Journal of Mathematical Extension
Vol. 17, No. 5, (2023) (3)1-12
URL: https://doi.org/10.30495/JME.2023.2562
ISSN: 1735-8299
Original Research Paper

*-Boundedness and *-Continuity of Non-Newtonian Superposition Operators on $\ell_{\infty,\alpha}$ and $\ell_{1,\beta}$

F. Erdoğan
Ondokuz Mayıs University

B. Sağır*
Ondokuz Mayıs University

Abstract. Sağır and Erdoğan [13] have defined non-Newtonian superposition operators ${}_N S_\phi$ where $\phi : \mathbb{N} \times \mathbb{R}_\alpha \to \mathbb{R}_\beta$ by ${}_N S_\phi(x) = (\phi(m, x_m))_{m=1}^{\infty}$ for all $\alpha-$sequences (x_m). In this study, we get the conditions for the *-boundedness, *-locally boundedness and *-uniform continuity of the non-Newtonian superposition operator ${}_N S_\phi : \ell_{\infty,\alpha} \to \ell_{1,\beta}$.

AMS Subject Classification: 47H30; 46A45;11U10
Keywords and Phrases: *-Boundedness, *-locally boundedness, *-continuity, *-uniform continuity, non-Newtonian superposition operator, non-Newtonian sequence spaces

1 Introduction and Preliminaries

Grossman and Katz were the first to introduce non-Newtonian calculus to mathematics. They published a book on the basics of non-Newtonian

Received: November 2022; Accepted: July 2023
*Corresponding Author

calculus [8]. Recently, many writers have studied on classical sequence spaces using non-Newtonian calculus [4, 5, 12, 19]. Kirişci [10] got some conclusions on non-Newtonian metric spaces. Yılmaz[20] worked on multiplicative calculus.

An injective function with its domain the set of real numbers $\mathbb{R}$ is described as generator and the range of generator is a subset of $\mathbb{R}$. Let's take any α generator with range $A = \mathbb{R}_\alpha$. Let's define $\alpha-$addition, $\alpha-$subtraction, $\alpha-$multiplication, $\alpha-$division and $\alpha-$order as follows;

$\alpha-$addition	$u\dot{+}v = \alpha\left(\alpha^{-1}(u) + \alpha^{-1}(v)\right)$
$\alpha-$subtraction	$u\dot{-}v = \alpha\left(\alpha^{-1}(u) - \alpha^{-1}(v)\right)$
$\alpha-$multiplication	$u\dot{\times}v = \alpha\left(\alpha^{-1}(u) \times \alpha^{-1}(v)\right)$
$\alpha-$division	$u\dot{/}v = \alpha\left(\alpha^{-1}(u) / \alpha^{-1}(v)\right) \quad (v \neq \dot{0})$
$\alpha-$order	$u\dot{<}v \ \left(u\dot{\leq}v\right) \Leftrightarrow \alpha^{-1}(u) < \alpha^{-1}(v) \ \left(\alpha^{-1}(u) \leq \alpha^{-1}(v)\right)$

for $u, v \in \mathbb{R}_\alpha$ [8].

$(\mathbb{R}_\alpha, \dot{+}, \dot{\times}, \dot{\leq})$ is totally ordered field [2].

The numbers $x\dot{>}\dot{0}$ are $\alpha-$positive numbers and the numbers $x\dot{<}\dot{0}$ are $\alpha-$negative numbers in $\mathbb{R}_\alpha$. $\alpha-$integers are obtained by successive $\alpha-$addition of $\dot{1}$ to $\dot{0}$ and successive $\alpha-$subtraction of $\dot{1}$ from $\dot{0}$. Also $\dot{k} = \alpha(k)$ for every integer k.

$\alpha-$absolute value of a number $x \in \mathbb{R}_\alpha$, $\sqrt[p]{x}^{\alpha}$ and x^{p_α} were defined by Grossman and Katz. They also described the $*-$calculus with the aid of two randomly selected generators. Let's take any generator α and β and denote the ordered arithmetic pair $*$("star") ($\alpha-$arithmetic, $\beta-$arithmetic).The following notations will be used in $*-$calculus.

	$\alpha-$arithmetic	$\beta-$ arithmetic
Realm	$A\,(=\mathbb{R}_\alpha)$	$B\,(=\mathbb{R}_\beta)$
Summation	$\dot{+}$	$\ddot{+}$
Subtraction	$\dot{-}$	$\ddot{-}$
Multiplication	$\dot{\times}$	$\ddot{\times}$
Division	$\dot{/}$	$\ddot{/}$
Ordering	$\dot{<}$	$\ddot{<}$

The isomorphism from $a-$arithmetic to $\beta-$arithmetic is determined by the unique function $\imath$(iota) having the following properties.

1. $\imath$ is one-to-one.

2. $\imath$ is on A and onto B.
3. For any numbers p and q in A,

$$\begin{aligned} \iota\left(p\dot{+}q\right) &= \iota\left(p\right)\ddot{+}\iota\left(q\right), \\ \iota\left(p\dot{-}q\right) &= \iota\left(p\right)\ddot{-}\iota\left(q\right), \\ \iota\left(p\dot{\times}q\right) &= \iota\left(p\right)\ddot{\times}\iota\left(q\right), \\ \iota\left(p\dot{/}q\right) &= \iota\left(p\right)\ddot{/}\iota\left(q\right),\ q\neq\dot{0} \\ p\dot{<}q &\Longleftrightarrow \iota\left(p\right)\ddot{<}\iota\left(q\right). \end{aligned}$$

It turns out that $\iota\left(p\right)=\beta\left\{\alpha^{-1}\left(p\right)\right\}$ for every number p in A and that $\iota\left(\dot{k}\right)=\ddot{k}$ for each integer k [8].

Let X be a vector space over the field $\mathbb{R}_\alpha$ and $\|.\|_{X,\alpha}$ be a function from X to ${\mathbb{R}^+}_\alpha\cup\left\{\dot{0}\right\}$ satisfying the following non-Newtonian norm axioms. For $z,t\in X$ and $\lambda\in\mathbb{R}_\alpha$,

(NN1) $\|z\|_{X,\alpha}=\dot{0}\Leftrightarrow z=\dot{0}$,
(NN2) $\left\|\lambda\dot{\times}z\right\|_{X,\alpha}=|\lambda|_\alpha\dot{\times}\|z\|_{X,\alpha}$,
(NN3) $\left\|z\dot{+}t\right\|_{X,\alpha}\dot{\leq}\|z\|_{X,\alpha}\dot{+}\|t\|_{X,\alpha}$.

Then $\left(X,\|.\|_{X,\alpha}\right)$ is a non-Newtonian normed space.

The non-Newtonian sequence spaces S_α, $\ell_{\infty,\alpha}$ and $\ell_{p,\alpha}$ on $\mathbb{R}_\alpha$ are defined as following:

$$\begin{aligned} S_\alpha &= \{x=(x_n):\forall n\in\mathbb{N},\ x_n\in\mathbb{R}_\alpha\} \\ \ell_{\infty,\alpha} &= \left\{x=(x_n)\in S_\alpha:\ {}^{\alpha}\sup_{n\in\mathbb{N}}|x_n|_\alpha\dot{<}\dot{+}\infty\right\}, \\ \ell_{p,\alpha} &= \left\{x=(x_n)\in S_\alpha:\ {}_{\alpha}\sum_{n=1}^{\infty}|x_n|_\alpha^{p_\alpha}\dot{<}\dot{+}\infty\right\}\quad(1\leq p<\infty). \end{aligned}$$

The sequence space $\ell_{\infty,\alpha}$ is non-Newtonian normed space with the non-Newtonian norm $\|.\|_{\ell_\infty,\alpha}$. In here, the norm is defined by $\|x\|_{\ell_\infty,\alpha}={}^{\alpha}\sup_{n\in\mathbb{N}}|x_n|_\alpha$ [2]. The $\alpha-$sequence $e_n^{(k)}$ is defined as

$$e_n^{(k)}=\begin{cases}\dot{1}, & k=n\\ \dot{0}, & k\neq n\end{cases}.$$

Let X_α be space of non-Newtonian real number sequences, Y_α be a sequence space on $\mathbb{R}_\alpha$ and Z_β be a sequence space on $\mathbb{R}_\beta$. A non-Newtonian superposition operator ${}_NS_\phi$ on Y_α is a mapping from Y_α to X_α which is defined by ${}_NS_\phi\left(x\right) = \left(\phi\left(m, x_m\right)\right)_{m=1}^{\infty}$ where $\phi : \mathbb{N} \times \mathbb{R}_\alpha \to \mathbb{R}_\beta$. In addition, the function ϕ satisfies following condition (NA_1).

(NA_1) $\phi(m, \dot{0}) = \ddot{0}$ for every $m \in \mathbb{N}$.

If ${}_NS_\phi(x) \in Z_\beta$ for all $x = (x_m) \in Y_\alpha$, ${}_NS_\phi$ acts from Y_α into Z_β and it is written that ${}_NS_\phi : Y_\alpha \to Z_\beta$ [13].

Also, we shall suppose the following conditions.

(NA_2) $\phi(m, .)$ is *-continuous for every $m \in \mathbb{N}$.

(NA_2') $\phi(m, .)$ is $\beta-$bounded on every $\alpha-$bounded subset of $\mathbb{R}_\alpha$ for every $m \in \mathbb{N}$.

Sağır and Erdoğan [13] have characterized the non-Newtonian superposition operator ${}_NS_\phi$ on $\ell_{\infty,\alpha}$ as the following.

Theorem 1.1. *Let us suppose that $\phi : \mathbb{N} \times \mathbb{R}_\alpha \to \mathbb{R}_\beta$ satisfies the condition (NA_2'). Then ${}_NS_\phi : \ell_{\infty,\alpha} \to \ell_{1,\beta}$ iff there exists $(u_m) \in \ell_{1,\beta}$ such that*

$$\left|\phi\left(m, t\right)\right|_\beta \ddot{\leq} u_m \quad \text{whenever} \quad \left|t\right|_\alpha \dot{\leq} \mu$$

for each $\alpha-$number $\mu \dot{>} \dot{0}$ and all $m \in \mathbb{N}$.

Theorem 1.2. *Non-Newtonian superposition operator ${}_NS_\phi : \ell_{\infty,\alpha} \to \ell_{1,\beta}$ is *-continuous on $\ell_{\infty,\alpha}$ iff the function $\phi\left(m, .\right)$ is *-continuous $\mathbb{R}_\alpha$ for all $m \in \mathbb{N}$.*

Prior to proving theorems about non-Newtonian superposition operators on $\ell_{\infty,\alpha}$ to $\ell_{1,\beta}$, we give the required definitions and theorems in the sense of *-calculus.

Definition 1.3. *Let (X_α, d_α) and $\left(Y_\beta, d'_\beta\right)$ be non-Newtonian sequence spaces. An operator $T : X_\alpha \to Y_\beta$ is *-bounded if $T\left(E\right)$ is $\beta-$bounded for all $\alpha-$bounded subset E of X_α .*

Definition 1.4. *Let (X_α, d_α) and $\left(Y_\beta, d'_\beta\right)$ be non-Newtonian sequence spaces. An operator $T : X_\alpha \to Y_\beta$ is *-locally bounded at $u_0 \in X_\alpha$ if there exist $\mu \dot{>} \dot{0}$ and $\eta \ddot{>} \ddot{0}$ such that $T\left(u\right) \in B_{d'_\beta}\left[T\left(u_0\right), \eta\right]$ for $u \in B_{d_\alpha}\left[u_0, \mu\right]$. T is *-locally bounded if it is *-locally bounded it is *-locally bounded for all $u \in X_\alpha$.*

Theorem 1.5. *Let (X_α, d_α) and $\left(Y_\beta, d'_\beta\right)$ be non-Newtonian metric sequence spaces. An operator $T : X_\alpha \to Y_\beta$ is *-locally bounded if T is *-bounded.*

Theorem 1.6. *If the function $\phi : \mathbb{N} \times \mathbb{R}_\alpha \to \mathbb{R}_\beta$ is *-locally bounded, it is satisfies the condition (NA'_2). [6]*

Definition 1.7. *Let $h : X \to \mathbb{R}_\beta$ with $X \subset \mathbb{R}_\alpha$. If for all $\varepsilon \ddot{>} \ddot{0}$, there exists an α-number $\delta = \delta(\varepsilon) \dot{>} \dot{0}$ such that*

$$\left|h(u_1) \ddot{-} h(u_2)\right|_\beta \ddot{<} \varepsilon \text{ when } \left|u_1 \dot{-} u_2\right|_\alpha \dot{<} \delta$$

*for every $u_1, u_2 \in X$, h is *-uniformly continuous on X. If $h : X \to \mathbb{R}_\beta$ is *-uniformly continuous on X, then h is *-continuous on X.*

Let $\left(X, \|\,.\,\|_{X,\alpha}\right)$ and $\left(Y, \|\,.\,\|_{Y,\beta}\right)$ be non-Newtonian normed spaces and let $T : X \to Y$ be an operator. If for all $\varepsilon \ddot{>} \ddot{0}$, there exists an α-number $\delta = \delta(\varepsilon) \dot{>} \dot{0}$ such that

$$\left\|T(x_1) \ddot{-} T(x_2)\right\|_{Y,\beta} \ddot{<} \varepsilon \text{ when } \left\|x_1 \dot{-} x_2\right\|_{X,\alpha} \dot{<} \delta$$

*for all $x_1, x_2 \in X$, T is *-uniformly continuous on X [7].*

Superposition operators were discussed according to classical arithmetic by several authors. Dedagich and Zabreiko [3] have found the conditions for the superposition operators on ℓ_p, ℓ_∞ and c_0. In addition, several features of the superposition operator, such as boundedness, continuity, compactness, were worked by Sama-ae[16], Sağır and Güngör[14, 15], Kolk and Raidjoe[11] and many others [1, 9, 17, 18].

Sağır and Erdoğan [13] defined a non-Newtonian superposition operator ${}_N S_\phi$ where $\phi : \mathbb{N} \times \mathbb{R}_\alpha \to \mathbb{R}_\beta$ with ${}_N S_\phi(x) = (\phi(m, x_m))_{m=1}^\infty$ for all non-Newtonian real sequence (x_m) and characterized non-Newtonian superposition operators on $\ell_{\infty,\alpha}$, c_α, $c_{0,\alpha}$ and $\ell_{p,\alpha}$ into $\ell_{1,\beta}$. In this article, we proof that the non-Newtonian superposition operator ${}_N S_\phi : \ell_{\infty,\alpha} \to \ell_{1,\beta}$ is *-locally bounded iff ϕ satisfies the condition (NA'_2). Also we obtain that ${}_N S_\phi : \ell_{\infty,\alpha} \to \ell_{1,\beta}$ is *-bounded iff ϕ satisfies the condition (NA'_2). Finally we show that the necessary and sufficient conditions for the *-uniform continuity of ${}_N S_\phi : \ell_{\infty,\alpha} \to \ell_{1,\beta}$.

2 Main Results

Theorem 2.1. *Let the function $\phi : \mathbb{N} \times \mathbb{R}_\alpha \to \mathbb{R}_\beta$ be given. The non-Newtonian superposition operator ${}_N S_\phi : \ell_{\infty,\alpha} \to \ell_{1,\beta}$ is *-locally bounded iff ϕ satisfies the condition (NA_2').*

Proof. Assume that the function ϕ satisfies (NA_2'). Let $x = (x_m) \in \ell_{\infty,\alpha}$, $\mu \dot{>} \dot{0}$ and $z = (z_m) \in \ell_{\infty,\alpha}$ such that $\left\| x \dot{-} z \right\|_{\ell_{\infty,\alpha}} \dot{\leq} \mu$. Then $|z_m|_\alpha \dot{\leq} \varphi$ for all $m \in \mathbb{N}$. By Theorem 1.1, there exists a $(u_m) \in \ell_{1,\beta}$ such that $|\phi(m, z_m)|_\beta \ddot{\leq} u_m$ for every $m \in \mathbb{N}$. Then

$$\left\| {}_N S_\phi(z) \right\|_{\ell_{1,\beta}} = {}_\beta\sum_{m=1}^{\infty} |\phi(m, z_m)|_\beta \ddot{\leq} {}_\beta\sum_{m=1}^{\infty} u_m = {}_\beta\sum_{m=1}^{\infty} |u_m|_\beta = \|(u_m)\|_{\ell_{1,\beta}} .$$

Since

$$\begin{aligned} \left\| {}_N S_\phi(z) \ddot{-} {}_N S_\phi(x) \right\|_{\ell_{1,\beta}} &\ddot{\leq} \left\| {}_N S_\phi(z) \right\|_{\ell_{1,\beta}} \ddot{+} \left\| {}_N S_\phi(x) \right\|_{\ell_{1,\beta}} \\ &\ddot{\leq} \|(u_m)\|_{\ell_{1,\beta}} \ddot{+} \left\| {}_N S_\phi(x) \right\|_{\ell_{1,\beta}} , \end{aligned}$$

we get $\left\| {}_N S_\phi(z) \ddot{-} {}_N S_\phi(x) \right\|_{\ell_{1,\beta}} \ddot{\leq} \gamma$ whenever $\gamma = \|(u_m)\|_{\ell_{1,\beta}} \ddot{+} \left\| {}_N S_\phi(x) \right\|_{\ell_{1,\beta}}$. Thus ${}_N S_\phi$ is *-locally bounded at $x \in \ell_{\infty,\alpha}$.

Conversely, let ${}_N S_\phi : \ell_{\infty,\alpha} \to \ell_{1,\beta}$ be *-locally bounded. Let $m \in \mathbb{N}$ and $d \in \mathbb{R}_\alpha$. Let $\omega = (\omega_n)$ be defined as

$$\omega_n = \begin{cases} d \ , & n = m \\ \dot{0} \ , & n \neq m \end{cases} .$$

It is obvious that $(\omega_n) \in \ell_{\infty,\alpha}$. Since ${}_N S_\phi$ is *-locally bounded at $\omega \in \ell_{\infty,\alpha}$, there are $\mu \dot{>} \dot{0}$ and $\eta \ddot{>} \ddot{0}$ such that

$$\left\| {}_N S_\phi(x) \ddot{-} {}_N S_\phi(\omega) \right\|_{\ell_{1,\beta}} \ddot{\leq} \eta \quad \text{where} \quad \left\| x \dot{-} \omega \right\|_{\ell_{\infty,\alpha}} \dot{\leq} \mu . \tag{1}$$

Let $x = (x_n)$ be defined as

$$x_n = \begin{cases} a \ , & n = m \\ \dot{0} \ , & n \neq m \end{cases}$$

with $a \in \mathbb{R}_\alpha$ and $\left| a \dot{-} d \right|_\alpha \dot{\leq} \mu$. Then $(x_n) \in \ell_{\infty,\alpha}$. Since

$$\left\| x \dot{-} \omega \right\|_{\ell_{\infty,\alpha}} = {}^\alpha\sup_{n \in \mathbb{N}} \left| x_n \dot{-} \omega_n \right|_\alpha = \left| a \dot{-} d \right|_\alpha \dot{\leq} \mu ,$$

by 1, we get $\left\|_N S_\phi(x) \ddot{-} {}_N S_\phi(\omega)\right\|_{\ell_{1,\beta}} \ddot{\le} \eta$. Then

$$\begin{aligned}\left|\phi(m,a) \ddot{-} \phi(m,d)\right|_\beta &\ddot{\le} {}_\beta\sum_{n=1}^{\infty}\left|\phi(n,x_n) \ddot{-} \phi(n,\omega_n)\right|_\beta \\ &= \left\|_N S_\phi(x) \ddot{-} {}_N S_\phi(\omega)\right\|_{\ell_{1,\beta}} \\ &\ddot{\le} \eta\end{aligned}$$

Thus $\phi(m,.)$ is *-locally bounded at d. Since $d \in \mathbb{R}_\alpha$ is randomly, $\phi(m,.)$ is *-locally bounded. Therefore $\phi(m,.)$ satisfies to (NA_2') by Theorem 1.6. □

Theorem 2.2. *Let the function $\phi : \mathbb{N} \times \mathbb{R}_\alpha \to \mathbb{R}_\beta$ be given. The non-Newtonian superposition operator ${}_N S_\phi : \ell_{\infty,\alpha} \to \ell_{1,\beta}$ is *-bounded iff ϕ satisfies the condition (NA_2').*

Proof. Suppose that function ϕ satisfies (NA_2'). By Theorem 1.1, there is a $(u_m) \in \ell_{1,\beta}$ such that

$$|\phi(m,t)|_\beta \ddot{\le} u_m \text{ where } |t|_\alpha \dot{\le} \mu \tag{2}$$

for every $m \in \mathbb{N}$ and $\mu \dot{>} \dot{0}$. Let $\sigma \dot{>} \dot{0}$ and $x \in \ell_{\infty,\alpha}$ with $\|x\|_{\ell_{\infty,\alpha}} \dot{\le} \sigma$. Then $|x_m|_\alpha \dot{\le} \sigma$ for all $m \in \mathbb{N}$. From 2, we get $|\phi(m,x_m)|_\beta \ddot{\le} u_m$ for every $m \in \mathbb{N}$ and thereby obtaining that

$$\|_N S_\phi(x)\|_{\ell_{1,\beta}} = {}_\beta\sum_{m=1}^{\infty}|\phi(m,x_m)|_\beta \ddot{\le} {}_\beta\sum_{m=1}^{\infty} u_m = {}_\beta\sum_{m=1}^{\infty}|u_m|_\beta = \|(u_m)\|_{\ell_{1,\beta}}.$$

Hence ${}_N S_\phi$ is *-bounded.

Conversely, suppose that ${}_N S_\phi$ is *-bounded. Let $m \in \mathbb{N}$ and A is an $\alpha-$bounded interval. Then there exists an $\alpha-$number $\varphi \dot{>} \dot{0}$ such that $|t|_\alpha \dot{\le} \varphi$ for all $t \in A$. Since ${}_N S_\phi$ *-bounded, there exists $\xi \ddot{>} \ddot{0}$ such that

$$\|_N S_\phi(z)\|_{\ell_{1,\beta}} \ddot{\le} \xi \text{ whenever } \|z\|_{\ell_{\infty,\alpha}} \dot{\le} \varphi. \tag{3}$$

Let $a \in A$ and let $x = (x_n)$ be defined as $x_n = \begin{cases} a\ , & n = m \\ \dot{0}\ , & n \neq m \end{cases}$. It is seen that $x \in \ell_{\infty,\alpha}$ since $\|x\|_{\ell_{\infty,\alpha}} = {}^\alpha\sup_{n\in\mathbb{N}}|x_n|_\alpha = |a|_\alpha \dot{\le} \varphi$. Then we obtain that $\left\|{}_{N}S_{\phi}\left(x\right)\right\|_{\ell_{1,\beta}} \ddot{\leq} \xi$ by 3. Since

$$\left|\phi\left(m,a\right)\right|_{\beta} \ddot{\leq} {}_{\beta}\sum_{n=1}^{\infty} \left|\phi\left(n,x_{n}\right)\right|_{\beta} = \left\|{}_{N}S_{\phi}\left(x\right)\right\|_{\ell_{1,\beta}} ,$$

we have that $\left|\phi\left(m,a\right)\right|_{\beta} \ddot{\leq} \xi$. Thus ϕ satisfies the condition (NA_2'). □

Corollary 2.3. *Let the function $\phi : \mathbb{N} \times \mathbb{R}_{\alpha} \to \mathbb{R}_{\beta}$ be given. The non-Newtonian superposition operator ${}_{N}S_{\phi} : \ell_{\infty,\alpha} \to \ell_{1,\beta}$ is *-bounded iff ${}_{N}S_{\phi}$ is *-locally bounded.*

Theorem 2.4. *Let ${}_{N}S_{\phi} : \ell_{\infty,\alpha} \to \ell_{1,\beta}$. The non-Newtonian superposition operator ${}_{N}S_{\phi}$ is *-uniformly continuous on every α-bounded subset of $\ell_{\infty,\alpha}$ iff the function $\phi\left(m,.\right)$ is *-continuous on $\mathbb{R}_{\alpha}$ for every $m \in \mathbb{N}$.*

Proof. Suppose that ${}_{N}S_{\phi}$ is *-uniformly continuous. In that case $\phi\left(m,.\right)$ is *-continuous by Theorem 1.2. Conversely, let $\phi\left(m,.\right)$ be *-continuous on $\mathbb{R}_{\alpha}$ for all $m \in \mathbb{N}$. It should be shown that ${}_{N}S_{\phi}$ is *-uniformly continuous on α-ball $B_{\alpha}\left[\dot{0},\varphi\right]$ for all $\varphi \dot{>} \dot{0}$. Let $\varphi \dot{>} \dot{0}$ and $\varepsilon \ddot{>} \ddot{0}$. Since ϕ satisfies the condition (NA_2), ϕ also satisfies the condition (NA_2'). Then, by Theorem 1.1, there exists a $(u_m) \in \ell_{1,\beta}$ such that

$$\left|\phi\left(m,t\right)\right|_{\beta} \ddot{\leq} u_{m} \text{ where } \left|t\right|_{\alpha} \dot{\leq} \varphi \tag{4}$$

for every $m \in \mathbb{N}$. There exists $N \in \mathbb{N}$ such that ${}_{\beta}\sum_{m=N}^{\infty} u_m \ddot{<} \frac{\varepsilon}{\ddot{3}}\beta$ because of $(u_m) \in \ell_{1,\beta}$. Since $\phi\left(m,.\right)$ is *-uniformly continuous on $\left[\dot{0}\dot{-}\varphi,\varphi\right]$, there is a $\delta \in \mathbb{R}_{\alpha}$ with $\dot{0} \dot{<} \delta \dot{<} \dot{1}$ such that

$$\left|\phi\left(m,t\right) \ddot{-} \phi\left(m,s\right)\right|_{\beta} \ddot{<} \frac{\varepsilon}{\ddot{3} \ddot{\times} \left(\ddot{N} \ddot{-} \ddot{1}\right)}\beta \text{ whenever } \left|t \dot{-} s\right|_{\alpha} \dot{<} \delta \tag{5}$$

for all $m \in \{1,2,...,N-1\}$ and $s,t \in \left[\dot{0}\dot{-}\varphi,\varphi\right]$. Let $x = (x_m), y = (y_m) \in B_{\alpha}\left[\dot{0},\varphi\right]$ with $\left\|x \dot{-} y\right\|_{\ell_{\infty,\alpha}} \dot{<} \delta$. In that case $\left|x_m\right|_{\alpha} \dot{\leq} \varphi$, $\left|y_m\right|_{\alpha} \dot{\leq} \varphi$ for every $m \in \mathbb{N}$. Accordingly, $\left|x_m \dot{-} y_m\right|_{\alpha} \dot{<} \delta$ for all $m \in \mathbb{N}$. From 5, we find

$$\left|\phi\left(m,x_m\right) \ddot{-} \phi\left(m,y_m\right)\right|_{\beta} \ddot{<} \frac{\varepsilon}{\ddot{3} \ddot{\times} \left(\ddot{N} \ddot{-} \ddot{1}\right)}\beta$$

for all $m \in \{1, 2, ..., N-1\}$. Thus

$$_{\beta}\sum_{m=1}^{N-1} \left|\phi(m, x_m) \ddot{-} \phi(m, y_m)\right|_{\beta} \ddot{<} \frac{\varepsilon}{\ddot{3}}\beta. \tag{6}$$

By 4, it is written that $|\phi(m, x_m)|_{\beta} \ddot{\leq} u_m$ and $|\phi(m, y_m)|_{\beta} \ddot{\leq} u_m$ for all $m \in \mathbb{N}$. Hence, we get

$$_{\beta}\sum_{m=N}^{\infty} |\phi(m, x_m)|_{\beta} \ddot{\leq} {}_{\beta}\sum_{m=N}^{\infty} u_m \ddot{<} \frac{\varepsilon}{\ddot{3}}\beta \tag{7}$$

and

$$_{\beta}\sum_{m=N}^{\infty} |\phi(m, y_m)|_{\beta} \ddot{\leq} {}_{\beta}\sum_{m=N}^{\infty} u_m \ddot{<} \frac{\varepsilon}{\ddot{3}}\beta. \tag{8}$$

From 6, 7 and 8,

$$\begin{aligned}
\|{}_N S_\phi(x) \ddot{-} {}_N S_\phi(y)\|_{\ell_{1,\beta}} &\ddot{\leq} {}_{\beta}\sum_{m=1}^{\infty} |\phi(m, x_m) \ddot{-} \phi(m, y_m)|_{\beta} \\
&\ddot{\leq} {}_{\beta}\sum_{m=1}^{N-1} |\phi(m, x_m) \ddot{-} \phi(m, y_m)|_{\beta} \ddot{+} {}_{\beta}\sum_{m=N}^{\infty} |\phi(m, x_m) \ddot{-} \phi(m, y_m)|_{\beta} \\
&\ddot{\leq} {}_{\beta}\sum_{m=1}^{N-1} |\phi(m, x_m) \ddot{-} \phi(m, y_m)|_{\beta} \ddot{+} {}_{\beta}\sum_{m=N}^{\infty} |\phi(m, x_m)|_{\beta} \ddot{+} {}_{\beta}\sum_{m=N}^{\infty} |\phi(m, y_m)|_{\beta} \\
&\ddot{<} \frac{\varepsilon}{\ddot{3}}\beta \ddot{+} \frac{\varepsilon}{\ddot{3}}\beta \ddot{+} \frac{\varepsilon}{\ddot{3}}\beta \\
&= \varepsilon.
\end{aligned}$$

Thus ${}_N S_\phi$ is *-uniformly continuous on every α-bounded subset of $\ell_{\infty,\alpha}$.
□

Example 2.5. Let $\phi : \mathbb{N} \times \mathbb{R}_\alpha \to \mathbb{R}_\beta$ be defined as $\phi(m, r) = \dfrac{\left|\iota(r) \ddot{-} \ddot{1}\right|_{\beta}}{\ddot{8}^{m_\beta}}\beta$ for each $r \in \mathbb{R}_\alpha$. The function $\phi(m, .)$ is *-continuous. So ϕ satisfies the condition (NA_2'). It is written that

$$\left|\iota(r) \ddot{-} \ddot{1}\right|_{\beta} \ddot{\leq} |\iota(r)|_{\beta} \ddot{+} \ddot{1}$$

for all $r \in \mathbb{R}_\alpha$. Let $\zeta \dot{>} \dot{0}$ with $|r|_\alpha \dot{<} \zeta$. Then

$$|\phi(m, r)|_{\beta} = \frac{\left|\iota(r) \ddot{-} \ddot{1}\right|_{\beta}}{\ddot{8}^{m_\beta}}\beta \ddot{\leq} \frac{|\iota(r)|_{\beta} \ddot{+} \ddot{1}}{\ddot{8}^{m_\beta}}\beta \ddot{\leq} \frac{\iota(\zeta) \ddot{+} \ddot{1}}{\ddot{8}^{m_\beta}}\beta.$$

Since

$$_{\beta}\sum_{m=1}^{\infty} \frac{\iota\left(\zeta\right)\ddot{+}\ddot{1}}{\ddot{8}^{m_{\beta}}}\beta = \left(\iota\left(\zeta\right)\ddot{+}\ddot{1}\right)\ddot{\times}\frac{\ddot{1}}{\ddot{8}}\beta\ddot{\times}\frac{\ddot{1}}{\ddot{1}\ddot{-}\frac{\ddot{1}}{\ddot{8}}\beta}\beta = \frac{\iota\left(\zeta\right)\ddot{+}\ddot{1}}{\ddot{7}}\beta,$$

we get $(u_m) \in \ell_{1,\beta}$ with $u_m = \frac{\iota\left(\zeta\right)\ddot{+}\ddot{1}}{\ddot{8}^{m_{\beta}}}\beta$ for all $m \in \mathbb{N}$. By Theorem 1.1, it is written that ${}_NS_{\phi} : \ell_{\infty,\alpha} \to \ell_{1,\beta}$. By Theorem 2.4 ${}_NS_{\phi}$ is *-uniformly continuous on every α-bounded subset of $\ell_{\infty,\alpha}$.

3 Conclusion

This article includes proofs of conditions that *-locally boundedness, *-boundedness and *-uniform continuity of non-Newtonian superposition operators which acts $\ell_{\infty,\alpha}$ to $\ell_{1,\beta}$.

Let the function $\phi : \mathbb{N} \times \mathbb{R}_{\alpha} \to \mathbb{R}_{\beta}$ be given. The non-Newtonian superposition operator ${}_NS_{\phi} : \ell_{\infty,\alpha} \to \ell_{1,\beta}$ is *-locally bounded iff ϕ satisfies the condition (NA_2'). Also, the non-Newtonian superposition operator ${}_NS_{\phi} : \ell_{\infty,\alpha} \to \ell_{1,\beta}$ is *-bounded iff ϕ satisfies the condition (NA_2').

The non-Newtonian superposition operator ${}_NS_{\phi}$ is *-uniformly continuous on every α-bounded subset of $\ell_{\infty,\alpha}$ iff the function $\phi\left(m,.\right)$ is *-continuous on $\mathbb{R}_{\alpha}$ for every $m \in \mathbb{N}$.

We think that our results will be presented new opinions for future works.

References

[1] J. Banaś, On the superposition operator and integrable solutions of some functional equations, *Nonlinear Analysis*, vol. 12, 8, 777-784, (1988).

[2] A. F. Çakmak and F. Başar, Some new results on sequence spaces with respect to non-Newtonian calculus, *Journal of Inequalities and Applications*, vol. 228, 1, 1-17, (2012).

[3] F. Dedagich and P.P. Zabreiko, Operator superpositions in the spaces l_p, *Sibirskii Matematicheskii Zhurnal*, 28, 86-98, (1987).

[4] C. Duyar, B. Sağır and O. Oğur, Some basic topological properties on non- Newtonian real line, *British Journal of Mathematics and Computer Science*, 9:4, 300-307, (2015).

[5] C. Duyar and M. Erdogan, On non-Newtonian real number series, *IOSR Journal of Mathematics*, 12, 6, ver. IV, 34–48, (2016).

[6] F. Erdoğan and B. Sağır, On *-boundedness and *-locally boundedness of non-Newtonian superposition operators in $c_{0,\alpha}$ and c_α to $l_{1,\beta}$, *Journal of Universal Mathematics*, 4:2, 241-251, (2021).

[7] F. Erdoğan and B. Sağır, On *-continuity and *-uniform continuity of some non-Newtonian superposition operators, *European Journal of Science and Technology*, 28, 959-967, (2021).

[8] M. Grossman and R. Katz, *Non-Newtonian Calculus*, 1st ed., Lee Press, Pigeon Cove Massachussets, (1972).

[9] E. Herawati and M. Mursaleen, Superposition operators on some new type of order modular spaces, *An. Univ. Craiova*, Ser. Mat. Inf. 47, No. 2, 285–293 (2020).

[10] M. Kirişci, Topological structures of non-Newtonian metric spaces, *Electron. J. Math. Anal. Appl.* 5, No. 2, 156–169 (2017).

[11] E. Kolk and A. Raidjoe, The continuity of superposition operators on some sequence spaces defined by moduli , *Czechoslovak Mathematical Journal*, 57, 777-792, (2007).

[12] N. Sager and B. Sağır, Some inequalities in quasi-Banach algebra of non-Newtonian bicomplex numbers, *Filomat*, 35(7), (2021).

[13] B. Sağır and F. Erdoğan, On characterization of non-Newtonian superposition operators in some sequence spaces, *Filomat*, 33:9, 2601-2612, (2019).

[14] B. Sağır and N. Güngör, Continuity of superposition operators on the double sequence spaces L_p, *Filomat*, 29:9, 2107-2118, (2015).

[15] B. Sağır and N. Güngör, Locally boundedness and continuity of superposition operators on the double sequence spaces C_{r0}, *Journal of Computational Analysis and Applications*,19:2, 365-377, (2015).

[16] A. Sama-ae, *Boundedness of Superposition Operators on the Sequence Spaces of Maddox*, Master of Sciences Dissertation, Graduate School of Chiang Mai University, 55, Thailand, (1997).

[17] P. Tainchai, *Boundedness of Superposition Operators on Some Sequence Spaces*, Master of Sciences Dissertation, Graduate School of Chiang Mai University, 40, Thailand, (1996).

[18] A. Thuangoon, *Continuity of Superposition Operators on Some Sequence Spaces of Maddox*, Master of Sciences Dissertation, Graduate School of Chiang Mai University, 66, Thailand, (1998).

[19] D. F. M. Torres, On a non-Newtonian calculus of variations, *Axioms*,10(3), 171, (2021).

[20] E. Yılmaz,Multiplicative Bessel equation and its spectral properties *Ricerche di Matematica*, doi.10.1007/s11587-021-00674-1 (2021).

Fatmanur Erdoğan
Department of Mathematics
Doctoral Student of Mathematics
Institute of Graduate Studies
Ondokuz Mayıs University
Samsun, TURKEY
E-mail: fatmanurkilic89@hotmail.com

Birsen Sağır
Department of Mathematics
Professor of Mathematics
Faculty of Science and Art
Ondokuz Mayıs University
Samsun, TURKEY
E-mail: bduyar@omu.edu.tr