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Original Research Paper

On Some Properties of e-Spaces

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Abstract. An open subset of a space is said to be e-open if its closure is also open and if a space has a base consisting of e-open sets, we call it an e-space. In this paper we first introduce e-spaces and compare them with relative spaces such as extremally disconnected and zero-dimensional spaces. Subspaces of e-spaces and product of e-spaces are investigated and we define the concept of e-compactness and characterize e-compact spaces via e-convergence of nets and filters. We introduce e-separation axioms $T_1^e - T_4^e$ and investigate the counterparts of results in the literature of topology concerning separation axioms. It is shown that a space is a $T_3 - e$-space if and only if it is zero-dimensional and a space is a T_4^e-space if and only if it is a strongly zero-dimensional T_4-space. In contrast to extremally disconnected spaces whose product is not necessarily an extremally disconnected space, we observe that any product of e-spaces is an e-space. Also we see that the e-closure of a set need not be e-closed, contrary to closure of a set which is closed.

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1 Introduction

Remind that a topological space X is called extremally disconnected if the closure of every open set in X is open. Extremally disconnected spaces were defined and studied in [4] and also in [3]. Extremally disconnected spaces are important in studying the Stone-$\check{C}$ech compactification of a product space (19I in [10]), as well as, more generally, in study of the Stone space of any complete Boolean algebra; see chapter 2 in [9]. Also, a topological space X is called basically disconnected provided that the closure of $\operatorname{coz} f$ for any $f \in C(X)$ is open, where $C(X)$ is the ring of all real-valued continuous functions on a space X and $\operatorname{coz} f = \{x \in X : f(x) \neq 0\}$. Basically disconnected spaces were studied in [6] and [8]; see also 1H and 3N in [3]. Clearly, every extremally disconnected space is basically disconnected and whenever X is a basically disconnected Tychonoff space, then (by Theorem 3.2 in [3]) X has a base consisting of (open) cozero-sets whose closures is open.

In this investigation, we generalize these two notions and call an open set in a topological space X, extremely open (briefly e-open) if its closure is open and whenever X has a base consisting of e-open sets, it is called an e-space. Clearly, every basically disconnected Tychonoff space and also every zero-dimensional space is an e-space. In the second section, we study e-spaces and their similarities with basically and extremally disconnected spaces. We first observe that the intersection of any two e-open sets is a an e-open set and this follows that the set $\mathcal{E}(X)$ consisting of e-open subsets of a space X forms a base for a topology on X. Also, we show that any open or dense subspace of an e-space is an e-space but not every subspace (even a closed subspace) is necessarily an e-space. We give examples of e-spaces which are not basically disconnected. Next, we introduce and investigate the counterparts of the compactness and Lindelöfness, namely e-compactness and e-Lindelöfness respectively. We characterize e-compact spaces via the e-convergence of nets and filters and we observe that every e-compact T_1-space is pseudocompact. Note that a space X is said to be pseudocompact if every real valued continuous function on X is bounded. It is natural to ask that: is a product of e-compact T_1 spaces an e-compact space? We could not settle this question but we have shown that whenever a product of T_1-spaces is e-compact then each factor space is e-compact.

Section 3 is devoted to separation axioms via e-open sets, namely e-separation axioms. For each separation axiom T_n, $1 \leq n \leq 4$ we define a similar axiom via e-open sets which we denote it by T_n^e. We observe that there are similar relations between e-compactness, e-Lindelöfness and e-separation axioms to those in the corresponding covering properties and separation axioms. Among others, we prove that e-compactness coincides with the compactness in the realm of e-Hausdorff spaces (T_2^e-spaces), and every e-compact T_2^e-space is a T_3^e-space. We also observe in this section that a space is a T_3-e-space if and only if it is zero-dimensional if and only if it is a T_3^e-space. Also we see that T_4^e-spaces coincide with strongly zero-dimensional spaces. Using the results of this section we give an example of e-space which is neither extremally disconnected nor zero-dimensional. Finally for undefined terms and notations in this article, we refer the readers to [2], [5], [9] and [10].

2 Spaces with Bases Consisting of e-Open Sets

We call a set G in a topological space X an extremely open (briefly e-open) if G and $\mathrm{cl}_X G$ are open subsets of X and we call a subset of a topological space an e-closed set if its complement is e-open. Equivalently, a set is e-closed if and only if it is closed and its interior is also closed. Clearly every closed-open (clopen) set in a topological space is an e-open set, but not conversely. For example $\mathbb{R} \setminus \{0\}$ is an e-open subset of $\mathbb{R}$ which is not a clopen set. Moreover, for each T_1-space X, the set $X \setminus \{x\}$ is an e-open set for each $x \in X$. In fact, if x is an isolated point of X, then $X \setminus \{x\}$ is clopen, so it is e-open. Otherwise $X \setminus \{x\}$ is open, since X is T_1 and $\mathrm{cl}_X(X \setminus \{x\}) = X$ is open, i.e., $X \setminus \{x\}$ is e-open. More generally, every dense open subset of a space is a non-empty e-open set. The convers is also true, when the space is connected, as the only clopen subsets of a connected space are the empty set and all of the space. In particular a nonempty open subset G of $\mathbb{R}$ is e-open in $\mathbb{R}$ if and only if $\mathbb{R} \setminus G$ has an empty interior. The family of all e-open subsets of a space X is denoted by $\mathcal{E}(X)$. Using the following lemma, for each topological space X, the collection $\mathcal{E}(X)$ may be a base for a topology on X.

Lemma 2.1. *The following statements hold.*

1. *If X is a dense subspace of T and V is an open subset of T, then $cl_T V = cl_T(V \cap X)$.*
2. *If U is an open set and V is an e-open subset of a space X, then $cl_X(U \cap V) = cl_X U \cap cl_X V$.*

Proof. (1) See Theorem 1.3.6 in [2].

(2) Let $x \in \mathrm{cl}_X U \cap \mathrm{cl}_X V$ and W be an open set containing x. Since $\mathrm{cl}_X V$ is an open set containing x and $x \in \mathrm{cl}_X U$, we have $(W \cap \mathrm{cl}_X V) \cap U \neq \emptyset$ because $W \cap \mathrm{cl}_X V$ is an open set containing x. Now take $y \in (W \cap \mathrm{cl}_X V) \cap U$, then $W \cap U$ is a neighborhood of $y \in \mathrm{cl}_X V$, hence $W \cap V \cap U \neq \emptyset$. Therefore $\mathrm{cl}_X(U \cap V) = \mathrm{cl}_X U \cap \mathrm{cl}_X V$. □

Corollary 2.2. *In any space, the intersection of every two e-open sets is an e-open set.*

By Corollary 2.2, the collection $\mathcal{E}(X)$ of all e-open subsets of X form a base for a topology τ_e on X. From now on, whenever (X, τ) is a topological space, τ_e stands for the topology on X generated by the set of all e-open subsets of (X, τ) and the corresponding space (X, τ_e) is denoted by X_e, for simplicity.

Definition 2.3. *A topological space (X, τ) is called e-space, if $\mathcal{E}(X)$ forms a base for its open sets, i.e., $\tau_e = \tau$.*

Remark 2.4. Corollary 2.2 is not true for an arbitrary intersection (union) of e-open sets. In fact an arbitrary intersection of e-open sets, even clopen sets, need not be even an open set. If X is a space and $x \in X$ is the lone non-isolated point of X, then for each $x \neq y \in X$, the set $X \setminus \{y\}$ is e-open (clopen). Now $G = \bigcap_{x \neq y \in X}(X \setminus \{y\}) = \{x\}$, which is not even open. Also, an arbitrary union of e-open sets need not be an e-open set. For example, if we take $X = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ as a subspace of $\mathbb{R}$, then $G_n = \{\frac{1}{2n}\}$, for each $n \in \mathbb{N}$, is clopen and hence e-open but $\bigcup_{n \in \mathbb{N}} G_n$ is not e-open because its closure is $\{\frac{1}{2n} : n \in \mathbb{N}\} \cup \{0\}$, which is not open in X.

Example 2.5. (a) Whenever every open subset of a space has an open closure, i.e., if a space is extremally disconnected, then clearly it is an e-space. In particular, every discrete space is an e-space.

(b) Every completely regular Hausdorff space X which is also basically disconnected, is an e-space. We recall from [3], that a space X is basically disconnected, if every cozero-set has an open closure. Also note that, a Hausdorff space is completely regular if and only if the family of all cozero-sets form a base for the open sets; see Theorem 3.2 in [3].

(c)*A non-basically disconnected e-space.* Consider the one-point compactification space $X = \{1, 2, \cdots, \frac{1}{n}, \cdots\} \cup \{0\}$ as a subspace of $\mathbb{R}$. The set $\mathfrak{B} = \{\{\frac{1}{n}\} : n \in \mathbb{N}\} \cup \{G \subseteq X : X \setminus G \text{ is finite and } 0 \in G\}$ consisting of e-open (clopen) sets is a base for X. Since the closure of the cozero-set $\{1, \frac{1}{3}, \frac{1}{5}, \cdots\}$ is not open, X is an e-space which is not basically disconnected.

(d) Every zero-dimensional space is an e-space. We recall from [2], that a T_1-space X is zero-dimensional if each point of X has a neighborhood base consisting of clopen sets.

In the next section we present an example of an e-space which is neither zero-dimensional nor extremally disconnected. In this section we study the subspaces, products, quotients, homeomorphic images of e-spaces and e-compactness. Also whenever S is a subspace of a space X, we identify conditions on S under which its e-open subsets are the intersection of e-open subsets of X with S. In the sequel, we define the counterpart of compactness via e-open sets under the name of e-compactness and study the relation between e-compact and compact spaces.

Definition 2.6. *Let A be a subset of a topological space X. An element $x \in X$ is called an e-cluster point of A if each e-open subset of X containing x meets A. The set of all e-cluster points of A is called the e-closure of A and we denote it by e-$\mathrm{cl}_X A$.*

In fact, a point $x \in X$ is an e-cluster point of A as a subset of X if and only if x is a cluster point of A as a subset of X_e, and e-$\mathrm{cl}_X A = \mathrm{cl}_{X_e} A$. Clearly for each subset A of a space X, we have $\mathrm{cl}_X A \subseteq e$-$\mathrm{cl}_X A$ and the inclusion may be proper. For instance if we consider the open interval $(0, 1)$ in $\mathbb{R}$, then $\mathrm{cl}_{\mathbb{R}}(0, 1) = [0, 1]$, but e-$\mathrm{cl}_{\mathbb{R}}(0, 1) = \mathbb{R}$. Let us call any open subset of X which is a union of e-open sets, an E-open set, and similarly call any closed subset of X an E-closed set if it is an intersection of e-closed subsets of X . Indeed, E-open and E-closed subsets of X, are in fact open and closed subsets of X_e, respectively. Therefore, as a closure of a set, the e-closure of a set A in a space X is an E-closed set and it is the intersection of all e-closed subsets of X containing A. Whenever

$$S = \{H \subseteq X : \ H \text{ is an } e\text{-closed set and } A \subseteq H\},$$

then $e\text{-cl}_X A = \bigcap_{H \in S} H$. In fact if $x \in \bigcap_{H \in S} H$ and $x \notin e\text{-cl}_X A$, then there is an e-open set G containing x such that $G \cap A = \emptyset$. Now $X \setminus G$ is an e-closed set containing A which does not contain x, a contradiction. The reverse inclusion is also routine. The e-interior is defined similarly and the e-interior of a set A is denoted by $e\text{-int}_X A$, indeed $e\text{-int}_X A = \text{int}_{X_e} A$.

In contrast to the closure of a set which is closed, the e-closure of a set need not be e-closed. To this end, we let $X = \{0, 1, \frac{1}{2}, \frac{1}{3}, \dots\}$ be a subspace of $\mathbb{R}$ with usual topology and $A = \{\frac{1}{2}, \frac{1}{4}, \dots\}$. Then e-$\text{cl}_X A = A \cup \{0\}$ which is not e-closed. The following lemma gives more examples of e-spaces. Whenever X is a topological space, we denote the set of all clopen subsets of X by $\mathcal{B}(X)$.

Lemma 2.7. *Let (X, τ) be a topological space. Then the following statements hold.*

1. $\mathcal{B}(X) = \mathcal{B}(X_e)$.
2. $\mathcal{E}(X) \subseteq \mathcal{E}(X_e)$.
3. *The space X_e is an e-space.*

Proof. (1) Let V be a clopen subset of X. Since any clopen subset of X is e-open, V and $X \setminus V$ are both open in X_e, i.e., $V \in \mathcal{B}(X_e)$. The convers is evident, as $\tau_e \subseteq \tau$.

(2) Let $V \in \mathcal{E}(X)$. Then $\text{cl}_X V$ is clopen in X, i.e., $\text{cl}_X V \in \mathcal{B}(X)$. Using part (1), $\text{cl}_X V \in \mathcal{B}(X_e)$ and therefore $\text{cl}_{X_e} V \subseteq \text{cl}_X V$. On the other hand, since $\tau_e \subseteq \tau$ we have $\text{cl}_X V \subseteq \text{cl}_{X_e} V$. Hence $\text{cl}_{X_e} V = \text{cl}_X V$ is clopen in X_e. This mean that V is e-open in X_e, i.e., $V \in \mathcal{E}(X_e)$.

(3) Note that τ_e is the topology on X generated by $\mathcal{E}(X)$, so the result is now evident by part (2). $\square$

The inclusion in part (2) in the previous lemma may be proper; see the following example.

Example 2.8. Let X be the real line, where the neighborhoods of all point outside of $(0,1)$ are as usual and all points in the interval $(0,1)$ are assumed to be isolated. It is clear that the open subset $(0,1)$ is not an e-open set in X, as $\mathrm{cl}_X(0,1) = [0,1]$ which is not open in X. On the other hand, since any clopen subset of X is e-open and the e-open sets of X form a base for the topology of X_e, the open interval $(0,1)$ is an open subset of X_e; it is a union of clopen subsets of X. Now, we show that $\mathrm{cl}_{X_e}(0,1) = X_e$ and therefore $(0,1)$ will be an e-open set in X_e. Let $p \in \mathbb{R} \setminus (0,1)$ and U be a basic neighborhood of p in X_e, i.e., U is an e-open set in X. If $U \cap (0,1) = \emptyset$, then $\mathrm{cl}_X U \cap (0,1) = \emptyset$, as $(0,1)$ is open in X. But, U is e-open in X, so $\mathrm{cl}_X U$ is clopen and thus $\mathrm{cl}_X U \cap (-\infty, 0]$ and $\mathrm{cl}_X U \cap [1,\infty)$ both are clopen subsets in the connected subspaces $(-\infty,0]$ and $[1,\infty)$, respectively. Hence $\mathrm{cl}_X U$ must be equal to either $(-\infty,0]$, $[1,\infty)$ or $(-\infty,0] \cup [1,\infty)$. But neither of the latter subsets is clopen in X. Hence $U \cap (0,1) \neq \emptyset$ and we are done.

The following corollary which is the counterpart of 1H.4 and 1H.5 in [3] concerning extremally disconnected spaces, states that some subspaces of an e-space are e-spaces; see also Proposition 2.4 in [1]. First we need the following proposition.

Proposition 2.9. *The trace of any e-open subset on an open or dense set is e-open.*

Proof. First, suppose that X is an open subspace of Y and let $V \subseteq Y$ be e-open. Let $V_0 = V \cap X$. Then using Lemma 2.1 we have

$$\mathrm{cl}_X V_0 = \mathrm{cl}_Y V_0 \cap X = \mathrm{cl}_Y V \cap \mathrm{cl}_Y X \cap X = \mathrm{cl}_Y V \cap X.$$

Since V is an e-open subset of Y, $\mathrm{cl}_Y V$ is clopen in Y and hence $\mathrm{cl}_X V_0$ is clopen in X. Next, suppose that X is a dense subspace of Y and let $V \subseteq Y$ be e-open. If we put again $V_0 = V \cap X$, using the previous lemma we have $\mathrm{cl}_Y V_0 = \mathrm{cl}_Y(V \cap X) = \mathrm{cl}_Y V$. Now $\mathrm{cl}_X V_0 = X \cap \mathrm{cl}_Y V$ implies that $\mathrm{cl}_X V_0$ is open. $\square$

Corollary 2.10. *Every open or dense subspace of an e-space is an e-space.*

Example 2.11. An arbitrary (even an e-closed) subspace of an e-space need not be an e-space. For example, let

$$\mathfrak{B} = \{U \subseteq \mathbb{R} : U \text{ is open in } \mathbb{R} \text{ with usual topology and } [0,\infty)\setminus U \text{ is finite}\}.$$

Clearly $\mathfrak{B}$ may be a base for a topology on $\mathbb{R}$, say τ. Since each member of $\mathfrak{B}$ is an e-open set with respect to topology τ, $(\mathbb{R},\tau)$ is an e-space. In fact for each $U \in \mathfrak{B}$, we have $\mathrm{cl}_X U = \mathbb{R}$ and hence $\mathrm{cl}_X U$ is open. Now consider the subspace $(-\infty, 0]$ of $(\mathbb{R},\tau)$ which is e-closed. The collection of all subsets of the form $U \cap (-\infty, 0]$ forms a base for $(-\infty, 0]$, where U is an open subset of $\mathbb{R}$ with usual topology. This implies that the space $(-\infty, 0]$ as a subspace of $(\mathbb{R},\tau)$ has the usual topology which is not an e-space.

The previous example also shows that Proposition 2.9 is not necessarily true for closed (e-closed) subspaces, and the trace of e-open sets on a closed (e-closed) subspace need not be e-open. However, the following result states that the e-open sets of an e-closed subspace are the intersection of the e-open sets of the space with the subspace.

Proposition 2.12. *Let X be a topological space and S be an e-closed subspace of X. If U is an e-open set in S, then there is an e-open set $\tilde{U}$ in X with $U = \tilde{U} \cap S$.*

Proof. Suppose that $U \subseteq S$ is an e-open set (in S). Then U and $\mathrm{cl}_S U$ both are open in S. Let $\tilde{U} = X \setminus (S \setminus U)$. Since $S \setminus U$ is closed in S and also in X, $\tilde{U}$ is an open subset of X and obviously $U = \tilde{U} \cap S$. Now, we need only to show that $\tilde{U}$ is an e-open set in X, i.e., $\mathrm{cl}_X \tilde{U}$ is clopen in X. Since $\tilde{U} = U \cup (X \setminus S)$, $\mathrm{cl}_X \tilde{U} = \mathrm{cl}_X U \cup \mathrm{cl}_X (X \setminus S)$. Note that, the subset $\mathrm{cl}_X (X \setminus S) = X \setminus \mathrm{int}_X S$ is clopen in X, as S is e-closed in X. Let $p \in \mathrm{cl}_X U$. Since S is a closed subspace, $\mathrm{cl}_X U = \mathrm{cl}_S U$ and then $p \in \mathrm{cl}_S U$. But, $\mathrm{cl}_S U$ is clopen in S, thus there exists an open neighborhood V of p in X such that $V \cap S \subseteq \mathrm{cl}_S U$. Therefore

$$V = (V \cap S) \cup (V \cap (X \setminus S)) \subseteq \mathrm{cl}_S U \cup (X \setminus S) \subseteq \mathrm{cl}_X U \cup \mathrm{cl}_X (X \setminus S) = \mathrm{cl}_X \tilde{U},$$

and we are done. □

The preceding proposition is not necessarily true for any subspace, not even for a dense (e-open) subspace. For example, let $S = \mathbb{R} \setminus \{0\}$ be as a subspace of $\mathbb{R}$ with the usual topology. Then S is a dense e-open subspace of $\mathbb{R}$ and $U = (0, \infty)$ is e-open (clopen) in S. But, since the only non-empty e-open sets in $\mathbb{R}$ are the dense open subsets, so there is no e-open set $\tilde{U}$ in $\mathbb{R}$ such that $U = \tilde{U} \cap S$.

Corollary 2.13. *Let X be a topological space and S be an e-closed set in X. If U is an E-open (E-closed) set in S, then there is an E-open (E-closed) set $\tilde{U}$ in X such that $U = \tilde{U} \cap S$.*

Proof. Suppose that $U \subseteq S$ is an E-open set (in S). Then there is a family $\{U_i\}_{i\in I}$ of e-open sets in S, such that $U = \bigcup_{i\in I} U_i$. Using Proposition 2.12, for any $i \in I$, there is an e-open set $\tilde{U}_i$ in X such that $U_i = \tilde{U}_i \cap S$. Thus $\tilde{U} = \bigcup_{i\in I} \tilde{U}_i$ is an E-open set in X and clearly $U = \tilde{U} \cap S$. On the other hand, if H is an E-closed set in S, then $S \setminus H$ is E-open. Therefore, by the first part, there is an E-open set G in X such that $S \setminus H = G \cap S$. Hence $X \setminus G$ is an E-closed set in X and it is easy to see that $H = (X \setminus G) \cap S$. □ We remind that, any closed subset of a closed subspace S of a space X, is closed in X. In the following corollary, using Proposition 2.12, we examine the counterpart of this well-known fact.

Corollary 2.14. *Let X be a topological space, S be an e-closed subspace of X and $A \subseteq S$. If A is an e-closed (E-closed) set in S, then it is e-closed (E-closed) in X.*

Proof. Suppose that, A is e-closed in S. Then $U = S \setminus A$ is e-open in S and by the argument in the proof of Proposition 2.12, $\tilde{U} = X \setminus (S \setminus U) = X \setminus A$ is e-open in X. It is clear that $(X \setminus \tilde{U}) \cap S = A$ and since an intersection of any two e-closed sets in X is e-closed, A is e-closed in X. In case A is E-closed, the proof is similar. □

The preceding result is not necessarily true for e-open sets. For instance, if we let $S = \mathbb{R} \setminus \{0\}$, then S is an e-open subspace of $\mathbb{R}$ and $U = (0, \infty)$ is e-open in S, but it is not e-open in $\mathbb{R}$.

Let (X, τ) be a topological space and $S \subseteq X$ be a subspace. Then what we have understood so far, we may define three different topologies on S. The first one is the relative topology inherits from (X, τ), which is denoted by $\tau|_S$. Second, the relative topology on S in $(X, \tau_e) = X_e$ and it is denoted by $\tau_e|_S$. The third one, is the topology on S generated by the e-open subsets of $(S, \tau|_S)$, which is denoted by $(\tau|_S)_e$. Now, using these notations and Proposition 2.9, whenever S is a dense or open subspace of X, then $\tau_e|_S \subsetneq (\tau|_S)_e$. While, if S is an e-closed subspace then $(\tau|_S)_e \subsetneq \tau_e|_S$, by Corollary 2.13. Then, whenever $S \subseteq X$ is a clopen subspace, we have $\tau_e|_S = (\tau|_S)_e$ and therefore the the topology generated by all e-open subsets of S as a subspace of X, coincide with the relative topology on S in X_e.

Proposition 2.15. *If S is a clopen subspace of a topological space X, then the topology of S_e coincide with the relative topology on S in X_e.*

Proposition 2.16. *Let S be a clopen subspace of a topological space X and $A \subseteq S$. Then A is e-open (e-closed) in S if and only if it is e-open (e-closed) in X.*

Proof. Suppose that A is e-open in S, then A is open in S and hence $\mathrm{cl}_S A$ is clopen in S. But, since S is clopen in X, A is open in X and $\mathrm{cl}_X A = \mathrm{cl}_S A$ is clopen in X; see Lemma 16.2 and Theorem 17.3 in [5]. The convers is obvious by Proposition 2.9. Now, suppose that A is e-closed in X, then $X \setminus A$ is e-open in X and therefore $U = S \cap (X \setminus A)$ is e-open in S by Proposition 2.9. Hence $S \setminus U = A$ is e-closed in S. The proof of the convers follows from Corollary 2.14. □

In contrast to extremally disconnected spaces (see 19I in [10]), any product of e-spaces is an e-space

Theorem 2.17. *Let X_α, for each $\alpha \in S$, be an e-space. Then $\prod_{\alpha \in S} X_\alpha$ is also an e-space.*

Proof. Let V be an open subset of $\prod_{\alpha \in S} X_\alpha$ containing a point (x_α). Then there is a neighborhood base $\pi^{-1}_{\alpha_1}(U_{\alpha_1}) \cap \cdots \cap \pi^{-1}_{\alpha_k}(U_{\alpha_k})$ containing (x_α) contained in V. Now using our hypothesis, there are e-open sets $G_{\alpha_1}, \cdots, G_{\alpha_k}$ such that $x_{\alpha_i} \in G_{\alpha_i} \subseteq U_{\alpha_i}$, $i = 1, \cdots, k$. So we have

$$(x_\alpha) \in \pi^{-1}_{\alpha_1}(G_{\alpha_1}) \cap \cdots \cap \pi^{-1}_{\alpha_k}(G_{\alpha_k}) \subseteq \pi^{-1}_{\alpha_1}(U_{\alpha_1}) \cap \cdots \cap \pi^{-1}_{\alpha_k}(U_{\alpha_k}) \subseteq V.$$

It is enough to show that $\pi_{\alpha_1}^{-1}(G_{\alpha_1}) \cap \cdots \cap \pi_{\alpha_k}^{-1}(G_{\alpha_k})$ is an e-open set. To this end, using 8D in [10], we have

$$\mathrm{cl}_X(\pi_{\alpha_1}^{-1}(G_{\alpha_1}) \cap \cdots \cap \pi_{\alpha_k}^{-1}(G_{\alpha_k})) = \pi_{\alpha_1}^{-1}(\mathrm{cl}_{X_{\alpha_1}} G_{\alpha_1}) \cap \cdots \cap \pi_{\alpha_k}^{-1}(\mathrm{cl}_{X_{\alpha_k}} G_{\alpha_k}).$$

Since π_{α_i} is continuous for each α and $\mathrm{cl}_{X_{\alpha_i}} G_{\alpha_i}$ is open for each $i = 1, \cdots, k$, the set $\pi_{\alpha_1}^{-1}(\mathrm{cl}_{X_{\alpha_1}} G_{\alpha_1}) \cap \cdots \cap \pi_{\alpha_k}^{-1}(\mathrm{cl}_{X_{\alpha_k}} G_{\alpha_k})$ is open which means that

$$\mathrm{cl}_X(\pi_{\alpha_1}^{-1}(G_{\alpha_1}) \cap \cdots \cap \pi_{\alpha_k}^{-1}(G_{\alpha_k}))$$

is open and therefore $\pi_{\alpha_1}^{-1}(G_{\alpha_1}) \cap \cdots \cap \pi_{\alpha_k}^{-1}(G_{\alpha_k})$ is e-open. □

The following example shows that a quotient space of an e-space, need not be an e-space.

Example 2.18. Let X be the space presented in Example 2.8 and let Y be the quotient space of X obtained by identifying all points of the interval $[0, \infty)$ with a single point. It is easy to see that the quotient space Y is homeomorphic to $(-\infty, 0]$ with the relative topology inherited from $\mathbb{R}$ with the usual topology. But it is clear that $(-\infty, 0]$ is not an e-space.

Proposition 2.19. *Every homeomorphic image of an e-space is an e-space.*

Proof. Let X and Y be two homeomorphic spaces, X be an e-space and $\varphi : X \to Y$ be an onto homeomorphism. Let V be an open subset of Y and $y \in V$. Then there is $x \in X$ such that $y = \varphi(x)$. Since φ is continuous $\varphi^{-1}(V)$ is an open subset of X containing x and hence there exists an e-open subset U of X such that $x \in U \subseteq \varphi^{-1}(V)$. But φ is an open function, so $\varphi(U)$ is open and $y \in \varphi(U) \subseteq V$. Now it is enough to show that $\mathrm{cl}_Y \varphi(U)$ is open, i.e., $\varphi(U)$ is an e-open set. Using Theorem 7.9 in [10], we have $\varphi(\mathrm{cl}_X U) = \mathrm{cl}_Y \varphi(U)$ and since φ is open, $\varphi(\mathrm{cl}_X U)$ is an open subset of Y because $\mathrm{cl}_X U$ is an open subset of X. This shows that $\mathrm{cl}_Y \varphi(U)$ is open and we are through. □

By a natural way similar to definition of compactness, a space is called e-compact if every e-open cover of the space has a finite subcover. Clearly every compact space is e-compact and it is shown in [1] that every e-compact T_1-space is countably compact. Recall from [10], that a space X is countably compact provided that each sequence has a cluster point. Using Theorem 11.5 in [10], this is equivalent to saying that each sequence with a cluster point has a convergent subnet to that cluster point. In this paper we are to study more properties of e-compact spaces via nets and filters. First we need the notion of e-convergence.

Definition 2.20. *Let $(x_\lambda)_{\lambda\in\Lambda}$ be a net in a space X. Then we say $(x_\lambda)_{\lambda\in\Lambda}$ e-converges to $x \in X$ provided that for each e-open set U containing x, there is some $\lambda_0 \in \Lambda$ such that $\lambda \geq \lambda_0$ implies $x_\lambda \in U$. Also, we say that $(x_\lambda)_{\lambda\in\Lambda}$ has $x \in X$ as an e-cluster point if for every e-open neighborhood U of x and every $\lambda_0 \in \Lambda$, there is some $\lambda \geq \lambda_0$ such that $x_\lambda \in U$.*

Thus, a net (x_λ) e-converges to $x \in X$ or has $x \in X$ as an e-cluster point if and only if (x_λ) converges to x in X_e or has x as a cluster point in X_e.

Definition 2.21. *Whenever $\mathcal{F}$ is a filter on a space X, we call $\mathcal{F}$ e-converges to $x \in X$ if any e-open neighborhood of x contains some member of $\mathcal{F}$. We say that a filter $\mathcal{F}$ of subsets of X has $x \in X$ as an e-cluster point if $x \in \bigcap_{F\in\mathcal{F}} e\text{-}cl_X F$.*

Note that, a filter $\mathcal{F}$ e-converges to $x \in X$ (has $x \in X$ as an e-cluster point) if and only if $\mathcal{F}$ converges to x in X_e (has x as a cluster point in X_e). Moreover, the notions of convergence and e-convergence coincide whenever the space is an e-space.

It is known and easy to see that, whenever X is a topological space and $\mathfrak{B}$ is a base for the topology, then X is compact if and only if any cover of X consisting of elements of $\mathfrak{B}$, has a finite subcover. Thus a space X is e-compact if and only if X_e is compact. Now, using this argument and also by a similar proof of Theorem 17.4 in [10], the following theorem is evident. The following theorem gives a characterization of e-compact spaces in terms of nets and filters and it is the counterpart of Theorem 17.4 in [10]

Theorem 2.22. *For any topological space X, the following statements are equivalent.*

1. *The space X is e-compact.*
2. *Each family $\mathfrak{E}$ of e-closed subsets of X with the finite intersection property has non-empty intersection.*
3. *Each filter on X has an e-cluster point.*
4. *Each net in X has an e-cluster point.*
5. *Each ultranet in X e-converges.*
6. *Each ultrafilter on X e-converges.*

It is known that any closed subspace of a compact space is compact. The following proposition deals with the counterpart of this fact.

Proposition 2.23. *If X is e-compact and A is an e-closed subset of X, then A is e-compact.*

Proof. Let $\mathcal{C} = \{G_\alpha : \alpha \in S\}$ be an e-open cover for A. Using Proposition 2.12, for any $\alpha \in S$, there is an e-open set $\tilde{G}_\alpha$ in X such that $G_\alpha = \tilde{G}_\alpha \cap A$. Let $\tilde{\mathcal{C}} = \{\tilde{G}_\alpha : G_\alpha \in \mathcal{C}\}$. Then $\tilde{\mathcal{C}} \cup \{X \setminus A\}$ is an e-open cover of X since $X \setminus A$ is an e-open set. But X is e-compact, then $\tilde{\mathcal{C}} \cup \{X \setminus A\}$ has a finite subcover $\tilde{\mathcal{F}}$. Now clearly $\mathcal{F} = \{G_\alpha : \tilde{G}_\alpha \in \tilde{\mathcal{F}}\}$ is a finite subcover of $\mathcal{C}$ for A and hence A is e-compact. □

We could not prove the counterpart of the Tychonoff Theorem for product of e-compact spaces. But using the above proposition and the following lemma, we may show that whenever a product of some T_1-spaces with product topology is e-compact, then each factor space is e-compact.

Lemma 2.24. *Suppose that $X = \prod_{\alpha\in S} X_\alpha$ is a non-empty product of a family of T_1-spaces. Then each factor space X_α, for any $\alpha \in S$, is homeomorphic to an e-closed subspace of X.*

Proof. Let $\beta \in S$. Since X is non-empty, there is a point $p = (p_\alpha)_{\alpha\in S} \in X$. Put

$$Y_\beta = \{(q_\alpha)_{\alpha\in S} \in X : q_\alpha = p_\alpha, \text{ unless } \alpha = \beta\}.$$

Then it is easy to see that Y_β is homeomorphic with X_β under the restriction to Y_β of the projection map π_β. Now it is enough to show that Y_β is e-closed or equivalently $X \setminus Y_\beta$ is e-open in X. Whenever $q = (q_\alpha)_{\alpha\in S} \in X \setminus Y_\beta$, there is $\gamma \neq \beta$ such that $q_\gamma \neq p_\gamma$. Since X_γ is T_1, there is an open neighborhood U of q_γ in X_γ not containing p_γ. Then $\pi_\gamma^{-1}(U) \subseteq X \setminus Y_\beta$, i.e., $X \setminus Y_\beta$ is open in X. Now we prove that $\mathrm{cl}_X(X \setminus Y_\beta)$ is clopen and therefore $X \setminus Y_\beta$ is e-open. We consider two cases and in each case we show that $\mathrm{cl}_X(X \setminus Y_\beta)$ is clopen.
Case1. Suppose that the index set S is finite. If p_α is an isolated point in X_α, for any $\alpha \neq \beta$, then $Y_\beta = X_\beta \times \prod_{\alpha\neq\beta}\{p_\alpha\}$ is an open subset of X and therefore $\mathrm{cl}_X(X \setminus Y_\beta) = X \setminus Y_\beta$ is clopen. Otherwise, assume that, there is $\gamma \neq \beta$ such that p_γ is not isolated in X_γ. Thus, whenever $q = (q_\alpha)_{\alpha\in S} \in Y_\beta$ and $U = \prod_{\alpha\in S} U_\alpha$ is an arbitrary neighborhood of q, then $U_\gamma \cap (X \setminus \{p_\gamma\}) \neq \emptyset$ and hence $U \cap (X \setminus Y_\beta) \neq \emptyset$. This means that $\mathrm{cl}_X(X \setminus Y_\beta) = X$ which is clopen.
Case2. Suppose that the index set S is infinite and let $F = \{\alpha \in S : |X_\alpha| > 1\}$ be the subset of S consisting of all index $\alpha \in S$ where X_α have more than one point. If F is an infinite set, then $\mathrm{cl}_X(X \setminus Y_\beta) = X$ which is clopen. In fact, in this case, whenever $q = (q_\alpha)_{\alpha\in S} \in Y_\beta$ and $U = \bigcap_{i=1}^n \pi_{\alpha_i}^{-1}(U_i)$ is an arbitrary neighborhood of q, then $F \setminus \{\beta, \alpha_1, \cdots, \alpha_n\} \neq \emptyset$ and so $U \cap (X \setminus Y_\beta) \neq \emptyset$. If the set F is finite, then it is easy to see that $X \simeq \prod_{\alpha\in F} X_\alpha$, and we return to the first case. □

The following result is an immediate consequence of the preceding lemma and Proposition 2.23.

Proposition 2.25. *If a non-empty product of a family of T_1-spaces is e-compact, then each factor space is e-compact.*

Using the fact that any e-compact T_1-space is countably compact (Proposition 2.12 in [1]) and every countably compact is pseudocompat (see 1.4 in [3]), the following result is now evident.

Proposition 2.26. *Every e-compact T_1-space is pseudocompact.*

Whenever S is a subspace of a topological space X and $E \subseteq S$, then E is compact (e-compact) as a subspace of X if and only if it is compact (e-compact) when it is regarded as a subspace of S. In fact, the topology of E as a subspace of X coincide with its topology as a subspace of S. On the other hand, if E is a subspace of X, then E is compact if and only if any cover of E by open subsets of X has a finite subcover. This is true, since by the definition of the relative topology, a subset U of E is open in E if and only if there is an open set $\tilde{U}$ in X such that $U = \tilde{U} \cap E$. But, this is not true in general for e-compactness, because, an intersection of an e-open subset of X with E need not be e-open in E and not every e-open subset of E is of this form; see Example 2.8. However, the following proposition shows that the e-compactness of the clopen subspaces of a space can be defined by the e-open subsets of the space.

Proposition 2.27. *A clopen subset E of X is e-compact if and only if any cover of E by e-open subsets of X has a finite subcover.*

Proof. Suppose that E is e-compact and $\tilde{\mathcal{U}}$ is a cover of E consisting of e-open subsets of X. Since E is e-open (clopen), Proposition 2.9 implies that $\mathcal{U} = \{U \cap E : U \in \tilde{\mathcal{U}}\}$ is a cover of E by e-open subsets of E. But E is e-compact, hence $\mathcal{U}$ has a finite subcover which yields that $\tilde{\mathcal{U}}$ has a finite subcover for E. Conversely, suppose that the condition hold and $\mathcal{U}$ is a cover of E by e-open subsets of E. Since E is e-closed (clopen), using Proposition 2.12, for every $U \in \mathcal{U}$ there is an e-open set $\tilde{U}$ in X such that $U = \tilde{U} \cap E$. Then $\tilde{\mathcal{U}} = \{\tilde{U} : U \in \mathcal{U}\}$ is a cover of E consisting of e-open subsets of X and therefore by hypothesis, it has a finite subcover. This means that $\mathcal{U}$ has a finite subcover, i.e., E is e-compact. □

3 e-Separation Axioms

Similar to the topological spaces which have useless structures when they are considered in a general mode without requiring any of the separation axioms, general e-spaces are also not much different from a single point space without considering separation and e-separation axioms (note that any space with trivial topology is an e-space). Therefore, the aim of this section is to define the counterparts of separation axioms on a space X via e-open sets to make sure that the topology of the space X contains enough e-open sets to distinguish between the points of X. In this section we investigate the relations between these axioms so called e-separation axioms with their corresponding usual axioms and give similar results.

Definition 3.1. *A topological space X is said to be an extremely T_1-space (briefly a T_1^e-space) if whenever x and y are distinct points in X, there is an e-open set containing each not the other.*

It is clear from the definition that a topological space X is T_1^e if and only if X_e is a T_1. In fact, the following result shows that the notions T_1 and T_1^e coincide in any topological space.

Proposition 3.2. *The following are equivalent, for a topological space X.*

1. *X is T_1^e.*
2. *X is T_1.*
3. *Each one-point set in X is e-closed.*
4. *Each subset of X is the intersection of the e-open sets containing the subset.*

Proof. (1) $\Rightarrow$ (2) Since any e-open set is open, it is evident.

(2) $\Rightarrow$ (3) If x is an isolated point of X, then $X \setminus \{x\}$ is clopen, so it is e-open. Otherwise $X \setminus \{x\}$ is open, since X is T_1 and $\mathrm{cl}_X(X \setminus \{x\}) = X$ is open, i.e., $X \setminus \{x\}$ is e-open. Hence in each case $X \setminus \{x\}$ is e-open and so $\{x\}$ is e-closed.

(3) $\Rightarrow$ (4) If $A \subseteq X$, then $A = \bigcap_{x \notin A}(X \setminus \{x\})$, where each $X \setminus \{x\}$ is e-open.

(4) $\Rightarrow$ (1) Whenever $x, y \in X$ and $x \neq y$, then each one of the sets $\{x\}$ and $\{y\}$ is the intersection of its e-open neighborhoods. Hence there is an e-open set containing one point but not the other. □

By the previous proposition, a space X is T_1^e if and only if it is T_1, i.e., every point is a closed set; see also Theorem 13.4 in [10]. Therefore, all results concerning T_1-spaces, also hold for T_1^e. More specifically, every subspace of a T_1^e-space is T_1^e and a non-empty product space is T_1^e if and only if each factor space is T_1^e also a quotient of a space X is T_1^e

if and only if each element of the corresponding decomposition is closed in X; see Problem 13B in [10]. Recall from [10], that a decomposition $\mathcal{D}$ of a space X is a collection of disjoint subsets whose union is X, and any decomposition of X will be a quotient space of X by Definition 9.5 and Theorem 9.6 in the same reference. The convers is also true, in the sense that whenever Y is a quotient space of X and $f : X \to Y$ is the quotient map, then Y is homeomorphic to the decomposition space $\mathcal{D}$ whose elements are the sets $f^{-1}(y)$, where $y \in Y$; see Theorem 9.7 in [10]. Also remember that, whenever $\mathcal{D}$ is a decomposition of a space X, an open set V in X is called saturated relative to $\mathcal{D}$ if V is a union of elements of $\mathcal{D}$, i.e., $V = P^{-1}(W)$ for some open set W in $\mathcal{D}$, where $P : X \to D$ is the natural quotient map.

Definition 3.3. *We call a space X an e-Hausdorff space (or T_2^e-space) if any two different points of X can be separated by two disjoint e-open sets.*

Clearly, every T_2^e-space is T_1^e. The real line $\mathbb{R}$ with the ususal topology, is a Hausdorff space which is not e-Hausdorff, as there are no two disjoint nonempty e-open sets in $\mathbb{R}$; see the argument preceding Corollary 2.2. Thus a Hausdorff space (also a T_1^e-space, by Proposition 3.2) need not be e-Hausdorff. Since every e-open set is open, any e-Hausdorff space is clearly Hausdorff. The following proposition shows that e-Hausdorff spaces are much stronger than Hausdorff spaces. First we recall from [7], that a space X is called ultra-Hausdorff if every two different points of X can be separated by two disjoint clopen sets. The following result states that the concept of T_2^e coincides with the concept of ultra-Hausdorff.

Proposition 3.4. *A topological space is e-Hausdorff if and only if it is ultra-Hausdorff.*

Proof. First we note that whenever U and V are two disjoint e-open subsets of a topological space X, then by Lemma 2.1, $\mathrm{cl}_X U \cap \mathrm{cl}_X V = \emptyset$. Next if a space X is e-Hausdorff, for each two different points x and y in X there exist two disjoint e-open subsets U and V of X containing x and y respectively. Therefore, $\mathrm{cl}_X U$ and $\mathrm{cl}_X V$ are two disjoint clopen sets containing x and y respectively, hence X is an ultra-Hausdorff. Since every clopen set is an e-open set, the proof of the converse is evident. □

Theorem 3.5. *The following statements hold.*

1. *Every subspace of a T_2^e-space is T_2^e.*
2. *A non-empty product space is T_2^e if and only if each factor space is T_2^e.*

Proof.

(1) Since the trace of every clopen set on any subspace is clopen, the result is evident by Proposition 3.4.

(2) Using Proposition 3.4 the proof is quite similar to the proof of part (b) in Theorem 13.8 in [10], by replacing open neighborhoods with clopen neighborhoods. □

The following example shwos that a quotient space of a T_2^e-space need not be T_2^e.

Example 3.6. **(a)** *A closed continuous image of a T_2^e-space need not be T_2^e.* Let $X = \mathbb{Q}$ be the space of rational numbers with neighborhoods of any nonzero point as usual, while neighborhoods of 0 have the form $U \setminus A$, where U is a neighborhood of 0 in the usual topology and $A = \{\frac{1}{n} : n = 1, 2, \dots\}$. Then X is e-Hausdorff and A is a closed subset of X, therefore the quotient space X/A, obtained by identifying A with a single point is a closed continuous image of X. But, the space X/A is not e-Hausdorff, as if $P : X \to X/A$ is the quotient map then there is no disjoint e-open sets separating $P(0)$ and $P(A)$.

(b) *An open continuous image of a T_2^e-space which is not T_2^e.* Let $X = \mathbb{Q} \times \{0, 1\}$, be the product of rational numbers and the two point discrete space $\{0, 1\}$ with the product topology. It is clear that X is e-Hausdorff (zero-dimensional). But, if we define a quotient space Y of X by identifying each point $(x, 0)$, where $x \neq 0$, with its corresponding point $(x, 1)$, then the quotient space thus we obtain is an open continuous image of X. However, Y is not e-Hausdorff, as any two neighborhoods (e-open sets) of $(0, 0)$ and $(0, 1)$ intersect each other.

As we observe previously, a Hausdorff space need not be e-Hausdorff. However, these two notions coincide in the realm of e-spaces. This is the subject of the following corollary which is evident.

Corollary 3.7. *Let X be an e-space. Then X is Hausdorff if and only if it is ultra-Hausdorff.*

Using the previous corollary we have the following results which are the counterparts of Theorems 17.5(b) and 17.6(a) in [10].

Proposition 3.8. *An e-compact subset of an e-Hausdorff space is E-closed.*

Proof. Let A be an e-compact subset of an e-Hausdorff space X. We show that any point not in A can be separated from A by a clopen set, whence we conclude that $A = e\text{-cl}_X A$. Let $x \in X \setminus A$. Since X is e-Hausdorff, for each $a \in A$ there exists a clopen set U_a containing a but not x, by Proposition 3.4. Now $\mathcal{C} = \{A \cap U_a : a \in A\}$ is an e-open cover of the e-compact subspace A and hence it has a finite subcover, say $\{A \cap U_{a_1}, \cdots, A \cap U_{a_n}\}$, so $A = \bigcup_{i=1}^n A \cap U_{a_i}$. But $U = \bigcup_{i=1}^n U_{a_i}$ is a clopen set not containing x and hence $X \setminus U$ is a clopen set containing x which does not meet A. □

Proposition 3.9. *Disjoint e-compact subsets of an e-Hausdorff space can be separated by disjoint e-open sets.*

Proof. Suppose that A and B are two disjoint e-compact subsets of X. Using the proof of Proposition 3.8, every $a \in A$ can be separated from B by a clopen set. Thus, for every $a \in A$, there is a clopen set U_a such that $a \in U_a \subseteq X \setminus B$. Now, a similar argument to the proof of the preceding proposition shows that there is a clopen set U in X containing A and disjoint from B. □

The notions of Hausdorff and ultra-Hausdorff not only coincide in the realm of e-spaces, but in finite decompositions of e-spaces as well. Recall from [10], that a decomposition $\mathcal{D}$ of a space X is called finite, whenever only finitely many elements of $\mathcal{D}$ have more than one point. Note that, any finite decomposition $\mathcal{D}$ of a space X, is technicality the same as a decomposition which has only one element with more than one point.

Proposition 3.10. *Let X be an e-space and Y be a quotient space of X obtained by identifying a closed subset A of X with a point. Then Y is Hausdorff if and only if it is ultra-Hausdorff.*

Proof. The sufficiency is evident, as any ultra-Hausdorff space is Hausdorff. To prove the necessity, suppose that Y is Hausdorff and let $P : X \to Y$ be the quotient map. If y_1 and y_2 are two distinct points of Y such that $\{y_1\}, \{y_2\} \neq P(A) = \{a\}$, then there are two distinct points $x_1, x_2 \in X \setminus A$ such that $P^{-1}(\{y_i\}) = \{x_i\}$, for $i = 1, 2$. Since Y is Hausdorff, there is an open set U in Y, such that $y_1 \in U$ and $a, y_2 \notin \mathrm{cl}_Y U$. Then, $W = P^{-1}(U)$ is an open subset of X containing x_1 but not x_2. Moreover, as P is continuous, $P^{-1}(\mathrm{cl}_Y U)$ is a closed set containing W and disjoint from A, thus $A \cap \mathrm{cl}_X W = \emptyset$. Now, since X is an e-space, there is an e-open neighborhood V of x_1 in X which is contained in W. Therefore, $\mathrm{cl}_X V$ is clopen in X and disjoint from A, which follows that $\mathrm{cl}_X V$ is a saturated clopen set in X. Hence $P(\mathrm{cl}_X V)$ is a clopen set in Y, which contains y_1 but not y_2. Whenever y is a point of Y different from a, a similar argument to the first part of our proof shows that, there is a clopen set in Y, which contains y but not a. □

The following theorem gives another characterization of e-Hausdorff spaces in terms of filters, nets and also via the e-closure of e-neighborhoods. We note by the definition, that a space X is e-Hausdorff if and only if X_e is Hausdorff. Although, using Lemma 2.7 and the previous corollary, this is equivalent to saying that X_e is ultra-Hausdorff.

Theorem 3.11. *The following statements are equivalent for a topological space X.*

1. *The space X is e-Hausdorff.*
2. *e-limits in X are unique, i.e., no net and no filter in X e-converges to more than one point.*
3. *Every point $x \in X$ is the intersection of the e-closure of its e-open neighborhoods.*

Proof. (1) ⇔ (2) Using Theorem 13.7 in [10] and by the argument preceding the theorem, X is e-Hausdorff if and only if limits in X_e are unique. But, by the argument after Definitions 2.20 and 2.21, this is equivalent to the uniqueness of e-limits in X.

(1) $\Leftrightarrow$ (3) Suppose that X is a T_2^e-space and let $x \in X$. For any $y \neq x$, there is disjoint e-open sets U_y and V_y containing x and y respectively. Thus, for every $y \neq x$, $y \notin e\text{-cl}_X U_y$ and so $y \notin \bigcap_{x \in W \in \mathcal{E}(X)} e\text{-cl}_X W$. Therefore $\bigcap_{x \in W \in \mathcal{E}(X)} e\text{-cl}_X W \subseteq \{x\}$, and hence $\bigcap_{x \in W \in \mathcal{E}(X)} e\text{-cl}_X W = \{x\}$. Conversely, suppose that $x \neq y$, then $y \notin \{x\} = \bigcap_{x \in W \in \mathcal{E}(X)} e\text{-cl}_X W$. Thus there is an e-open neighborhood W of x such that $y \notin e\text{-cl}_X W$. But, this means that there is an e-open set V containing y and disjoint from W, i.e., x and y can be separated by disjoint e-open sets. □

Remark 3.12. One can think that a similar condition such as the third part of Theorem 13.7 in [10], i.e., the E-closedness of the diagonal in $X \times X$, may be equivalent to other parts of Theorem 3.11. Whenever Y is a topological space, then the closed sets in Y_e are the E-closed subsets of Y (the sets which are an intersection of some e-closed sets). Thus here, the counterpart of the third condition of Theorem 13.7 in [10] could be as follows: the diagonal $\Delta = \{(x, x) : x \in X\}$ is E-closed in $X \times X$. Indeed, a similar argument to the proof of $(b) \Rightarrow (c)$ in Theorem 13.7 in [10] shows that, if X is e-Hausdorff then the aforementioned condition holds. But the convers is not necessarily true. For example, if $X = \mathbb{R}$ be the real line with usual topology, then it is clear that $\mathbb{R}^2 \setminus \Delta$ is e-open (it is open and its closure is equal to $\mathbb{R}^2$) so the diagonal Δ is e-closed in $X \times X$, but X is not e-Hausdorff.

In contrast to the basic open subsets of $X \times X$, the above example shows that the e-open subsets of $X \times X$ are not necessarily of the form $U \times V$, where U and V are e-open sets in X, i.e., $(X \times X)_e \neq X_e \times X_e$. But using Lemma 2.7, every e-open set in X is e-open in X_e and since $\mathcal{E}(X)$ form a base for the topology of X_e we conclude that every open subset of $X_e \times X_e$ is an E-open set in $X \times X$. Thus, we can add a fourth equivalent item to the previous theorem as follows:

"(4) The diagonal $\Delta = \{(x, x) : x \in X\}$ is closed in $X_e \times X_e$."

Definition 3.13. *A topological space X is said to be e-regular, whenever for any e-closed set $F \subseteq X$ and every point x not in F, there is disjoint*

e-open sets U and V such that $x \in U$ and $F \subseteq V$. An e-regular space which is also T_1 (or T_1^e) is called a T_3^e-space.

The trivial topology on a set X with at least two points, is an example of an e-regular space which is not T_2^e and therefore an e-regular space need not be e-Hausdorff. However, it is clear that any T_3^e-space is a T_2^e. Also, a regular (T_3) space need not be e-regular (T_3^e), for instance the space $\mathbb{R}$ with the usual topology is a regular (T_3) space. But, $\mathbb{R}$ is not e-regular (T_3^e), as any two non-empty e-open sets in $\mathbb{R}$ have non-empty intersection. We will see in the sequel that every e-regular space is regular. However, these two notions coincide in the realm of the e-spaces. To this end, we need the following lemma and propositions.

Lemma 3.14. *A space X is e-regular (T_3^e) if and only if X_e is a regular (T_3) space.*

Proof. Suppose that X is e-regular. Let $A \subseteq X_e$ be a closed set and $p \in X_e \setminus A$. Then A is an E-closed subset of X, i.e., an intersection of e-closed sets of X, say $A = \bigcap_{i\in I} H_i$. Therefore, there is $i \in I$ such that $p \notin H_i$ and thus by our hypothesis there exists two disjoint e-open subsets U and V of X which separate p and H_i. But, since $A \subseteq H_i$ and e-open subsets of X are basic open subsets of X_e, we are through.

Conversely, suppose that H is an e-closed set of X and $p \in X \setminus H$. Thus H is a closed subset of X_e and the hypothesis implies that there are two disjoint open subsets W and V of X_e, such that $p \in W$ and $A \subseteq V$. Since e-open sets of X are basic open subsets of X_e, there exists an e-open set U of X with $p \in U \subseteq W$. On the other hand, any open subset of X_e is open in X, so Lemma 2.1 implies that $\mathrm{cl}_X U \cap \mathrm{cl}_X V = \mathrm{cl}_X(U \cap V) = \emptyset$. Therefore, $V \subseteq X \setminus \mathrm{cl}_X U$. But, U is an e-open set in X, so $X \setminus \mathrm{cl}_X U$ is also e-open (clopen) and these two disjoint e-open sets separate p and A. □

From [2], recall that a T_1-space X is zero-dimensional if each point of X has a neighborhood base consisting of clopen sets. Equivalently, a T_1-space X is zero-dimensional if and only if for each $x \in X$ and each closed set A not containing x, there exists a clopen set containing x which does not meet A. So every zero-dimensional space is a completely regular Hausdorff e-space. The converse is also true by the following proposition.

Proposition 3.15. *A space is a T_3-e-space if and only if it is zero-dimensional.*

Proof. Let X be a T_3-e-space. Let G be an open set in X and $x \in G$. Using the regularity of the e-space X, there exists an open set H such that $x \in H \subseteq \mathrm{cl}_X H \subseteq G$. Now, since X is an e-space, there is an e-open set K in X such that $x \in K \subseteq H$ and hence $x \in K \subseteq \mathrm{cl}_X K \subseteq \mathrm{cl}_X H \subseteq G$, where $\mathrm{cl}_X K$ is clopen, so X is zero-dimensional. The convers is obvious, as a zero-dimensional space has a base consisting of clopen sets. □

Example 3.16. (**a**) *A T_1-e-space need not be zero-dimensional.* Whenever X is an infinite set with cofinite topology, then clearly X is a T_1-e-space which is not even T_2, so it is not zero-dimensional.

(**b**) *A T_2-e-space which is not zero-dimensional.* Let X be the space presented in Example 3.6(a). Then X is a T_2-space which is not T_3 and

$$\begin{aligned}\mathfrak{B} = &\{(\alpha,\beta)\cap\mathbb{Q} : 0 \notin (\alpha,\beta) \text{ and } \alpha,\beta \in \mathbb{R}\setminus\mathbb{Q}\}\\ &\cup \{((\alpha,\beta)\setminus A)\cap\mathbb{Q} : 0 \in (\alpha,\beta) \text{ and } \alpha,\beta \in \mathbb{R}\setminus\mathbb{Q}\}\end{aligned}$$

is a base for X consisting of e-open sets. Thus X is an e-space and since X is not T_3, it is not zero-dimensional, by the previous proposition.

Proposition 3.17. *For a space X, the following statements are equivalent.*

1. *The space X is e-regular.*
2. *If U is an e-open set and $x \in U$, then there is an e-open set V containing x such that $\mathrm{cl}_X V \subseteq U$.*
3. *Each point $x \in X$, has a neighborhood base consisting of clopen sets.*

Proof. (1) ⇒ (2). Let X be an e-regular space, U be an e-open set and $x \in U$. Then $X \setminus U$ is an e-closed set not containing x and so there are disjoint e-open sets V and W in X, such that $x \in V$ and $X \setminus U \subseteq W$. Using Lemma 2.1, $\mathrm{cl}_X V \cap \mathrm{cl}_X W = \emptyset$ and therefore

$$\mathrm{cl}_X V \subseteq X \setminus \mathrm{cl}_X W \subseteq X \setminus W \subseteq U.$$

(2) $\Rightarrow$ (3). Let $x \in X$ and U be an arbitrary neighborhood of x in X. Since $W = U \cup \mathrm{int}_X(X \setminus U)$ is a dense open set in X, it is an e-open set in X and $x \in W$. Then, by the hypothesis, there is an e-open set V in X containing x with $\mathrm{cl}_X V \subseteq W$. Let $G = \mathrm{cl}_X V$. Then G is clopen and $G \cap \mathrm{cl}_X U = G \cap U$, as $G \subseteq W$ and $\mathrm{cl}_X U \cap \mathrm{int}_X(X \setminus U) = \emptyset$. Thus using Lemma 2.1 and the fact that G is e-open (clopen), we have

$$\mathrm{cl}_X(G \cap U) = \mathrm{cl}_X G \cap \mathrm{cl}_X U = G \cap \mathrm{cl}_X U = G \cap U.$$

This means that $G \cap U$ is a clopen set in X which contains x and $G \cap U \subseteq U$.
(3) $\Rightarrow$ (1). It is evident. □

The equivalence of parts (1) and (3) implies the following corollary.

Corollary 3.18. *Every e-regular space is regular.*

Using Propositions 3.15 and 3.17, the following result is now evident.

Theorem 3.19. *For a topological space X, the following are equivalent.*

1. *The space X is T_3^e.*
2. *The space X is a T_3-e-space.*
3. *The space X is zero-dimensional.*

Theorems 6.2.11 and 6.2.14 in [2] and our Theorem 3.19 yield the following result.

Theorem 3.20. *The following statements hold.*

1. *Any subspace of an e-regular (T_3^e) space is e-regular (T_3^e).*
2. *A non-empty product space is e-regular (T_3^e) if and only if each factor space is e-regular (T_3^e).*

Example 3.21. *A quotient of an e-regular (T_3^e) space need not be e-regular (T_3^e).* Let X be the space introduced in Example 3.6(b), then it is easy to see that X is a zero-dimensional space. Thus X is a T_3^e-space by Theorem 3.19. But, the quotient space Y constructed in the same example, is not even T_2^e.

By Proposition 3.17, any e-regular space has a base consisting of clopen sets. In particular, every e-regular space is an e-space, as it contains a base consisting of e-open sets. Thus using Theorem 3.20, we have the following result.

Corollary 3.22. *Any subspace of an e-regular space is an e-space.*

It is clear that every compact space is e-compact. Although, the validity of the convers is not yet clear. But, since an e-space is e-compact if and only if it is compact, then Corollary 3.22 follows that an e-regular space (or any subspace of an e-regular space) is e-compact if and only if it is compact. However, Proposition 3.17 and a similar argument to the proof of Proposition 3.9, yield the following result; see also Theorem 17.6 in [10].

Proposition 3.23. *Every e-compact set in an e-regular space can be separated by a clopen set from each disjoint closed set.*

Proposition 3.24. *Every e-compact, e-Hausdorff space is T_3^e.*

Proof. Let X be an e-compact, e-Hausdorff space. Then X is a T_1-space. Now, let $p \notin F$, where F is an e-closed subset of X. Using Proposition 2.23, F is e-compact and therefore there is a clopen set which separates p from F, by Proposition 3.8. □

In the previous result we observe that an e-compact, e-Hausdorff space is T_3^e. In the sequel (Theorem 3.31), we will see that e-compact e-Hausdorff spaces are too stronger than T_3^e-spaces. However, using the previous proposition, Corollary 3.22 and the fact that an e-space is e-compact if and only if it is compact, we have the following theorem.

Theorem 3.25. *Let X be an e-Hausdorff space. Then X is e-compact if and only if it is compact.*

The quotient space obtained by identifying the subset $A = \{\frac{1}{n} : n \in \mathbb{N}\}$ with a single point in Example 3.6(a), is not T_2^e because the space X was not T_3^e. In case X is a T_3^e-space, then such quotient spaces will be T_2^e.

Theorem 3.26. *If X is a T_3^e-space and Y is a quotient space of X obtained by identifying a closed set A in X with a point, then Y is T_2^e.*

Proof. Using Theorem 3.19, whenever X is T_3^e, then it is a T_3-e-space. Therefore, Theorem 14.7 in [10] implies that the space Y is Hausdorff. Now, the result is evident by Proposition 3.10. □

Definition 3.27. *A topological space X is called e-normal provided that every two disjoint E-closed subsets of X can be separated by two disjoint e-open sets. An e-normal T_1-space is called T_4^e.*

Clearly every e-normal (T_4^e) space is e-regular (T_3^e). Therefore, Corollary 3.22 implies that any e-normal space is an e-space and thus every closed set is E-closed. On the other hand, Lemma 2.1, implies that any two disjoint e-open sets have disjoint closures. Now, since the closure of any e-open set is clopen we conclude that, a space X is e-normal if and only if whenever A and B are two disjoint closed sets, there is a clopen set U such that $A \subseteq U \subseteq X \setminus B$. We summarize this in the following proposition.

Proposition 3.28. *A space X is e-normal if and only if whenever A is a closed set and V is an open neighborhood of A, then there is a clopen subset U of X such that $A \subseteq U \subseteq V$.*

By the preceding proposition, every e-normal (T_4^e) space is normal (T_4). However, the converse is not necessarily true; for instance the space $\mathbb{R}$ with the usual topology is a normal space which is not e-normal, since the space $\mathbb{R}$ is connected and thus it contains no non-empty proper clopen subset.

Example 3.29. **(a)** *The Sorjenfrey line* $\mathbf{E}$*, is a T_4^e-space.* The Sorjenfrey line, $\mathbf{E}$, is the real line $\mathbb{R}$ with the topology in which basic neighborhoods of any $x \in \mathbb{R}$ are the sets $[x, z)$, for $z > x$; see Example 4.6 in [10]. Clearly, $\mathbf{E}$ is T_1. To prove that $\mathbf{E}$ is e-normal, we use the previous proposition. So, let $A \subseteq \mathbf{E}$ be a closed subset and V be an arbitrary open subset containing A. For every $a \in A$, let $[a, z_a)$ be a basic neighborhood of a in $\mathbf{E}$ which is contained in V and put $U = \bigcup_{a \in A}[a, z_a)$. Thus $A \subseteq U \subseteq V$, and it is enouth to show that U is clopen. Suppose that $b \in \mathbf{E} \setminus U$, then $b \notin A$ and therefore there is a basic neighborhood $[b, z)$ of b disjoint from A. We claim that $[b, z) \cap U = \emptyset$; if it is the case, we conclude that $\mathbf{E} \setminus U$ is open, i.e., U is clopen. Suppose on the contrary, that $[b, z) \cap U \neq \emptyset$, then there is $a \in A$ such that $[b, z) \cap [a, z_a) \neq \emptyset$. This is impossible, because in this case we have either $a \in [b, z)$ which implies $[b, z) \cap A \neq \emptyset$ or $b \in [a, z_a)$ which implies $b \in U$, and in both cases we get a contradiction.

(b) *Any well-ordered set with the order topology is a T_4^e-space.* Suppose that X is a well-ordered set regarded with the order topology. Note that, for every $x_0 \neq x \in X$ any subset of the form $(y, x]$ is open, where x_0 is the smallest element of X. Whenever x is the largest element of X (if exists), then $(y, x]$ is clearly a basic open neighborhood of x. If x is not the largest element of X, there exists an immediate successor x^+ of x since X is well-ordered and then $(y, x] = (y, x^+)$ is an open interval. Also, note that, the set $\{x_0\}$ is clopen in X. Moreover, it is clear that the set $B = \{(y, x] : x, y \in X \text{ and } y < x\} \cup \{\{x_0\}\}$ forms a base for X. Now, suppose that A is a closed subset of X such that $x_0 \notin A$ and V is an arbitrary open subset of X containing A. Then an argument similar to part(a) shows that there is a clopen set U in X such that $A \subseteq U \subseteq V$. Whenever $x_0 \in A$, since $\{x_0\}$ is clopen in X, $A \setminus \{x_0\}$ is also closed. Thus, by the preceding argument there is a clopen set U in X such that $A \setminus \{x_0\} \subseteq U \subseteq V$. Hence, $U \cup \{x_0\}$ is clopen and $A \subseteq U \cup \{x_0\} \subseteq V$.

The following theorem gives a characterization of T_4^e-spaces. First, we recall from [2], that a Tychonoff space X is called strongly zero-dimensional if any cover $\{U_i\}_{i=1}^n$ of X consisting of cozero-sets, has a finite open refinement $\{V_i\}_{i=1}^m$ such that $V_i \cap V_j = \emptyset$, whenever $i \neq j$. Remember that, a cover $\mathcal{V}$ of a space X is a refinement of another cover $\mathcal{U}$, if for every $V \in \mathcal{V}$ there is $U \in \mathcal{U}$ such that $V \subseteq U$. Using Theorem 6.4.2 in [2], a non-empty Tychonoff space X is strongly zero-dimensional if and only if for every pair A, B of completely separated subsets of the space X there exists a clopen set $U \subseteq X$ such that $A \subseteq U \subseteq X \setminus B$.

Theorem 3.30. *A space is T_4^e if and only if it is a strongly zero-dimensional T_4-space.*

Proof. The necessity is evident by the argument preceding the theorem. To prove the sufficiency, let A be a closed subset of X and V be an arbitrary open set containing A. Then A and $B = X \setminus V$ are two disjoint closed sets. Since X is T_4, the Urysohn's lemma (15.6 in [10]) follows that A and B are completely separated. Now, as X is also strongly zero-dimensional, by the argument preceding the theorem, there exists a clopen set $U \subseteq X$ such that $A \subseteq U \subseteq X \setminus B$. This means that $A \subseteq U \subseteq V$, and therefore X is T_4^e by Proposition 3.28. □

The following theorem is the counterpart of Theorem 17.10 in [10].

Theorem 3.31. *Every e-compact e-Hausdorff space is T_4^e.*

Proof. Using Proposition 3.24, every e-compact e-Hausdorff space is T_3^e or equivalently zero-dimensional, by Theorem 3.19. On the other hand, every e-compact e-Hausdorff space is compact by Theorem 3.25. Now, since any compact Hausdorff space is T_4 (by Theorem 17.10 in [10]) and any compact (Lindelöf) zero-dimensional space is strongly zero-dimensional (by Theorem 6.2.7 in [2]), the result is evident by the previous theorem. □

Proposition 3.32. *The following statements hold.*

1. *Closed subspaces of e-normal (T_4^e) spaces are e-normal (T_4^e).*
2. *Products of (even two) e-normal (T_4^e) spaces need not be e-normal (T_4^e).*
3. *Quotients of e-normal (T_4^e) spaces need not be e-normal (T_4^e).*

Proof.
(1) Let S be a closed subspace of an e-normal space X. Moreover, suppose that A is a closed subset of S and V is an open subset of S which contains A. Then there is an open set $\tilde{V}$ in X such that $V = \tilde{V} \cap S$. Also, since A is closed in S, it is closed in X too. Now, using Proposition 3.28, there exists a clopen set U in X such that $A \subseteq U \subseteq \tilde{V}$. Hence $A \subseteq U \cap S \subseteq \tilde{V} \cap S = V$, and since the trace of every clopen set on any

subspace is clopen, we are through by Proposition 3.28. Note that, any subspace of a T_1-space is T_1.

(2) See Example 3.33(b).

(3) See Example 3.33(c). $\square$

Example 3.33. **(a)** *Arbitrary subspaces of T_4^e-spaces need not be T_4^e.* Let $X = \mathbb{W}^* \times \mathbb{W}^*$, where $\mathbb{W}$ is the space of all countable ordinals and $\mathbb{W}^* = \beta\mathbb{W}$ is the Stone-Čech compactification of $\mathbb{W}$; see 5.12 in [3] for more details of the space $\mathbb{W}$. Clearly, X is e-compact (compact) and e-Hausdorff, so Theorem 3.31 implies that X is T_4^e. Example 2 of Section 32 in [5], shows that the (open) subspace $S = \mathbb{W} \times \mathbb{W}^*$ of X is not T_4, thus it is not T_4^e as well, by Theorem 3.30.

(b) *Product of T_4^e-spaces need not be T_4^e.* The Sorjenfrey line $\mathbf{E}$ is a T_4^e-space by Example 3.29(a). But, as we observe in Example 15.5(b) in [10], the product space $\mathbf{E} \times \mathbf{E}$ is not T_4. Thus $\mathbf{E} \times \mathbf{E}$ is not T_4^e, by Theorem 3.30.

(c) *Quotients of T_4^e-spaces need not be T_4^e.* Let X be the space introduced in Example 3.6(b). It is clear that X is countable T_4-space, so Corollary 6.2.8 in [2] implies that X is strongly zero-dimensional. Then X is a T_4^e-space, by Theorem 3.30. However, the quotien space Y of X which is constructed in Example 3.6(b) is not even T_2^e.

In the preceding example we observe that a quotient of a T_4^e-space may fail to be T_4^e. However, the following proposition show that a finite decomposition with closed elements of a T_4^e-space is T_4^e.

Proposition 3.34. *If X is an e-normal (T_4^e) space and Y is obtained from X by identifying a closed set A in X with a point, then Y is e-normal (T_4^e).*

Proof. Let $P : X \to Y$ be the quotient map and $P(A) = \{a\}$, for some $a \in Y$. Moreover, suppose that $H \subseteq Y$ is a closed set and $V \subseteq Y$ is an open set containing H. First, assume that $a \notin H$. Then $A \cap P^{-1}(H) = \emptyset$ and $P^{-1}(H)$ is a closed subset of X which is contained in the open set $P^{-1}(V)$. Using e-normality of X, there is a clopen set U in X such that $P^{-1}(H) \subseteq U \subseteq X \setminus (A \cup (X \setminus P^{-1}(V)))$. Thus, U is a saturated clopen subset of X and therefore $P(U)$ is a clopen subset of Y and $H \subseteq P(U) \subseteq V$. Now, if $a \in H$ then $A \subseteq P^{-1}(H)$ and by a similar argument as in the first case, there is a clopen set $U \subseteq X$ such that $P(U)$ is clopen in Y and $H \subseteq P(U) \subseteq V$. In either case, we are through by Proposition 3.28. Also note that, the quotient space Y of X so constructed is essentially T_1; see also Problem 13B in [10]. □

Definition 3.35. *A space X is called e-Lindelöf if every e-open cover of X has a countable subcover.*

Clearly, every e-compact space is e-Lindelöf. Also, any Lindelöf space is e-Lindelöf and in the realm of e-spaces, a space is Lindelöf if and only if it is e-Lindelöf. In particular, whenever a space X is e-regular, then it is an e-space, by Corollary 3.22. Now, by a similar argument to the proof of Theorem 6.2.7 in [2], using Proposition 3.28 we have the following result.

Theorem 3.36. *An e-regular, e-Lindelöf space is e-normal.*

Also, by a similar argument to the proof of Proposition 2.23, we have the following result concerning subspaces of e-Lindelöf spaces.

Proposition 3.37. *An e-closed subspace of an e-Lindelöf space is e-Lindelöf.*

Example 3.38. **(a)** *Products of e-Lindelöf spaces need not be e-Lindelöf.* The Sorjenfrey line $\mathbf{E}$ is -Lindelöf and therefore it is e-Lindelöf; see Example 3 of section 30 in [5]. But, the product space $\mathbf{E} \times \mathbf{E}$, is not e-Lindelöf; otherwise, the e-closed relatively discrete (uncountable) subspace $L = \{(x, -x) : x \in \mathbf{E}\}$ of $\mathbf{E} \times \mathbf{E}$ will be e-Lindelöf by the previous proposition, which is a contradiction.

(b) *An arbitrary subspace of an e-Lindelöf space need not be e-Lindelöf.* The space $\mathbb{W}^* = \beta\mathbb{W}$, the Stone-Čech compactification of $\mathbb{W}$ (space of all countable ordinals) is e-Lindelöf (compact). However, the (e-open) subspace $\mathbb{W}$ is not e-Lindelöf, as $\mathcal{U} = \{[1, \alpha) : \alpha \in \mathbb{W}\}$ is an e-open cover of $\mathbb{W}$ with no countable subcover.

The preivious example shows that any products (even two) e-Lindelöf spaces is not necessarily e-Lindelöf. But, whenever a non-empty product of T_1-spaces is e-Lindelöf, then each factor spaces is e-Lindelöf, by Proposition 3.37 and Lemma 2.24.

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