & \text{if } u = u_{i,j} \in P_m, i = 1, 2, \cdots, n \ \& \ j = 2, 3, \cdots, m-1. \end{cases}$$

### 3.2 Betweenness centrality of corona graph $P_n \odot K_m$

The corona graph $P_n \odot K_m$ having a path graph $P_n$ and $n$ copies of complete graph $K_m$ is obtained by the joining of the $i$th node point of $P_n$ by an edge with each node point of the corresponding copy of $K_m$. The number of vertices of the corona graph $P_n \odot K_m$ is $n + nm$. Let $\{u_i : i = 1, 2, \cdots, n\}$ and $\{u_{i,j} : j = 1, 2, \cdots, m\}$ be the set of vertices of $P_n$ and $K_m$ corresponding to the vertex $u_i, i = 1, 2, \cdots, n$ respectively. A corona graph $P_4 \odot K_3$ is shown in Figure 2.

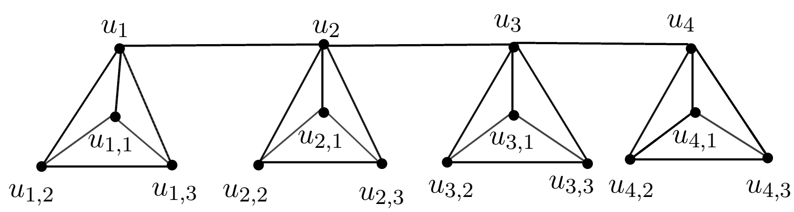


**Figure 2:** Corona graph $P_4 \odot K_3$

.

**Theorem 3.3.** *The $B_C(u)$ of any vertex $u$ of $P_n \odot K_m$ is*

$$B_C(u) = \begin{cases} 0, \text{ if } u = u_{i,j} \in K_m, i = 1, 2, \cdots, n \text{ and } j = 1, 2, \cdots, m \\ m(m+1)(n-1) + (i-1)(n-i)(m+1)^2, \text{ if } u = u_i \in P_n \\ \qquad , i = 1, 2, \cdots, n. \end{cases}$$

**Proof.** Let us consider $\{u_i : i = 1, 2, \cdots, n\}$ be the set of vertices of $P_n$ and $\{u_{i,j} : j = 1, 2, \cdots, m\}$ be the set of vertices of $K_m$ corresponding to the vertex $u_i, i = 1, 2, \cdots, n$. If $u = u_{i,j}$ where $i = 1, 2, \cdots, n; j = 1, 2, \cdots, m$, then no shortest path between any pairs of vertices of $P_n \odot K_m$ (except $u$) pass through $u$. Therefore, $B_C(u) = 0$.

If $u = u_1$, then there exist only one shortest path between a vertex of $\{u_{1,j} : j = 1, 2, \cdots, m\}$ and a vertex of $\{u_i, u_{i,j} : j = 1, 2, \cdots, m; i = 2, 3, \cdots, n\}$ and which path pass through $u$. The total number of pairs of vertices is $m(m+1)(n-1)$ to $B_C(u)$. So, these pairs of vertices of $P_n \odot K_m$ contribute the value $m(m+1)(n-1)$ to $B_C(u)$. Therefore,

$B_C(u) = m(m+1)(n-1)$. If $u = u_n$, then in similar way we can show that, $B_C(u) = m(m+1)(n-1)$.

If $u = u_p, i = 2, 3, \cdots, n-1$, then from the above result, the pair of vertices whose one vertex in $\{u_{p,j} : j = 1, 2, \cdots, m\}$ and other vertex in $\{u_i, u_{i,j} : i = 1, 2, \cdots, p-1, p+1, p+2 \cdots, n; j = 1, 2, \cdots, m\}$ contribute the centrality $m(m+1)(n-1)$ to $u$. Again, the shortest path between a vertex of $\{u_i, u_{i,j} : i = 1, 2, \cdots, p-1; j = 1, 2, \cdots, m\}$ and a vertex of $\{u_i, u_{i,j} : i = p+1, p+2, \cdots, n; j = 1, 2, \cdots, m\}$ is also pass through $u_p$ and the total number of such pairs of vertices is $(p-1)(n-p)(m+1)^2$. These pairs of vertices contribute the centrality $(p-1)(n-p)(m+1)^2$ to $B_C(u)$. Therefore,

$$B_C(u_p) = (p-1)(n-p)(m+1)^2 + m(m+1)(n-1).$$

If $u = u_i, i = 1, 2, \cdots, n$, then

$$B_C(u_i) = m(m+1)(n-1) + (i-1)(n-i)(m+1)^2. \qquad \square$$

**Corollary 3.4.** *The relative betweenness centrality of any vertex $u$ of $P_n \odot K_m$ is given by*

$$B'_C(u) = \begin{cases} 0, \text{ if } u = u_{i,j} \in K_m, i = 1, 2, \cdots, n \text{ and } j = 1, 2, \cdots, m \\ \frac{2m(m+1)(n-1)}{(nm+n-1)(nm+n-2)} + \frac{2(i-1)(n-i)(m+1)^2}{(nm+n-1)(nm+n-2)}, \quad \text{if } u = u_i \in P_n \\ \qquad , i = 1, 2, \cdots, n. \end{cases}$$

## 3.3 Betweenness centrality of corona graph $C_n \odot K_m$

The corona graph $C_n \odot K_m$ having a cyclic graph $C_n$ and $n$ copies of complete graph $K_m$ is obtained by the joining of the $i$th node point of $C_n$ by an edge with each node point of the corresponding copy of $K_m$. The number of vertices of the corona graph $C_n \odot K_m$ is $n + nm$. Let the vertex set of $C_n$ be $\{u_i : i = 1, 2, \cdots, n\}$ and the vertex set of $K_m$ corresponding to the vertex $u_i, i = 1, 2, \cdots, n$ be $\{u_{i,j} : j = 1, 2, \cdots, m\}$. A corona graph $C_4 \odot K_3$ is shown in Figure 3.

**Theorem 3.5.** *The $B_C(u)$ of any vertex $u$ of $C_n \odot K_m$ is*

$$B_C(u) = \begin{cases} 0, \ \textit{if } u = u_{i,j} \in K_m, i = 1, 2, \cdots, n \textit{ and } j = 1, 2, \cdots, m, \\ m(m+1)(n-2) + \frac{m(m+1)}{2} + \frac{(n-2)^2(m+1)^2}{8}, \textit{if } u = u_i \in C_n, \\ \qquad i = 1, 2, \cdots, n, \ n \textit{ is even} \\ m(m+1)(n-1) + \frac{(n-1)(n-3)(m+1)^2}{8}, \qquad \textit{if } u = u_i \in C_n, \\ \qquad i = 1, 2, \cdots, n \textit{ and } n \textit{ is odd.} \end{cases}$$

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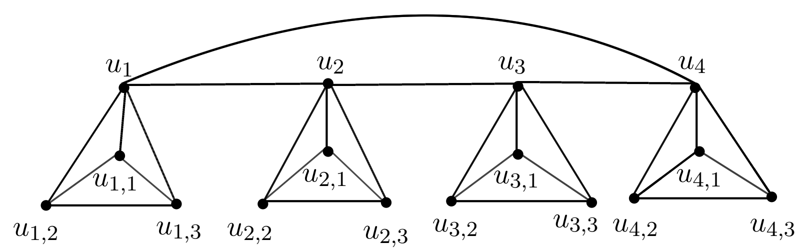


**Figure 3:** Corona graph $C_4 \odot K_3$

**Proof.** Let us consider $\{u_i : i = 1, 2, \cdots, n\}$ be the set of vertices of $C_n$ and $\{u_{i,j} : j = 1, 2, \cdots, m\}$ be the set of vertices of $K_m$ corresponding to the vertex $u_i, i = 1, 2, \cdots, n$. If $u = u_{i,j}$ where $i = 1, 2, \cdots, n; j = 1, 2, \cdots, m$, then no shortest path between any pairs of vertices of $C_n \odot K_m$ (except $u$) pass through $u$. Therefore, $B_C(u) = 0$.

If $n$ is odd and $u = u_1$, then there exist only one shortest path between a vertex of $\{u_{1,j} : j = 1, 2, \cdots, m\}$ and a vertex of $\{u_i, u_{i,j} : j = 1, 2, \cdots, m; i = 2, 3, \cdots, n\}$ and that path passes through $u$. The total number of pairs of such vertices is $m(m+1)(n-1)$. So, these pairs of vertices of $C_n \odot K_m$ contribute the value $m(m+1)(n-1)$ to $B_C(u)$. Let us consider $n$ is even and $u_p$ be the vertex of $C_n$ situated at the opposite of $u_1$. If $u = u_1$, then there exist only one shortest path between a vertex of $\{u_{1,j} : j = 1, 2, \cdots, m\}$ and a vertex of $\{u_i, u_{i,j} : j = 1, 2, \cdots, m; i = 2, 3, \cdots, n\} - \{u_p, u_{p,j} : j = 1, 2, \cdots, m\}$ pass through $u$. The total number of pairs of such vertices is $m(m+1)(n-2)$ and these pairs of vertices of $C_n \odot K_m$ contribute the value $m(m+1)(n-2)$ to $B_C(u)$. There exist two shortest path between a vertex of $\{u_{1,j} : j = 1, 2, \cdots, m\}$ and a vertex of $\{u_p, u_{p,j} : j = 1, 2, \cdots, m; i = 2, 3, \cdots, n\}$ and one of them pass through $u$. Therefore, these pairs of such vertices of $C_n \odot K_m$ contribute the value $m(m+1) \cdot \frac{1}{2} = \frac{m(m+1)}{2}$ to $B_C(u)$. Again the B-centrality of each vertex of the cycle graph $C_n$ is $\frac{(n-2)^2}{8}$, if $n$ is even and $\frac{(n-1)(n-3)}{8}$, if $n$ is odd [51]. Therefore, the vertices of $C_n$ in $C_n \odot K_m$ contributes the value $\frac{(n-2)^2}{8}$, if $n$ is even and $\frac{(n-1)(n-3)}{8}$, if $n$ is odd to $B_C(u)$. As $m$ vertices of $K_m$ are attached to each vertex of $C_n$, therefore there are $n$ sets of $m+1$ vertices. The total number of pairs between the vertices of such two sets, each having $m+1$ vertices, is $(m+1)^2$. Therefore the vertices $\{u_i, u_{i,j} : i = 2, 3, \cdots, n; j = 1, 2, \cdots, m\}$ contribute the centrality $\frac{(n-2)^2(m+1)^2}{8}$, if $n$ is even and $\frac{(n-1)(n-3)(m+1)^2}{8}$, if $n$ is odd to $B_C(u)$. So, $B_C(u) = m(m+1)(n-2) + \frac{m(m+1)}{2} + \frac{(n-2)^2(m+1)^2}{8}$, if $n$ is even and $m(m+1)(n-1) + \frac{(n-1)(n-3)(m+1)^2}{8}$, if $n$ is odd. Similarly, if $u = u_i, i = 2, 3, \cdots, n$ then $B_C(u_i) = m(m+1)(n-2) + \frac{m(m+1)}{2} + \frac{(n-2)^2(m+1)^2}{8}$, if $n$ is even and $m(m+1)(n-1) + \frac{(n-1)(n-3)(m+1)^2}{8}$, if $n$ is odd. Hence the result is proved. □

**Corollary 3.6.** *The relative betweenness centrality $B'_C(u)$ of any vertex $u$ of $C_n \odot K_m$ is*

$$\begin{cases} 0, \text{ if } u = u_{i,j} \in K_m, i = 1, 2, \cdots, n \text{ and } j = 1, 2, \cdots, m \\ \frac{2m(m+1)(n-2)}{(nm+n-1)(nm+n-2)} + \frac{m(m+1)}{(nm+n-1)(nm+n-2)} + \frac{(n-2)^2(m+1)^2}{4(nm+n-1)(nm+n-2)}, \\ \quad \text{if } u = u_i \in C_n,\ i = 1, 2, \cdots, n \text{ and } n \text{ is even} \\ \frac{m(m+1)(n-1)}{(nm+n-1)(nm+n-2)} + \frac{(n-1)(n-3)(m+1)^2}{4(nm+n-1)(nm+n-2)}, \quad \text{if } u = u_i \in C_n, \\ \quad i = 1, 2, \cdots, n,\ n \text{ is odd.} \end{cases}$$

### 3.4 Betweenness centrality of corona graph $C_n \odot P_m$

The corona graph $C_n \odot P_m$ having a cyclic graph $C_n$ and $n$ copies of path graph $P_m$ is obtained by the joining of the $i$th node point of $C_n$ by an edge with each node point of the corresponding copy of $P_m$. The cardinality of the vertex set of the corona graph $C_n \odot P_m$ is $n+nm$. Let $\{u_i : i = 1, 2, \cdots, n\}$ and $\{u_{i,j} : j = 1, 2, \cdots, m\}$ be the set of vertices of $C_n$ and $P_m$ corresponding to the vertex $u_i, i = 1, 2, \cdots, n$ respectively. A corona graph $C_3 \odot P_4$ is shown in Figure 4.

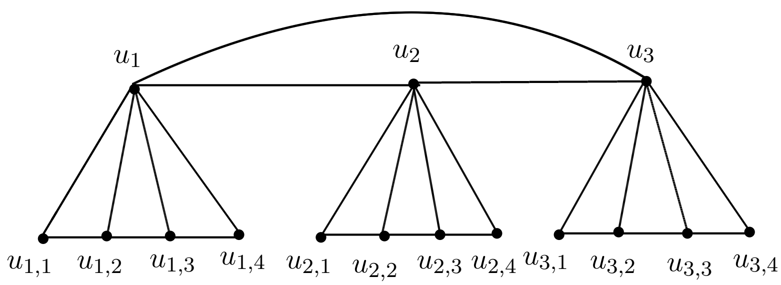


**Figure 4:** Corona graph $C_3 \odot P_4$

.

**Theorem 3.7.** *The $B_C(u)$ of any vertex $u$ of $C_n \odot P_m$ is given by*

$$B_C(u) = \begin{cases} 0, & \text{if } u = u_{i,j} \in P_m, i = 1, 2, \cdots, n \text{ and } j = 1, m \\ \frac{1}{2}, & \text{if } u = u_{i,j} \in P_m, i = 1, 2, \cdots, n \ \& \ j = 2, 3, \cdots, m-1 \\ m(m+1)(n-2) + \frac{m(m+1)}{2} + \frac{(n-2)^2(m+1)^2}{8} + \frac{(m-2)^2}{2}, & \text{if } u = u_i \in C_n, \ i = 1, 2, \cdots, n \text{ and } n \text{ is even} \\ m(m+1)(n-1) + \frac{(n-1)(n-3)(m+1)^2}{8} + \frac{(m-2)^2}{2}, & \text{if } u = u_i \in C_n, i = 1, 2, \cdots, n \text{ and } n \text{ is odd.} \end{cases}$$

**Proof.** Let us consider $\{u_i : i = 1, 2, \cdots, n\}$ be the set of vertices of $C_n$. Also, let $\{u_{i,j} : j = 1, 2, \cdots, m\}$ be the set of vertices of $P_m$ corresponding to the vertex $u_i, i = 1, 2, \cdots, n$. If $u = u_{i,1}$ or $u = u_{i,m}$, $i = 1, 2, \cdots, n$, then no shortest path between any pairs of vertices of $C_n \odot P_m$ (except $u$) pass through $u$. Therefore, $B_C(u) = 0$.

If $u = u_{i,j}, i = 1, 2, \cdots, n; j = 2, 3, \cdots, m-1$, then there exist two shortest path between the pairs of vertices $u_{i,j-1}$ and $u_{i,j+1}$ - one of

them pass through $u$, and no other shortest path between any pairs of vertices of $C_n \odot P_m$ pass through $u$. Therefore, $B_C(u) = \frac{1}{2}$.
If $n$ is odd and $u = u_1$, then the pair of vertices whose one vertex in $\{u_{1,j} : j = 1, 2, \cdots, m\}$ and other in $\{u_i, u_{i,j} : j = 1, 2, \cdots, m; i = 2, 3, \cdots, n\}$ contribute the centrality $\frac{(m-2)^2}{2}$ to $u$ (see the proof of the theorem 3.1) and the pairs of vertices between a vertex of $\{u_{1,j} : j = 1, 2, \cdots, m\}$ and a vertex of $\{u_i, u_{i,j} : j = 1, 2, \cdots, m; i = 2, 3, \cdots, n\}$ contribute centrality $m(m+1)(n-1)$ to $u$ (see the proof of the theorem 3.5). Let us consider $n$ is even and $u_p$ be the vertex of $C_n$ situated at the opposite of $u_1$. If $u = u_1$, then there exist only one shortest path between a vertex of $\{u_{1,j} : j = 1, 2, \cdots, m\}$ and a vertex of $\{u_i, u_{i,j} : j = 1, 2, \cdots, m; i = 2, 3, \cdots, n\} - \{u_p, u_{p,j} : j = 1, 2, \cdots, m\}$ pass through $u$. The total number of pair of vertices is $m(m+1)(n-2)$ and these pair of vertices of $C_n \odot P_m$ contribute the value $m(m+1)(n-2)$ to $B_C(u)$. There exist two shortest path between a vertex of $\{u_{1,j} : j = 1, 2, \cdots, m\}$ and a vertex of $\{u_p, u_{p,j} : j = 1, 2, \cdots, m\}$ and one of them pass through $u$. Therefore, these pairs of vertices of $C_n \odot P_m$ contribute the value $m(m+1) \cdot \frac{1}{2} = \frac{m(m+1)}{2}$ to $B_C(u)$. The pair between the vertices of $\{u_{1,j} : j = 1, 2, \cdots, m\}$ contribute the centrality $\frac{(m-2)^2}{2}$ to $u$ (see the proof of the theorem 3.1). The set of vertices $\{u_i, u_{ij} : i = 2, 3, \cdots, n; j = 1, 2, \cdots, m\}$ contributes the centrality $\frac{(n-2)^2(m+1)^2}{8}$, if $n$ is even and $\frac{(n-1)(n-3)(m+1)^2}{8}$, if $n$ is odd to $u$ (see the proof of the theorem 3.5). Therefore, $B_C(u) = m(m+1)(n-2) + \frac{m(m+1)}{2} + \frac{(m-2)^2}{2} + \frac{(n-2)^2(m+1)^2}{8}$, if $n$ is even and $B_C(u) = m(m+1)(n-1) + \frac{(m-2)^2}{2} + \frac{(n-1)(n-3)(m+1)^2}{8}$, if $n$ is odd. Similarly, if $u = u_i, i = 2, 3, \cdots, n$, then we get the same result. □

**Relative betweenness centrality for the corona graph $P_n \odot P_m$.**
We know, $B'_C(u) = \dfrac{B_C(u)}{MaxB_C(u)} = \dfrac{2B_C(u)}{(nm+n-1)(nm+n-2)}$, where $MaxB_C(u) = \dfrac{(nm+n-1)(nm+n-2)}{2}$ [using the result for the star graph with $nm + n$ vertices].

**Corollary 3.8.** *The relative betweenness centrality $B'_C(u)$ of any vertex $u$ of $C_n \odot P_m$ is*

$$\begin{cases} 0, & \textit{if } u = u_{i,j} \in P_m, i = 1,2,\cdots,n \textit{ and } j = 1,m \\ \dfrac{1}{(nm+n-1)(nm+n-2)}, & \textit{if } u = u_{i,j} \in P_m, i = 1,2,\cdots,n \\ & \textit{and } j = 2,3,\cdots,m-1 \\ \frac{8m(m+1)(n-2)+4m(m+1)+(n-2)^2(m+1)^2+4(m-2)^2}{4(nm+n-1)(nm+n-2)}, & \textit{if } u = u_i \in C_n, \\ & i = 1,2,\cdots,n \textit{ and } n \textit{ is even} \\ \frac{2m(m+1)(n-1)}{(nm+n-1)(nm+n-2)} + \frac{(n-1)(n-3)(m+1)^2}{4(nm+n-1)(nm+n-2)} + \frac{(m-2)^2}{(nm+n-1)(nm+n-2)}, & \textit{if} \\ & u = u_i \in C_n, i = 1,2,\cdots,n \textit{ and } n \textit{ is odd.} \end{cases}$$

### 3.5 Betweenness centrality of corona graph $C_n \odot C_m$

The corona graph $C_n \odot C_m$ having a cyclic graph $C_n$ and $n$ copies of cyclic graph $C_m$ is obtained by the joining of the $i$th node point of $C_n$ by an edge with each node point of the corresponding copy of $C_m$. The vertex cardinality of the corona graph $C_n \odot C_m$ in $n + nm$. Let the vertex set of $C_n$ and $C_m$ (corresponding to the vertex $u_i, i = 1,2,\cdots,n$) be $\{u_i : i = 1,2,\cdots,n\}$ and $\{u_{i,j} : j = 1,2,\cdots,m\}$ respectively. A corona graph $C_4 \odot C_4$ is shown in Figure 5.

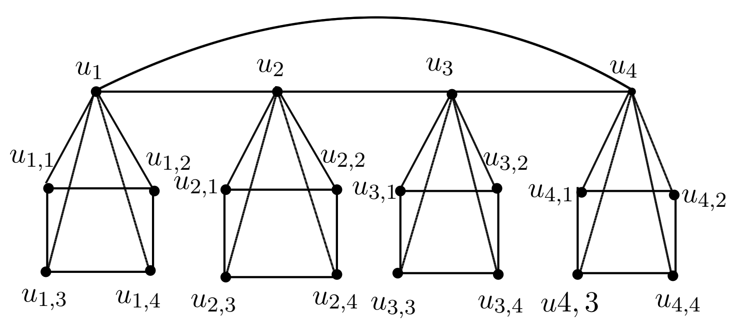


**Figure 5:** Corona graph $C_4 \odot C_4$

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**Theorem 3.9.** *The $B_C(u)$ of any vertex $u$ of corona graph $C_n \odot C_m$ is*

$$\begin{cases} \frac{1}{2}, \text{ if } u = u_{i,j} \in C_m, i = 1,2,\cdots,n, j = 1,2,\cdots,m \text{ and } m > 4 \\ m(m+1)(n-2) + \frac{m(m+1)}{2} + \frac{(n-2)^2(m+1)^2}{8} + \frac{m(m-4)}{2}, \quad \text{if } \\ \qquad u = u_i \in C_n,\ i = 1,2,\cdots,n,\ \ n \text{ is even and } m > 4 \\ m(m+1)(n-1) + \frac{(n-1)(n-3)(m+1)^2}{8} + \frac{m(m-4)}{2}, \quad \text{if } \\ \qquad u = u_i \in C_n, i = 1,2,\cdots,n,\ \ n \text{ is odd and } m > 4. \end{cases}$$

**Proof.** Let us consider $\{u_i : i = 1,2,\cdots,n\}$ be the set of vertices of $C_n$. Also, let $\{u_{i,j} : j = 1,2,\cdots,m, m > 4\}$ be the set of vertices of $C_m$ corresponding to the vertex $u_i, i = 1,2,\cdots,n$. If $u = u_{i,j},$, then there exist two shortest paths between the pairs of vertices of $u_{i,j-1}$ and $u_{i,j+1}$ of $C_m$ - one of them pass through $u$ and no shortest path between any pairs of vertices of $C_n \odot C_m$ pass through $u$. Therefore, $B_C(u) = \frac{1}{2}$.

If $u = u_p, p = 1,2,\cdots,n$ be any vertex of $C_n$ where $n$ is odd, then there exist only one shortest path between a vertex of $\{u_{p,j} : j = 1,2,\cdots,m\}$ and a vertex of $\{u_i, u_{i,j} : j = 1,2,\cdots,m; i = 1,2,\cdots,p-1,p+1,\cdots,n\}$ and that path passes through $u$. The total number of such pairs of vertices is $m(m+1)(n-1)$. So, these pairs of vertices of $C_n \odot C_m$ contribute the value $m(m+1)(n-1)$ for $B_C(u)$. Let us consider $n$ is even and $u_q$ be the vertex of $C_n$ situated at the opposite of $u_p$. If $u = u_p$, then there exist only one shortest path between a vertex of $\{u_{p,j} : j = 1,2,\cdots,m\}$ and a vertex of $\{u_i, u_{i,j} : j = 1,2,\cdots,m; i = 1,2,\cdots,p-1,p+1,\cdots,n\} - \{u_q, u_{q,j} : j = 1,2,\cdots,m\}$ pass through $u$. The total number of such pairs of vertices is $m(m+1)(n-2)$ and these pair of vertices of $C_n \odot C_m$ contribute the value $m(m+1)(n-2)$ to $B_C(u)$. There exist two shortest path between a vertex of $\{u_{p,j} : j = 1,2,\cdots,m\}$ and a vertex of $\{u_q, u_{q,j} : j = 1,2,\cdots,m\}$ and one of them pass through $u$. Therefore, these pairs of vertices of $C_n \odot C_m$ contribute the value $m(m+1) \cdot \frac{1}{2} = \frac{m(m+1)}{2}$ to $B_C(u)$. The pairs of vertices of the set $\{u_i, u_{i,j} : i = 1,2,\cdots,p-1,p+1,\cdots,n; j = 1,2,\cdots,m\}$ contribute the centrality $\frac{(n-2)^2(m+1)^2}{8}$, if $n$ is even and $\frac{(n-1)(n-3)(m+1)^2}{8}$, if $n$ is odd to $u$ (see the proof of the theorem 3.5). Now, we calculate the contribution of the pairs of vertices of $\{u_{p,j} : j = 1, 2, \cdots, m\}$ to $B_C(u_p)$. The number of pairs of vertices of $C_m$ is $\binom{m}{2}$. The length of the shortest path between each pair of vertices of $C_m$ is either 1 or 2 or greater than 2. The pairs of adjacent vertices (length of shortest path 1) contribute the value 0 to $B_C(u)$, and the number of such pairs is $m$. The pairs of vertices (situated at a distance of 2) contribute the value $\frac{1}{2}$ to $B_C(u)$, and the number of such pairs is $m$. Therefore, the number of pairs whose shortest distance greater than 2 is $\binom{m}{2} - m - m = \frac{m(m-1)}{2} - 2m = \frac{m(m-5)}{2}$ and these pairs of vertices contribute centrality 1 to $u$. Therefore, the pairs of vertices of $C_m$ contribute centrality $m \cdot 0 + m \cdot \frac{1}{2} + \frac{m(m-5)}{2} \cdot 1 = \frac{m(m-4)}{2}$ to $u$. So, the $B_C(u)$ is $m(m+1)(n-2) + \frac{m(m+1)}{2} + \frac{(n-2)^2(m+1)^2}{8} + \frac{m(m-4)}{2}$, if $n$ is even, and $m(m+1)(n-1) + \frac{(n-1)(n-3)(m+1)^2}{8} + \frac{m(m-4)}{2}$, if $n$ is odd. □

**Corollary 3.10.** *If $m = 4$, then*

$$B_C(u) = \begin{cases} \frac{1}{3}, & \text{if } u = u_{i,j} \in C_m, i = 1, 2, \cdots, n; j = 1, 2, \cdots, m \\ m(m+1)(n-2) + \frac{m(m+1)}{2} + \frac{(n-2)^2(m+1)^2}{8} + \frac{2}{3}, & \text{if } u = u_i \in C_n, i = 1, 2, \cdots, n \text{ and } n \text{ is even} \\ m(m+1)(n-1) + \frac{(n-1)(n-3)(m+1)^2}{8} + \frac{2}{3}, & \text{if } u = u_i \in C_n, i = 1, 2, \cdots, n \text{ and } n \text{ is odd.} \end{cases}$$

**Corollary 3.11.** *If $m = 3$, then*

$$B_C(u) = \begin{cases} 0, & \text{if } u = u_{i,j} \in C_m, i = 1, 2, \cdots, n; j = 1, 2, \cdots, m \\ m(m+1)(n-2) + \frac{m(m+1)}{2} + \frac{(n-2)^2(m+1)^2}{8}, & \text{if } u = u_i \in C_n, i = 1, 2, \cdots, n \text{ and } n \text{ is even} \\ m(m+1)(n-1) + \frac{(n-1)(n-3)(m+1)^2}{8}, & \text{if } u = u_i \in C_n, i = 1, 2, \cdots, n \text{ and } n \text{ is odd.} \end{cases}$$

**Corollary 3.12.** *The relative betweenness centrality $B'_C(u)$ of any vertex $u$ of $C_n \odot C_m$ is*

$$\begin{cases} \dfrac{1}{(nm+n-1)(nm+n-2)}, & \text{if } u = u_{i,j} \in C_m, i = 1,2,\cdots,n, \\ & j = 1,2,\cdots,m \text{ and } m > 4 \\ \frac{8m(m+1)(n-2)+4m(m+1)+(n-2)^2(m+1)^2+4m(m-4)}{4(nm+n-1)(nm+n-2)}, & \text{if } u = u_i \in C_n, \\ & i = 1,2,\cdots,n,\ n \text{ is even and } m > 4 \\ \frac{2m(m+1)(n-1)}{(nm+n-1)(nm+n-2)} + \frac{(n-1)(n-3)(m+1)^2}{4(nm+n-1)(nm+n-2)} + \frac{m(m-4)}{(nm+n-1)(nm+n-2)}, & \text{if} \\ & u = u_i \in C_n, i = 1,2,\cdots,n, n \text{ is odd and } m > 4. \end{cases}$$

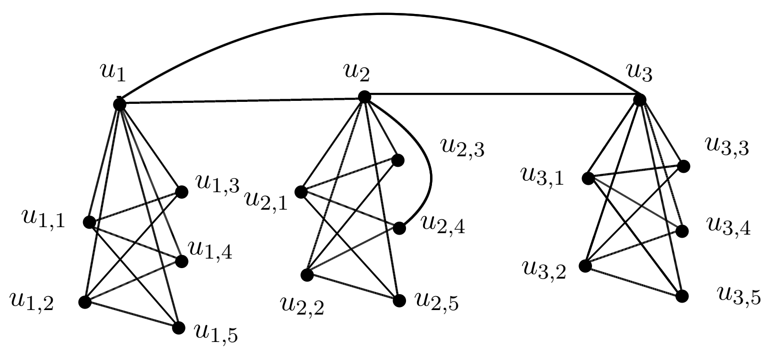


**Figure 6:** Corona graph $C_3 \odot K_{2,3}$

## 3.6 Betweenness centrality of corona graph $C_n \odot K_{l,m}$

The corona graph $C_n \odot K_{l,m}$ having a cyclic graph $C_n$ and $n$ copies of complete bipartite graph $K_{l,m}$ is obtained by the joining of $i$th vertex of $C_n$ by an edge with each vertex of the corresponding copy of $K_{l,m}$. The number of vertices of the corona graph $C_n \odot K_{l,m}$ is $n+n(l+m)$. Let the vertex set of $C_n$ and $K_{l,m}$(corresponding to the vertex $u_i, i = 1,2,\cdots,n$)

be $\{u_i : i = 1, 2, \cdots, n\}$ and $\{u_{i,j} : j = 1, 2, \cdots, l, l+1, \cdots, l+m\}$, respectively. A corona graph $C_3 \odot K_{2,3}$ is shown in Figure 6.

**Theorem 3.13.** *The $B_C(u)$ of any vertex $u$ of the corona graph $C_n \odot K_{l,m}$ is*

$$\begin{cases} [(l+m)(l+m+1)(n-2) + \frac{(l+m)(l+m+1)}{2} + \frac{1}{l+1}\binom{m}{2} + \frac{1}{m+1}\binom{l}{2} + \\ \quad \frac{(n-2)^2(l+m+1)^2}{8}], \quad \text{if } u = u_i, i = 1, 2, \cdots, n \text{ and } n \text{ is even} \\ (l+m)(l+m+1)(n-1) + \frac{(n-1)(n-3)(l+m+1)^2}{8} + \frac{1}{l+1}\binom{m}{2} + \frac{1}{m+1}\binom{l}{2}, \\ \quad \text{if } u = u_i, i = 1, 2, \cdots, n \text{ and } n \text{ is odd} \\ \frac{1}{l+1}\binom{m}{2}, \quad \text{if } u = u_{i,j} \in K_{l,m}, i = 1, 2, \cdots, n \text{ and } \\ \quad j = 1, 2, \cdots, l \\ \frac{1}{m+1}\binom{l}{2}, \quad \text{if } u = u_{i,j} \in K_{l,m}, i = 1, 2, \cdots, n \text{ and } j = l+1, l+2, \cdots, \\ \quad l+m. \end{cases}$$

**Proof.** Let $\{u_i : i = 1, 2, \cdots, n\}$ be the set of vertices of $C_n$ and $\{u_{i,j} : j = 1, 2, \cdots, l, l+1, \cdots, l+m\}$ be the set of vertices of the *ith* copy of $K_{l,m}$. If $u = u_{p,j}, j = 1, 2, \cdots, l$ be any vertex of the *pth* copy $K_{l,m}$, then there exit $l+1$ shortest paths between each pairs of vertices of $\{u_{p,j} : j = l+1, l+2, \cdots, l+m\}$ - one of them pass through $u$ and the total number of such pairs of vertices is $\binom{m}{2}$. Therefore,

$$B_C(u) = \frac{1}{l+1}\binom{m}{2}.$$

So, if $u = u_{i,j}, i = 1, 2, \cdots, n; j = 1, 2, \cdots, l$, then

$$B_C(u) = \frac{1}{l+1}\binom{m}{2}.$$

Similarly, if $u = u_{i,j}, i = 1, 2, \cdots, n; j = l+1, l+2, \cdots, l+m$, then

$$B_C(u) = \frac{1}{m+1}\binom{l}{2}.$$

If $u = u_p$ be any vertex of $C_n, n$ is odd, then there exist only one shortest path between a vertex of $\{u_{p,j} : j = 1, 2, \cdots, l+m\}$ and a vertex of $\{u_i, u_{i,j} : i = 1, 2, \cdots, p-1, p+1, \cdots, n; j = 1, 2, \cdots, l+m\}$ and that path passes through $u$. The total number of such pairs of vertices is $(l+m)(l+m+1)(n-1)$ and these pairs contribute the value $(l+m)(l+m+1)(n-1)$ to $B_C(u)$. Let us consider $n$ is even and $u_q$ be the vertex of $C_n$ situated at the opposite of $u_p$. If $u = u_p$ then there exist only one shortest path between a vertex of $\{u_{p,j} : j = 1, 2, \cdots, l+m\}$ and a vertex of $\{u_i, u_{i,j} : j = 1, 2, \cdots, l+m; i = 1, 2, \cdots, p-1, p+1, \cdots, n\} - \{u_q, u_{q,j} : j = 1, 2, \cdots, l+m\}$ and that path passes through $u$. The total number of such pairs of vertices is $(l+m)(l+m+1)(n-2)$ and these pairs of vertices of $C_n \odot K_{l,m}$ contribute the value $(l+m)(l+m+1)(n-2)$ to $B_C(u)$. There exist two shortest path between a vertex of $\{u_{p,j} : j = 1, 2, \cdots, l+m\}$ and a vertex of $\{u_q, u_{q,j} : j = 1, 2, \cdots, l+m\}$ - one of them pass through $u$. Therefore, these pairs of vertices of $C_n \odot K_{l,m}$ contribute the value $(l+m)(l+m+1) \cdot \frac{1}{2} = \frac{(l+m)(l+m+1)}{2}$ to $B_C(u)$. There exist $m+1$ shortest paths between each pairs of vertices of $\{u_{p,j} : j = 1, 2, \cdots, l\}$ - one of them pass through $u$ and the number of pairs is $\binom{l}{2}$. Therefore, these pairs of vertices contribute centrality $\frac{1}{m+1}\binom{l}{2}$ to $u$. Similarly, the pair of vertices of $\{u_{p,j} : j = l+1, l+2, \cdots, l+m\}$ contribute the value $\frac{1}{l+1}\binom{m}{2}$ to $B_C(u)$. As $l+m$ vertices of $K_{l,m}$ are attached with each vertex of $C_n$ so, there are $n$ sets of $l+m+1$ vertices. The vertices of $C_n$ contribute the value $\frac{(n-2)^2}{8}$, if $n$ is even and $\frac{(n-1)(n-3)}{8}$, if $n$ is odd [51] to $B_C(u)$. The total number of pairs between the vertices of such two sets (each having $l+m+1$ vertices) is $(l+m+1)^2$. Therefore the vertices $\{u_i, u_{i,j} : i = 1, 2, \cdots, p-1, p+1, \cdots, n; j = 1, 2, \cdots, l+m\}$ contribute the centrality $\frac{(n-2)^2(l+m+1)^2}{8}$, if $n$ is even and $\frac{(n-1)(n-3)(l+m+1)^2}{8}$, if $n$ is odd to $u$. Therefore, if $n$ is even, then

$$B_C(u_p) = (l+m)(l+m+1)(n-2) + \frac{(l+m)(l+m+1)}{2} + \frac{(n-2)^2(l+m+1)^2}{8} + \frac{1}{m+1}\binom{l}{2} + \frac{1}{m+1}\binom{l}{2},$$

and if $n$ is odd, then

$$B_C(u_p) = (l+m)(l+m+1)(n-1) + \frac{(n-1)(n-3)(l+m+1)^2}{8} + \frac{1}{m+1}\binom{l}{2} + \frac{1}{m+1}\binom{l}{2}.$$

□

**Relative betweenness centrality for the corona graph $C_n \odot K_{l,m}$.** We know

$B'_C(u) = \dfrac{B_C(u)}{MaxB_C(u)} = \dfrac{2B_C(u)}{[n(l+m+1)-1][n(l+m+1)-2]}$ [using the result for the star graph with $n(l+m)+n$ vertices].

**Corollary 3.14.** *The relative betweenness centrality $B'_C(u)$ of any vertex $u$ of $C_n \odot K_{l,m}$ is*

$$\begin{cases} [\frac{2(l+m)(l+m+1)(n-2)}{[n(l+m+1)-1][n(l+m+1)-2]} + \frac{(n-2)^2(l+m+1)^2}{4[n(l+m+1)-1][n(l+m+1)-2]} \\ \quad + \frac{(l+m)(l+m+1)}{[n(l+m+1)-1][n(l+m+1)-2]} + \frac{2}{(l+1)[n(l+m+1)-1][n(l+m+1)-2]}\binom{m}{2} \\ \quad + \frac{2}{(m+1)[n(l+m+1)-1][n(l+m+1)-2]}\binom{l}{2}],\ if\ \ u = u_i, \\ \qquad\qquad i = 1, 2, \cdots, n\ and\ n\ is\ even \\ \frac{2(l+m)(l+m+1)(n-1)}{[n(l+m+1)-1][n(l+m+1)-2]} + \frac{2}{(l+1)[n(l+m+1)-1][n(l+m+1)-2]}\binom{m}{2} \\ \quad + \frac{(n-1)(n-3)(l+m+1)^2}{4[n(l+m+1)-1][n(l+m+1)-2]} + \frac{2}{(m+1)[n(l+m+1)-1][n(l+m+1)-2]}\binom{l}{2}, \\ \qquad\qquad if\ \ u = u_i, i = 1, 2, \cdots, n\ and\ n\ is\ odd \\ \frac{2}{(l+1)[n(l+m+1)-1][n(l+m+1)-2]}\binom{m}{2},\quad if\ \ u = u_{i,j}, i = 1, 2, \cdots, n \\ \qquad\qquad and\ j = 1, 2, \cdots, l \\ \frac{2}{(m+1)[n(l+m+1)-1][n(l+m+1)-2]}\binom{l}{2}, if\ \ u = u_{i,j}, i = 1, 2, \cdots, n \\ \qquad\qquad and\ j = l+1, \cdots, l+m. \end{cases}$$

# 4 Betweenness Centrality Of Unicyclic Graph

A unicyclic graph is a connected graph that has exactly one cycle. In other words, we can say that a unicyclic graph is a connected graph with paths or trees attached to a cycle. The number of vertices and edges of the unicyclic graph are equal. A unicyclic graph $G_{n,k}$ is obtained by the joining of a cycle $C_n$ of length $n$ and one end of path graph $P_k$ or the central vertex of star graph $S_k$ by a bridge. -2.5cm

## 4.1 Betweenness centrality of a unicyclic graph $A(n,k,l)$

$A(n,k,l)$ is a unicyclic graph of order $n$ having a cycle $C_n$ and $l$ copies of path $P_k$ with $k$ vertices attached with a unique vertex of $C_n$, where $n \geq 3, k \geq 1$ and $l \geq 1$. The number of vertices of the unicyclic graph $A(n,k,l)$ is $n + kl$. Let $\{u_1, u_2, \cdots, u_{\frac{n}{2}-1}, u_{\frac{n}{2}}, u_{\frac{n}{2}+1}, \cdots, u_{n-1}, u_n\}$ be the vertices of $C_n$, where $n$ is even. From the vertex $u_n$ of $C_n$, the vertices $u_1$ and $u_{n-1}$, $u_2$ and $u_{n-2}$, $u_{\frac{n}{2}-1}$ and $u_{\frac{n}{2}+1}$ and so on lies at the same distances. Therefore, the B-centrality of $u_1$ and $u_{n-1}$ are equal. Similarly, the B-centrality of each pair of vertices(the distance of both vertices is equal from $u_n$) is equal. If $n$ is odd, then let the vertex set of $C_n$ be $\{u_1, u_2, \cdots, u_{\frac{n-1}{2}}, u_{\frac{n+1}{2}}, \cdots, u_{n-1}, u_n\}$. In this case, the B-centrality of $u_1$ and $u_{n-1}$, $u_2$ and $u_{n-2}, \cdots$, $u_{\frac{n-1}{2}}$ and $u_{\frac{n+1}{2}}$ are same. Also let $\{v_{i,1}, v_{i,2}, \cdots, v_{i,k}\}$, where $i = 1, 2, \cdots, l$ be the set of vertices of the $i$th copy of $P_k$. The betweenness centrality of each vertex of the $i$th copy of $P_k$ is equal to the corresponding vertex of the $j$th copy of $P_k$. A unicyclic graph $A(4, 3, 2)$ is shown in Figure 7.

**Theorem 4.1.** *The $B_C(u)$ of any vertex $u$ of unicyclic graph $A(n, k, l)$, where $l$ copies of $P_k$ attached with $u_n \in C_n$ is given by*

$$B_C(u) = \begin{cases} \frac{(n-1)(n-3)}{8} + kl(\frac{n-1}{2} - i), & \text{if } u = u_i \in C_n,\ n \text{ is odd} \\ & \text{and } i = 1, 2, \cdots, \frac{n-1}{2} \text{ and } B_C(u_i) = B_C(u_{n-i}) \\ \frac{(n-2)^2}{8} + kl(\frac{n-1}{2} - i), & \text{if } u = u_i \in C_n,\ n \text{ is even} \\ & \text{and } i = 1, 2, \cdots, \frac{n}{2} - 1 \text{ and } B_C(u_i) = B_C(u_{n-i}) \\ \frac{(n-2)^2}{8}, & \text{if } u = u_i \in C_n,\ n \text{ is even and } i = \frac{n}{2} \\ \frac{(n-1)(n-3)}{8} + kl(n-1) + k^2\binom{l}{2}, & \text{if } u = u_n \in C_n,\ n \text{ is odd} \\ \frac{(n-2)^2}{8} + kl(n-1) + k^2\binom{l}{2}, & \text{if } u = u_n \in C_n,\ n \text{ is even} \\ (j-1)(n-j+kl), & \text{if } u = v_{i,j} \in P_k,\ i = 1, 2, \cdots, l \\ & \text{and } j = 1, 2, \cdots, k. \end{cases}$$

.

**Proof.** Let us consider $\{u_1, u_2, \cdots, u_n\}$ be the node points of the cycle $C_n$ of $A(n, k, l)$ and $\{v_{i,1}, v_{i,2}, \cdots, v_{i,k}\}, i = 1, 2, \cdots, l$ be the node points of the $i$th copy of the path graph $P_k$ such that $v_{i,k}$ is attached with $u_n$ by an edge. First, we calculate the B-centrality of each node point of

$C_n$ where $n$ is either even or odd.

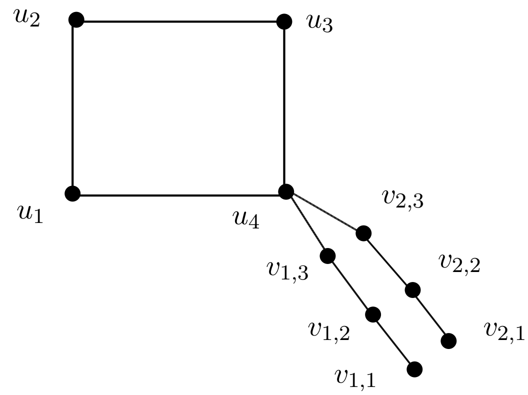


**Figure 7:** Unicyclic graph $A(4, 3, 2)$

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When $n$ is even:

Let $l$ copies of path graph $P_k$ attached with the vertex $u_n$ of $C_n$. We label the remaining vertices of $C_n$ by $u_1, u_2, \cdots, u_{\frac{n}{2}-1}, u_{\frac{n}{2}}, u_{\frac{n}{2}+1}, \cdots, u_{n-1}$ according to the clockwise direction of the vertex $u_n$. If $u = u_i, i = 1, 2, \cdots, \frac{n}{2} - 1$ be any vertex of $C_n$, then the pairs of vertices of $C_n$ contribute the value $\frac{(n-2)^2}{8}$ [51] to $B_C(u)$. Now for each $i = 1, 2, \cdots, \frac{n}{2} - 1$ the shortest paths between each pair whose one vertex in $\{v_{i,j} : i = 1, 2, \cdots, l; j = 1, 2, \cdots, k\}$ and other vertex in $\{u_1, u_2, \cdots, u_{\frac{n}{2}}\} - \{u_s : s = 1, 2, \cdots, i\}$ passes through $u$. Each of such pair contributes the value 1 to $B_C(u)$ and total number of such pair is $kl(\frac{n}{2}-1-i)$. Again each pair of vertices between a vertex of $\{v_{i,j} : i = 1, 2, \cdots, l; j = 1, 2, \cdots, k\}$ and the vertex $u_{\frac{n}{2}}$ contributes the value $\frac{1}{2}$ to $B_C(u)$ and the total number of such pairs is $kl$.

Therefore, $B_C(u) = \frac{(n-2)^2}{8} + kl(\frac{n}{2} - 1 - i) \cdot 1 + kl \cdot \frac{1}{2}$
$= \frac{(n-2)^2}{8} + kl(\frac{n}{2} - 1 - i) + \frac{kl}{2}$
$= kl(\frac{n-1}{2} - i)$

Since $u_i$ and $u_{n-i}$ are at symmetric position, for $i = 1, 2, \cdots, \frac{n}{2} - 1$, so, $B_C(u_i) = B_C(u_{n-i})$.

If $u = u_{\frac{n}{2}}$, then the vertices of $C_n$ contribute the value $\frac{(n-2)^2}{8}$ [51] to $B_C(u)$. No shortest path between a vertex $\{v_{i,j} : i = 1, 2, \cdots, l; j =$

$1, 2, \cdots, k\}$ and a vertex of $\{u_1, u_2, \cdots, u_n\} - \{u_{\frac{n}{2}}\}$ passes through $u$. Therefore, $B_C(u) = \frac{(n-2)^2}{8}$.
If $u = u_n$, then the vertices of $C_n$ contribute the value $\frac{(n-2)^2}{8}$ [51] to $B_C(u)$. The shortest path between the pairs whose two vertices lie in different $P_k$'s must pass through $u$ and the number of such pairs is $k^2\binom{l}{2}$. As there exist only one shortest path between each such pair, so, these pairs contribute the value $k^2\binom{l}{2}$ to $B_C(u)$. Again, the shortest path between a vertex of $\{v_{i,j} : i = 1, 2, \cdots, l; j = 1, 2, \cdots, k\}$ and a vertex of $\{u_1, u_2, \cdots, u_{n-1}\}$ passes through $u_n$ and the number of such pair is $kl(n-1)$. Therefore, $B_C(u) = \frac{(n-2)^2}{8} + k^2\binom{l}{2} + kl(n-1)$.
<u>When $n$ is odd:</u>
Let $l$ copies of path graph $P_k$) attached with the $u_n$ of $C_n$. We label the remaining vertices of $C_n$ by $u_1, u_2, \cdots, u_{\frac{n-1}{2}}, u_{\frac{n+1}{2}}, \cdots, u_{n-1}$ according to the clockwise direction of the vertex $u_n$. If $u = u_i, i = 1, 2, \cdots, \frac{n-1}{2}$ be any vertex of $C_n$, then the vertices of $C_n$ contribute the value $\frac{(n-1)(n-3)}{8}$ [51] to $B_C(u)$. Now, for each $i = 1, 2, \cdots, \frac{n-1}{2}$ the shortest paths between the pairs of a vertex of $\{v_{i,j} : i = 1, 2, \cdots, l; j = 1, 2, \cdots, k\}$ and a vertex of $\{u_1, u_2, \cdots, u_{\frac{n-1}{2}}\} - \{u_s : s = 1, 2, \cdots, i\}$ pass through $u$ and each pairs of vertices contribute the value 1 to $B_C(u)$. The total number of such pairs is $kl(\frac{n-1}{2} - i)$. Therefore, $B_C(u) = \frac{(n-1)(n-3)}{8} + kl(\frac{n-1}{2} - i)$.
If $u = u_n$, then the pairs of vertices of $C_n$ contribute the value $\frac{(n-1)(n-3)}{8}$ [51] to $B_C(u)$. The pairs whose vertices lie in different $P_k$'s contribute the value $k^2\binom{l}{2}$ to $B_C(u)$ (see the proof of even case). Also, from the even case, the pairs between a vertex of $\{v_{i,j} : i = 1, 2, \cdots, l; j = 1, 2, \cdots, k\}$ and a vertex of $\{u_1, u_2, \cdots, u_{n-1}\}$ contribute the value $kl(n-1)$ to $B_C(u)$. Therefore, $B_C(u) = \frac{(n-1)(n-3)}{8} + k^2\binom{l}{2} + kl(n-1)$.
Now we calculate the betweenness centrality of each vertex of $P_k$'s. Let $u = v_{p,q}$ be any vertex of the $p$th copy of $P_k$. There exist only one shortest path between a vertex of $S_1 = \{v_{p,1}, v_{p,2}, \cdots, v_{p,q-1}\}$ and a vertex of $V(A(n, k, l)) - S_1 \cup \{v_{p,q}\}$ and that path passes through $u$. The number of such pairs is $(q-1)(n + kl - q)$. So, $B_C(u) = (q-1)(n + kl - q)$. So if $u = v_{i,j}, i = 1, 2, \cdots, l; j = 1, 2, \cdots, k$, $B_C(u) = (j-1)(n - j + kl)$. □

**Relative betweenness centrality for the graph $A(n, k, l)$:**

$B'_C(u) = \dfrac{B_C(u)}{MaxB_C(u)} = \dfrac{2B_C(u)}{(n+kl-1)(n+kl-2)}$ [using the result for the star graph with $n+kl$ vertices].

**Corollary 4.2.** *The relative betweenness centrality $B'_C(u)$ of any vertex $u$ of $A(n,k,l)$ is*

$$\begin{cases} \frac{(n-1)(n-3)}{4(n+kl-1)(n+kl-2)} + \frac{2kl(\frac{n-1}{2}-i)}{(n+kl-1)(n+kl-2)}, & \text{if } u = u_i \in C_n,\ n \text{ is odd and } i = 1,2,\cdots,\frac{n-1}{2} \text{ and } B_C(u_i) = B_C(u_{n-i}) \\ \frac{(n-2)^2}{4(n+kl-1)(n+kl-2)} + \frac{2kl(\frac{n-1}{2}-i)}{(n+kl-1)(n+kl-2)}, & \text{if } u = u_i \in C_n,\ n \text{ is even and } i = 1,2,\cdots,\frac{n}{2}-1 \text{ and } B_C(u_i) = B_C(u_{n-i}) \\ \frac{(n-2)^2}{4(n+kl-1)(n+kl-2)}, & \text{if } u = u_i \in C_n,\ n \text{ is even and } i = \frac{n}{2} \\ \frac{(n-1)(n-3)}{4(n+kl-1)(n+kl-2)} + \frac{2\{kl(n-1)+k^2\binom{l}{2}\}}{(n+kl-1)(n+kl-2)}, & \text{if } u = u_n \in C_n,\ n \text{ is odd} \\ \frac{(n-2)^2}{4(n+kl-1)(n+kl-2)} + \frac{2\{kl(n-1)+k^2\binom{l}{2}\}}{(n+kl-1)(n+kl-2)}, & \text{if } u = u_n \in C_n,\ n \text{ is even} \\ \frac{2(j-1)(n-j+kl)}{(n+kl-1)(n+kl-2)}, & \text{if } u = v_{i,j} \in P_k,\ i = 1,2,\cdots,l \text{ and } j = 1,2,\cdots,k. \end{cases}$$

## 4.2 Betweenness centrality of a unicyclic graph $B(n,k,l)$

$B(n,k,l)$ is a unicyclic graph of order $n$ having a cycle $C_n$ and each of $l$ copies of path $P_k$ attached by an edge with each vertex of the cycle $C_n$, where $n \geq 3, k \geq 1$ and $l \geq 1$. i.e., $kl$ vertices are attached with each vertex of $C_n$. The number of node points of unicyclic graph $B(n,k,l)$ is $nkl + n = n(kl+1)$. Let the vertices of $C_n$ and the $i$th copy of $P_k$(corresponding to any vertex $u_p$ of $C_n$) be $\{u_1, u_2, \cdots, u_n\}$ and $\{v_{i,1}^{(p)}, v_{i,2}^{(p)}, \cdots, v_{i,k}^{(p)}\}$ respectively, where $i = 1,2,\cdots,l$ and $p = 1,2,\cdots,n$. The betweenness centrality of each vertex of the $i$th copy of $P_k$ is the same as that of the corresponding vertex of the $j$th copy of $P_k$ for all branches. The Figure 8 shows a unicyclic graph $B(4,2,2)$.

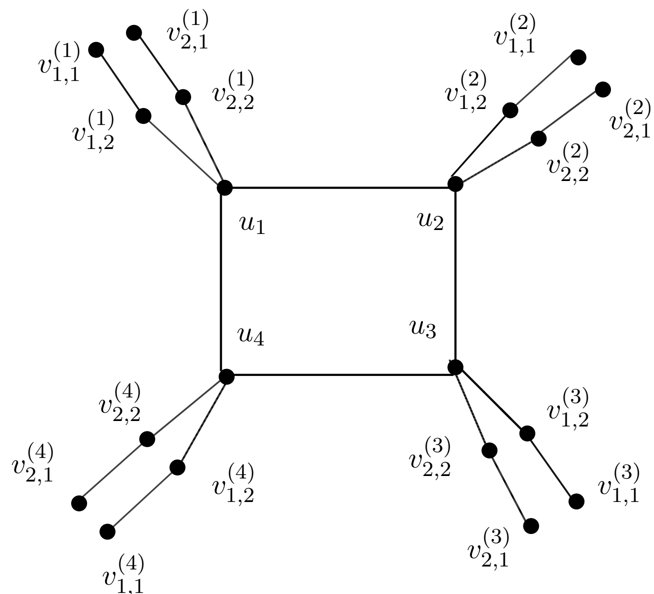


**Figure 8:** Unicyclic graph $B(4, 2, 2)$

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**Theorem 4.3.** *The $B_C(u)$ of any vertex $u$ of unicyclic graph $B(n, k, l)$ is*

$$\begin{cases} \frac{(n-1)(n-3)(kl+1)^2}{8} + kl(kl+1)(n-1) + k^2\binom{l}{2}, & \text{if } u = u_i \in C_n, \\ & n \text{ is odd and } i = 1, 2, \cdots, n \\ \frac{(n-2)^2(kl+1)^2}{8} + \frac{kl(kl+1)(2n-3)}{2} + k^2\binom{l}{2}, & \text{if } u = u_i \in C_n, \\ & n \text{ is even and } i = 1, 2, \cdots, n \\ (j-1)\{n(kl+1) - j\}, & \text{if } u = v_{i,j}^{(p)} \in P_k,\ i = 1, 2, \cdots, l \text{ and} \\ & j = 1, 2, \cdots, k \text{ and } p = 1, 2, \cdots, n. \end{cases}$$

**Proof.** Let $\{u_1, u_2, \cdots, u_n\}$ be the node points of the cycle $C_n$ of $B(n, k, l)$ and $\{v_{i,j}^{(p)} : i = 1, 2, \cdots, l; j = 1, 2, \cdots, k\}$ be the node points of the $i$th copy of the path graph $P_k$ attached by an edge with $u_p$ of $C_n$. First, we calculate the B-centrality of each node point of $P_k$. Let

$u = v_{p,q}^{(1)}, p = 1, 2, \cdots, l; q = 1, 2, \cdots, k$ be any node point of the $p$th copy of $P_k$ attached by an edge with $u_1$ of $C_n$. There exist $(q-1)(n+k-q)$ pairs of vertices between a vertex of $\{v_{p,1}^{(1)}, v_{p,2}^{(1)}, \cdots, v_{p,q-1}^{(1)}\}$ and a vertex of $\{v_{p,q+1}^{(1)}, v_{p,q+2}^{(1)}, \cdots, v_{pk}^{(1)}, u_1, u_2, \cdots, u_n\}$ and each pair contribute the value 1 to $B_C(u)$. As each vertex of $C_n$ attached $l$ copies of $P_k$ so, there are total $nl$ copies of $P_k$ in $B(n, k, l)$. The shortest path between a vertex of $\{v_{p,1}^{(1)}, v_{p,2}^{(1)}, \cdots, v_{p,q-1}^{(1)}\}$ and a vertex from the remaining $(nl-1)$ copies of $P_k$ in $B(n, k, l)$ pass through $u$. The number of pairs of vertices is $(q-1)k(nl-1)$ and each pair contribute the value 1 to $B_C(u)$.

$$\begin{aligned} \text{Therefore, } B_C(u) &= (q-1)(n+k-q) + (q-1)k(nl-1) \\ &= (q-1)(n-q+knl) \\ &= (q-1)\{n(kl+1)-q\}. \end{aligned}$$

If $u = v_{p,j}^{(1)}, j = 1, 2, \cdots, k$, then $B_C(u) = (j-1)\{n(kl+1)-j\}$. Therefore, if $u = v_{i,j}^{(r)}, i = 1, 2, \cdots, l; j = 1, 2, \cdots, k; r = 1, 2, \cdots, n$, then $B_C(u) = (j-1)\{n(kl+1)-j\}$.

We know that the vertices of $C_n$ contribute the value $\frac{(n-2)^2}{8}$, if $n$ is even and $\frac{(n-1)(n-3)}{8}$, if $n$ is odd [51]. As each vertex of $C_n$ attached $kl$ vertices so, there are $n$ sets of $(kl+1)$ vertices. If $u = u_p$, then these $n$ sets of $kl+1$ vertices of $C_n$ contribute the value $\frac{(n-2)^2(kl+1)^2}{8}$, if $n$ is even and $\frac{(n-1)(n-3)(kl+1)^2}{8}$, if $n$ is odd to $u$. If $n$ is odd and $u = u_p$ be any vertex of $C_n$, then the shortest path between a vertex from $kl$ vertices (attached with $u$) and a vertex of $\{u_1, u_2, \cdots, u_{p-1}, u_{p+1}, \cdots, u_n\}$ pass through $u$. The total number of pair is $kl(n-1)$ and each pair contribute the value 1 to $B_C(u)$. Again, the each pairs of vertices between a vertex of $\{v_{i,j}^{(p)} : i = 1, 2, \cdots, l; j = 1, 2, \cdots, k\}$ (attached with $u_p$) and a vertex from $kl(n-1)$ ($kl$ vertices are attached with $n-1$ vertices, other than the vertex $u_p$ of $C_n$) vertices of $B(n, k, l)$ contribute the centrality 1 to $u$. In this case the total number of pairs is $kl \cdot kl(n-1) = k^2l^2(n-1)$.

Let us consider $n$ is even and $u_q$ be the vertex of $C_n$ situated at the opposite of $u_p$. If $u = u_p$ then there exist only one shortest path between a vertex from $kl$ vertices (attached with $u$) and a vertex of $\{u_1, u_2, \cdots, u_{p-1}, u_{p+1}, \cdots, u_n\} - \{u_q\}$ pass through $u$. The total number of pair is $kl(n-2)$ and each pair contribute the value 1 to $B_C(u)$. Two shortest paths exist between a vertex from $kl$ vertices (attached with $u$)

and the vertex $u_q$ - one of them passes through $u$. Therefore, these pairs of vertices of $B(n,k,l)$ contribute the value $kl \cdot \frac{1}{2} = \frac{kl}{2}$ to $u$. Again, the each pairs of vertices between a vertex of $\{v_{i,j}^{(p)} : i = 1, 2, \cdots, l; j = 1, 2, \cdots, k\}$ (attached with $u_p$) and a vertex from $kl(n-2)$ ($kl$ vertices are attached with $n-2$ vertices, other than the vertex $u_p$ and $u_q$ of $C_n$) contribute the centrality 1 to $u$. In this case the total number of pairs is $kl \cdot kl(n-2) = k^2l^2(n-2)$. Also, the each pair between a vertex from $kl$ vertices (attached with $u$) and a vertex from $kl$ vertices (attached with $u_q$) contribute the value $\frac{1}{2}$ to $B_C(u)$ and the number of pair is $kl \cdot kl = k^2l^2$.
Again, the shortest path between the pairs whose two vertices lie in different $P_k$'s (attached with $u_p$) must pass through $u$. As the length of each copy is $k$ and there exist $\binom{l}{2}$ pairs between the $l$ copies of $P_k$ so, in this case the total number of pairs is $k \cdot k\binom{l}{2} = k^2\binom{l}{2}$ and each pair contribute the value 1 to $B_C(u)$.
Therefore, if $n$ is even, then

$$\begin{aligned} B_C(u) &= \tfrac{(n-2)^2(kl+1)^2}{8} + kl(n-2)\cdot 1 + \tfrac{kl}{2} + k^2l^2(n-2)\cdot 1 + k^2l^2 \cdot \tfrac{1}{2} + k^2\tbinom{l}{2}\cdot 1 \\ &= \tfrac{(n-2)^2(kl+1)^2}{8} + kl(n-2) + k^2l^2(n-2) + \tfrac{kl}{2} + \tfrac{k^2l^2}{2} + k^2\tbinom{l}{2} \\ &= \tfrac{(n-2)^2(kl+1)^2}{8} + kl(kl+1)(n-2) + \tfrac{kl(kl+1)}{2} + k^2\tbinom{l}{2} \\ &= \tfrac{(n-2)^2(kl+1)^2}{8} + \tfrac{kl(kl+1)(2n-3)}{2} + k^2\tbinom{l}{2}. \end{aligned}$$

and if $n$ is odd, then

$$\begin{aligned} B_C(u) &= \tfrac{(n-1)(n-3)(kl+1)^2}{8} + kl(n-1)\cdot 1 + k^2l^2(n-1)\cdot 1 + k^2\tbinom{l}{2}\cdot 1 \\ &= \tfrac{(n-1)(n-3)(kl+1)^2}{8} + kl(n-1) + k^2l^2(n-1) + k^2\tbinom{l}{2} \\ &= \tfrac{(n-1)(n-3)(kl+1)^2}{8} + kl(kl+1)(n-1) + k^2\tbinom{l}{2}. \end{aligned}$$ □

**Relative betweenness centrality for the graph** $B(n,k,l)$ .
We know

$$B'_C(u) = \frac{B_C(u)}{MaxB_C(u)}.$$

Now, using the result for the star graph with $n + nkl$ vertices, we can write

$$MaxB_C(u) = \frac{(n + nkl - 1)(n + nkl - 2)}{2}.$$

Therefore,

$$B'_C(u) = \frac{2B_C(u)}{(n + nkl - 1)(n + nkl - 2)}.$$

**Corollary 4.4.** *The relative betweenness centrality $B'_C(u)$ of any vertex $u$ of $B(n,k,l)$ is given by*

$$B_C(u) = \begin{cases} \frac{(n-1)(n-3)(kl+1)^2}{4(n+nkl-1)(n+nkl-2)} + \frac{2\{kl(kl+1)(n-1)+k^2\binom{l}{2}\}}{(n+nkl-1)(n+nkl-2)}, \text{if } u = u_i \in C_n, \\ \qquad n \text{ is odd and } i = 1,2,\cdots,n \\ \frac{(n-2)^2(kl+1)^2}{4(n+nkl-1)(n+nkl-2)} + \frac{2\{\frac{kl(kl+1)(2n-3)}{2}+k^2\binom{l}{2}\}}{(n+nkl-1)(n+nkl-2)}, \text{if } u = u_i \in C_n, \\ \qquad n \text{ is even and } i = 1,2,\cdots,n \\ \frac{2(j-1)\{n(kl+1)-j\}}{(n+nkl-1)(n+nkl-2)}, \text{if } u = v_{i,j}^{(p)} \in P_k,\ i = 1,2,\cdots,l; j = 1, \\ \qquad 2,\cdots,k;\ p = 1,2,\cdots,n. \end{cases}$$

## 4.3 Betweenness centrality of a unicyclic graph $D(n,k,l)$

Another type of unicyclic graph is $D(n,k,l)$. It is a unicyclic graph of order $n$ having a cycle $C_n$ and each of $l$ copies of star graph $S_k$ (having $k$ vertices) attached by an edge with only one vertex of the cycle $C_n$, where $n \geq 3, k \geq 1$ and $l \geq 1$. The number of vertices of the unicyclic graph $D(n,k,l)$ is $n+kl$. Let $\{u_1, u_2, \cdots, u_{\frac{n}{2}-1}, u_{\frac{n}{2}}, u_{\frac{n}{2}+1}, \cdots, u_{n-1}, u_n\}$ be the vertices of $C_n$, where $n$ is even. From the vertex $u_n$ of $C_n$, the vertices $u_1$ and $u_{n-1}$, $u_2$ and $u_{n-2}, \cdots, u_{\frac{n}{2}-1}$ and $u_{\frac{n}{2}+1}$ lie at the same distances. Therefore, the betweenness centrality of $u_1$ and $u_{n-1}$ are equal. Similarly, the betweenness centrality of each pair of vertices(the distance of both vertices that are situated at the same distance from $u_n$) are equal. If $n$ is odd, then let the vertex set of $C_n$ be $\{u_1, u_2, \cdots, u_{\frac{n-1}{2}}, u_{\frac{n+1}{2}}, \cdots, u_{n-1}, u_n\}$. In this case, the betweenness centrality of $u_1$ and $u_{n-1}$, $u_2$ and $u_{n-2}, \cdots$, $u_{\frac{n-1}{2}}$ and $u_{\frac{n+1}{2}}$ are equal. And also let $\{v_i, v_{i,1}, v_{i,2}, \cdots, v_{i,k-1}\}, i = 1,2,\cdots,l$ be the set of vertices of the $i$th copy of star graph $S_k$ (where $v_i$ is the central vertex of $S_k$) attached with $u_n \in C_n$. A unicyclic graph $D(3,4,2)$ is shown in Figure 9.

**Theorem 4.5.** *The $B_C(u)$ of any vertex $u$ of unicyclic graph $D(n,k,l)$, where $l$ copies of $S_k$ attached with $u_n \in C_n$ is given by*

$$B_C(u) = \begin{cases} \frac{(n-1)(n-3)}{8} + kl(\frac{n-1}{2} - i), & \textit{if } u = u_i \in C_n,\ n \textit{ is odd and } i = 1,2,\cdots,\frac{n-1}{2} \textit{ and } B_C(u_i) = B_C(u_{n-i}) \\ \frac{(n-2)^2}{8} + kl(\frac{n-1}{2} - i), & \textit{if } u = u_i \in C_n,\ n \textit{ is even and } i = 1,2,\cdots,\frac{n}{2}-1 \textit{ and } B_C(u_i) = B_C(u_{n-i}) \\ \frac{(n-2)^2}{8}, & \textit{if } u = u_i \in C_n,\ n \textit{ is even and } i = \frac{n}{2} \\ \frac{(n-1)(n-3)}{8} + kl(n-1) + k^2\binom{l}{2}, & \textit{if } u = u_n \in C_n,\ n \textit{ is odd} \\ \frac{(n-2)^2}{8} + kl(n-1) + k^2\binom{l}{2}, & \textit{if } u = u_n \in C_n,\ n \textit{ is even} \\ \binom{k-1}{2} + (k-1)\{n + k(l-1)\}, & \textit{if } u \textit{ is central vertex of any copy of } S_k \\ 0, & \textit{if } u \textit{ is pendant vertex of the star graph } S_k. \end{cases}$$

**Proof.** Let us consider $\{u_1, u_2, \cdots, u_n\}$ be the node points of the cycle $C_n$ of $D(n,k,l)$ and $\{v_i, v_{i,1}, v_{i,2}, \cdots, v_{i,k-1}\}, i = 1, 2, \cdots, l$ be the node points of the $i$th copy of the star graph $S_k$ (where $v_i$ is the central vertex of $S_k$) attached with $u_n \in C_n$. First, we calculate the B-centrality of each vertex of $C_n$ where $n$ is either even or odd.

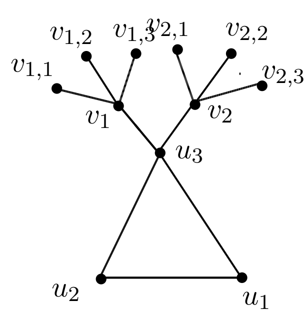


**Figure 9:** Unicyclic graph $D(3, 4, 2)$

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When $n$ is even:
Let $l$ copies of star graph $S_k$ attached with the vertex $u_n$ of $C_n$. We label the remaining vertices of $C_n$ by $u_1, u_2, \cdots, u_{\frac{n}{2}-1}, u_{\frac{n}{2}}, u_{\frac{n}{2}+1}, \cdots, u_{n-1}$ according to the clockwise direction of the vertex $u_n$. If $u = u_i, i = 1, 2, \cdots, \frac{n}{2} - 1$ be any vertex of $C_n$, then the vertices of $C_n$ contribute the value $\frac{(n-2)^2}{8}$ [51] to $B_C(u)$. Now for each $i = 1, 2, \cdots, \frac{n}{2} - 1$ the shortest paths between each pair whose one vertex in $\{v_i, v_{i,j} : i = 1, 2, \cdots, l; j = 1, 2, \cdots, k-1\}$ and other vertex in $\{u_1, u_2, \cdots, u_{\frac{n}{2}}\} - \{u_s : s = 1, 2, \cdots, i\}$ passes through $u$. Each of such pair contributes the value 1 to $B_C(u)$ and total number of such pair is $kl(\frac{n}{2} - 1 - i)$. Again each pair of vertices between a vertex of $\{v_i, v_{i,j} : i = 1, 2, \cdots, l; j = 1, 2, \cdots, k-1\}$ and the vertex $u_{\frac{n}{2}}$ contribute the value $\frac{1}{2}$ to $B_C(u)$ and the total number of such pairs is $kl$.
Therefore, $B_C(u) = \frac{(n-2)^2}{8} + kl(\frac{n}{2} - 1 - i) \cdot 1 + kl \cdot \frac{1}{2}$
$= \frac{(n-2)^2}{8} + kl(\frac{n}{2} - 1 - i) + \frac{kl}{2}$
$= kl(\frac{n-1}{2} - i)$.
Since $u_i$ and $u_{n-i}, i = 1, 2, \cdots, \frac{n}{2} - 1$ are situated at the same distance from $u_n$, so,

$$B_C(u_i) = B_C(u_{n-i}).$$

If $u = u_n$, then the vertices of $C_n$ contribute the value $\frac{(n-2)^2}{8}$ [51] to $B_C(u)$. The shortest path between the pairs whose two vertices lie in different $S_k$'s must pass through $u$ and the number of such pairs is $k^2\binom{l}{2}$. As there exist only one shortest path between each such pair, so, these pairs contribute the value $k^2\binom{l}{2}$ to $B_C(u)$. Again, the shortest path between a vertex of $\{v_i, v_{i,j} : i = 1, 2, \cdots, l; j = 1, 2, \cdots, k-1\}$ and a vertex of $\{u_1, u_2, \cdots, u_{n-1}\}$ passes through $u_n$ and the number of such pair is $kl(n-1)$. Therefore, $B_C(u) = \frac{(n-2)^2}{8} + k^2\binom{l}{2} + kl(n-1)$.
If $u = u_{\frac{n}{2}}$, then the vertices of $C_n$ contribute the value $\frac{(n-2)^2}{8}$ [51] to $B_C(u)$. No shortest path between a vertex $\{v_i, v_{i,j} : i = 1, 2, \cdots, l; j = 1, 2, \cdots, k\}$ and a vertex of $\{u_1, u_2, \cdots, u_n\} - \{u_{\frac{n}{2}}\}$ passes through $u$. Therefore, $B_C(u) = \frac{(n-2)^2}{8}$.
When $n$ is odd:
Let $l$ copies of $S_k$ attached with $u_n \in C_n$. We label the remaining vertices of $C_n$ by $u_1, u_2, \cdots,$
$u_{\frac{n-1}{2}}, u_{\frac{n+1}{2}}, \cdots, u_{n-1}$ according to the clockwise direction of the vertex

$u_n$. Since $u_i$ and $u_{n-i}, i = 1, 2, \cdots, \frac{n}{2} - 1$ are situated at the same distance from $u_n$, so, $B_C(u_i) = B_C(u_{n-i})$. If $u = u_i, i = 1, 2, \cdots, \frac{n-1}{2}$ be any vertex of $C_n$, then the vertices of $C_n$ contribute the value $\frac{(n-1)(n-3)}{8}$ [51] to $B_C(u)$. Now, for each $i = 1, 2, \cdots, \frac{n-1}{2}$ the shortest paths between the pairs of a vertex of $\{v_i, v_{i,j} : i = 1, 2, \cdots, l; j = 1, 2, \cdots, k-1\}$ and a vertex of $\{u_1, u_2, \cdots, u_{\frac{n-1}{2}}\} - \{u_s : s = 1, 2, \cdots, i\}$ pass through $u$ and each pairs of vertices contribute the value 1 to $B_C(u)$. The total number of such pairs is $kl(\frac{n-1}{2} - i)$. Therefore,

$$B_C(u) = \frac{(n-1)(n-3)}{8} + kl(\frac{n-1}{2} - i).$$

If $u = u_n$, then the pairs of vertices of $C_n$ contribute the value $\frac{(n-1)(n-3)}{8}$ [51] to $B_C(u)$. The pairs whose vertices lie in different $S_k$'s contribute the value $k^2\binom{l}{2}$ to $B_C(u)$ (see the proof of even case). Also, from the even case, the pairs between a vertex of $\{v_i, v_{i,j} : i = 1, 2, \cdots, l; j = 1, 2, \cdots, k\}$ and a vertex of $\{u_1, u_2, \cdots, u_{n-1}\}$ contribute the value $kl(n-1)$ to $B_C(u)$. Therefore, $B_C(u) = \frac{(n-1)(n-3)}{8} + k^2\binom{l}{2} + kl(n-1)$.

Now, we calculate the betweenness centrality of the central vertex of any copy of the star graph $S_k$. Let $u = v_p$ be the central vertex of the $p$th copy of star graph $S_k$. There exist only one shortest path between a vertex of $\{v_{p,j} : j = 1, 2, \cdots, k-1\}$ and a vertex of $\{u_1, u_2, \cdots, u_n\}$ which pass through $u$. The total number of such pairs is $n(k-1)$ and each pair contributes the value 1 to $B_C(u)$. Also, the shortest path between the pairs of vertices of $\{v_{p,j} : j = 1, 2, \cdots, k-1\}$ pass through $u$ and the number of pairs is $\binom{k-1}{2}$. Each pair contribute the value 1 to $B_C(u)$. Again, the shortest path between a vertex of $\{v_{p,j} : j = 1, 2, \cdots, k-1\}$ and a vertex of $\{v_i, v_{i,j} : i = 1, 2, \cdots, p-1, p+1, \cdots, l; j = 1, 2, \cdots, k-1\}$ pass through $u$ and in this case the total number of such pairs is $(k-1)k(l-1)$. These pairs of vertices contribute the value $(k-1)k(l-1)$ to $B_C(u)$. Therefore, $B_C(u) = n(k-1) + \binom{k-1}{2} + (k-1)k(l-1) = \binom{k-1}{2} + (k-1)\{n + k(l-1)\}$. If $u$ is pendant vertices of $D(n, k, l)$, then it is obvious that the betweenness centrality of $u$ is 0.

**Relative betweenness centrality for the graph $D(n, k, l)$:**

Here, $B'_C(u) = \dfrac{B_C(u)}{MaxB_C(u)} = \dfrac{2B_C(u)}{(n + kl - 1)(n + kl - 2)}$ [using the result for the star graph with $n + kl$ vertices].

**Corollary 4.6.** *The relative betweenness centrality $B'_C(u)$ of any vertex* *u of $D(n,k,l)$ is*

$$\begin{cases} \frac{(n-1)(n-3)}{4(n+kl-1)(n+kl-2)} + \frac{2kl(\frac{n-1}{2}-i)}{(n+kl-1)(n+kl-2)}, & \text{if } u = u_i \in C_n,\ n \text{ is odd and } i = 1,2,\cdots,\frac{n-1}{2} \text{ and } B_C(u_i) = B_C(u_{n-i}) \\ \frac{(n-2)^2}{4(n+kl-1)(n+kl-2)} + \frac{2kl(\frac{n-1}{2}-i)}{(n+kl-1)(n+kl-2)}, & \text{if } u = u_i \in C_n,\ n \text{ is even and } i = 1,2,\cdots,\frac{n}{2}-1 \text{ and } B_C(u_i) = B_C(u_{n-i}) \\ \frac{(n-2)^2}{4(n+kl-1)(n+kl-2)}, & \text{if } u = u_i \in C_n,\ n \text{ is even and } i = \frac{n}{2} \\ \frac{(n-1)(n-3)}{4(n+kl-1)(n+kl-2)} + \frac{2\{kl(n-1)+k^2\binom{l}{2}\}}{(n+kl-1)(n+kl-2)}, & \text{if } u = u_n \in C_n,\ n \text{ is odd} \\ \frac{(n-2)^2}{4(n+kl-1)(n+kl-2)} + \frac{2\{kl(n-1)+k^2\binom{l}{2}\}}{(n+kl-1)(n+kl-2)}, & \text{if } u = u_n \in C_n,\ n \text{ is even} \\ \frac{2\left[\binom{k-1}{2}+(k-1)\{n+k(l-1)\}\right]}{(n+kl-1)(n+kl-2)}, & \text{if } u \text{ is central vertex of any copy of } S_k \\ 0, & \text{if } u \text{ is pendant vertex of the star graph } S_k. \end{cases}$$

### 4.4 Betweenness centrality of a unicyclic graph $E(n,k,l)$

The unicyclic graph $E(n,k,l)$ , $n \geq 3, k \geq 1$ and $l \geq 1$ having a cycle $C_n$ and each of $l$ copies of star graph $S_k$ attached by an edge with each node points of $C_n$. The number of node points of unicyclic graph $E(n,k,l)$ is $nkl + n = n(kl+1)$. Let the vertices of $C_n$ be $\{u_1, u_2, \cdots, u_n\}$. And also let $\{v_i^{(p)}, v_{i,1}^{(p)}, v_{i,2}^{(p)}, \cdots, v_{i,k-1}^{(p)}\}, i = 1, 2, \cdots, l$ be the set of vertices of the $i$th copy of star graph $S_k$ attached to the vertex $u_p$ of $C_n$, where $v_i^{(p)}$ is the central vertex of $S_k$. The Figure 10 shows the unicyclic graph $E(3,4,1)$.

**Theorem 4.7.** *The $B_C(u)$ of any vertex $u$ of unicyclic graph $E(n,k,l)$ is*

$$B_C(u) = \begin{cases} \frac{(n-1)(n-3)(kl+1)^2}{8} + kl(kl+1)(n-1) + k^2\binom{l}{2}, \quad if \\ \qquad u = u_i \in C_n,\ n\ is\ odd\ and\ i = 1, 2, \cdots, n \\ \frac{(n-2)^2(kl+1)^2}{8} + \frac{kl(kl+1)(2n-3)}{2} + k^2\binom{l}{2}, \quad if \\ \qquad u = u_i \in C_n,\ n\ is\ even\ and\ i = 1, 2, \cdots, n \\ \binom{k-1}{2} + (k-1)\{n + k(nl-1)\}, \qquad if\ u\ is\ central\ vertex \\ \qquad of\ any\ copy\ of\ S_k \\ 0,\ if\ u\ is\ pendant\ vertex\ of\ the\ star\ graph\ S_k. \end{cases}$$

.

**Proof.** Let $\{u_1, u_2, \cdots, u_n\}$ be the node point of the cycle $C_n$ of $E(n, k, l)$ and $\{v_i^{(r)}, v_{i,j}^{(r)} : i = 1, 2, \cdots, l; j = 1, 2, \cdots, k-1\}$ be the node points of the $i$th copy of the star graph $S_k$ (where $v_i^{(r)}$ is the central node of $S_k$) attached with $u_r \in C_n$. First, we calculate the betweenness centrality of the central vertex of any copy of $S_k$ attached at $u_r$. Let $u = v_p^{(r)}$ then there is only one shortest path between each pair of vertices of $\{v_{p,j}^{(r)} : j = 1, 2, \cdots, k-1\}$ and passes through $v_p^{(r)}$ and the number of such pairs is $\binom{k-1}{2}$. Each pair of vertices contribute the value 1 to $B_C(u)$. Again, the shortest path between a vertex of $\{v_{p,j}^{(r)} : j = 1, 2, \cdots, k-1\}$ and a vertex of $\{u_1, u_2, \cdots, u_n\}$ pass also through $u$. In this case the number of such pairs is $n(k-1)$ and each pair contribute the value 1 to $B_C(u)$. As each vertex of $C_n$ attached $l$ copies of $S_k$ so, there are total $nl$ copies of $S_k$ in $E(n, k, l)$. The shortest path between a vertex of $\{v_{p,j}^{(r)} : j = 1, 2, \cdots, k-1\}$ and a vertex from the remaining $(nl-1)$ copies of $S_k$ of $E(n, k, l)$ pass through $u$. In this case, the number of pairs of vertices is $(k-1)k(nl-1)$ and each pair contribute the value 1 to $B_C(u)$. Therefore, $B_C(u) = \binom{k-1}{2} \cdot 1 + n(k-1) \cdot 1 + (k-1)k(nl-1) \cdot 1$

$$= \binom{k-1}{2} + n(k-1) + (k-1)k(nl-1)$$
$$= \binom{k-1}{2} + (k-1)\{n + k(nl-1)\}.$$

If $u$ is pendant vertices of $S_k$, then it is obvious that $B_C(u) = 0$.

Now we calculate the betweenness centrality of any vertex $u = u_p$ of

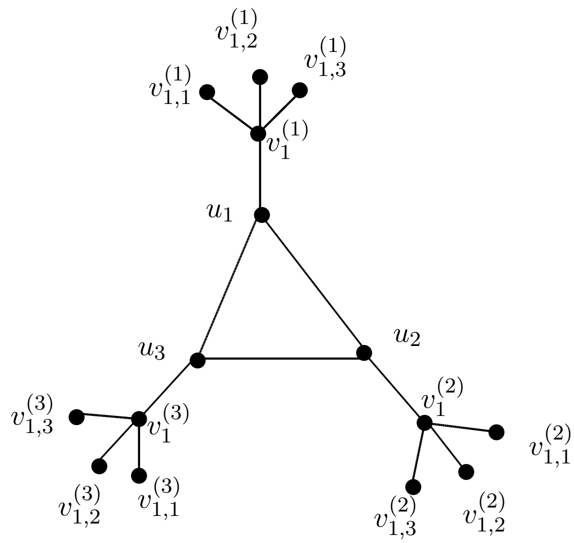


**Figure 10:** Unicyclic graph $E(3, 4, 1)$

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$C_n$. If $n$ is odd and $u = u_p$ be any vertex of $C_n$ then the shortest path between a vertex from $kl$ vertices (attached with $u$) and a vertex of $\{u_1, u_2, \cdots, u_{p-1}, u_{p+1}, \cdots, u_n\}$ pass through $u$. The total number of pairs is $kl(n-1)$ and each pair contribute the value 1 to $B_C(u)$. Again, the each pairs of vertices between a vertex of $\{v_i^{(p)}, v_{i,j}^{(p)} : i = 1, 2, \cdots, l; j = 1, 2, \cdots, k-1\}$ (attached with $u_p$) and a vertex from $kl(n-1)$ ($kl$ vertices are attached with $n-1$ vertices, other than the vertex $u_p$ of $C_n$) contribute the centrality 1 to $u$. In this case the total number of pairs is $kl \cdot kl(n-1) = k^2l^2(n-1)$.
Let $n$ is even and $u_q$ be the vertex of $C_n$ situated at the opposite of $u_p$. If $u = u_p$, then there exist only one shortest path between a vertex from $kl$ vertices (attached with $u$) and a vertex of
$\{u_1, u_2, \cdots, u_{p-1}, u_{p+1}, \cdots, u_n\} - \{u_q\}$ pass through $u$. The total number of pair is $kl(n-2)$ and each pair contribute the value 1 to $B_C(u)$. There exist two shortest paths between a vertex from $kl$ vertices (attached with $u$) and the vertex $u_q$ - one of them passes through $u$. Therefore, these pairs of vertices of $E(n, k, l)$ contribute the value $kl \cdot \frac{1}{2} = \frac{kl}{2}$

to $u$. Again, the each pairs of vertices between a vertex in $\{v_i^{(p)}, v_{i,j}^{(p)} : i = 1, 2, \cdots, l; j = 1, 2, \cdots, k-1\}$ (attached with $u_p$) and a vertex from $kl(n-2)$ ($kl$ vertices are attached with $n-2$ vertices, other than the vertex $u_p$ and $u_q$ of $C_n$) contribute the centrality 1 to $u$. In this case the total number of pairs is $kl \cdot kl(n-2) = k^2l^2(n-2)$. Also, the each pair between a vertex from $kl$ vertices (attached with $u$) and a vertex from $kl$ vertices (attached with $u_q$) contribute the value $\frac{1}{2}$ to $B_C(u)$ and the total number of pair is $kl \cdot kl = k^2l^2$.

We know that the vertices of $C_n$ contribute the value $\frac{(n-2)^2}{8}$, if $n$ is even and $\frac{(n-1)(n-3)}{8}$, if $n$ is odd [51]. As each vertex of $C_n$ attached $kl$ vertices so, there are $n$ sets of $(kl+1)$ vertices. If $u = u_p$ then these $n$ sets of $kl+1$ vertices of $C_n$ contribute the value $\frac{(n-2)^2(kl+1)^2}{8}$, if $n$ is even and $\frac{(n-1)(n-3)(kl+1)^2}{8}$, if $n$ is odd to $u$. Again, the shortest path between the pairs whose two vertices lie in different $S_k$'s (attached with $u_p$) must pass through $u_p$ and number of such pairs is $k^2\binom{l}{2}$ and each pair contribute the value 1 to $B_C(u)$.

Therefore, if $n$ is even, then

$$\begin{aligned} B_C(u) &= \tfrac{(n-2)^2(kl+1)^2}{8} + kl(n-2)\cdot 1 + \tfrac{kl}{2} + k^2l^2(n-2)\cdot 1 + k^2l^2 \cdot \tfrac{1}{2} + k^2\tbinom{l}{2}\cdot 1 \\ &= \tfrac{(n-2)^2(kl+1)^2}{8} + kl(n-2) + k^2l^2(n-2) + \tfrac{kl}{2} + \tfrac{k^2l^2}{2} + k^2\tbinom{l}{2} \\ &= \tfrac{(n-2)^2(kl+1)^2}{8} + kl(kl+1)(n-2) + \tfrac{kl(kl+1)}{2} + k^2\tbinom{l}{2} \\ &= \tfrac{(n-2)^2(kl+1)^2}{8} + \tfrac{kl(kl+1)(2n-3)}{2} + k^2\tbinom{l}{2}. \end{aligned}$$

and if $n$ is odd then

$$\begin{aligned} B_C(u) &= \tfrac{(n-1)(n-3)(kl+1)^2}{8} + kl(n-1)\cdot 1 + k^2l^2(n-1)\cdot 1 + k^2\tbinom{l}{2}\cdot 1 \\ &= \tfrac{(n-1)(n-3)(kl+1)^2}{8} + kl(n-1) + k^2l^2(n-1) + k^2\tbinom{l}{2} \\ &= \tfrac{(n-1)(n-3)(kl+1)^2}{8} + kl(kl+1)(n-1) + k^2\tbinom{l}{2}. \end{aligned}$$ □

**Relative betweenness centrality for the graph $E(n, k, l)$:**

We know

$$B'_C(u) = \frac{B_C(u)}{MaxB_C(u)}.$$

Now, using the result for the star graph with $n + nkl$ vertices, we can write

$$MaxB_C(u) = \frac{(n + nkl - 1)(n + nkl - 2)}{2}.$$

Therefore,

$$B'_C(u) = \frac{2B_C(u)}{(n+nkl-1)(n+nkl-2)}.$$

**Corollary 4.8.** *The relative betweenness centrality of any vertex $u$ of $E(n,k,l)$ is given by*

$$B_C(u) = \begin{cases} \frac{(n-1)(n-3)(kl+1)^2}{4(n+nkl-1)(n+nkl-2)} + \frac{2\{kl(kl+1)(n-1)+k^2\binom{l}{2}\}}{(n+nkl-1)(n+nkl-2)}, \text{if } u = u_i \in C_n, \\ \qquad n \text{ is odd and } i = 1,2,\cdots,n \\ \frac{(n-2)^2(kl+1)^2}{4(n+nkl-1)(n+nkl-2)} + \frac{2\{\frac{kl(kl+1)(2n-3)}{2}+k^2\binom{l}{2}\}}{(n+nkl-1)(n+nkl-2)}, \text{if } u = u_i \in C_n, \\ \qquad n \text{ is even and } i = 1,2,\cdots,n \\ \frac{2\left[\binom{k-1}{2}+(k-1)\{n+k(nl-1)\}\right]}{(n+nkl-1)(n+nkl-2)}, \text{if } u \text{ is central vertex of any copy} \\ \qquad \text{of the star graph } S_k \\ 0, \text{ if } u \text{ is pendant vertex of the star graph } S_k. \end{cases}$$

## 5 Conclusion

There are different centrality measurements to identify the critical vertices in networks. Betweenness centrality is an important variant of centrality measurement for recognizing a network's vertex characteristic. It is used to determine the important vertex in biological networks, sexual networks and AIDS, social networks, computer networks, urban networks, transportation networks, food web networks, supply chain networks, drug targets, organizational behavior, and terrorist networks. In this paper, we state and prove some theorems related to the betweenness centrality of corona graphs and unicyclic graphs. We also determine the relative betweenness centrality of these graphs. In the future, we shall try to determine the betweenness centrality of bicyclic graphs and cactus graphs based on the results of unicyclic graphs.

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