Fuzzy BIBO Stability of Linear Control Systems

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Abstract. In this paper, the concept of stability for fuzzy control systems is discussed from a mathematical point of view. Our main objective is to investigate the stability of fuzzy linear systems with bounded input and bounded output. Necessary and sufficient conditions for this kind of stability are presented by a theorem and it is illustrated by solving an example.

1. Introduction

After Zadeh's pioneering work on fuzzy sets[25], the theory of fuzzy logic began to growth. In 1974, Mamdani [13] introduced the first fuzzy controller. Till 1990 most of the researchers were interested in application of fuzzy control systems rather than it's theory. It was about 1990 that analytical structures for fuzzy controllers began to develop. Fuzzy controllers are rule-based controllers that have great potential in controlling highly-uncertain and poorly-modeled

processes. As fuzzy controllers are nonlinear, it seems that their main application should be control of nonlinear systems. However, nonlinear systems can be linearized around the operating points and generally the obtained linear system is a good approximation for the original nonlinear one. Therefore studying fuzzy control of linear systems seems interesting and useful.

Stability is one of the most important factors for any control system. In fact, an unstable control system is typically useless and potentially dangerous. Questions about stability arise in almost every control problem. As a consequence, stability is one of the most extensively studied subjects in system's theory [10],[9],[23],[6],[16],[12]. In the last years fuzzy control engineers have investigated stability analysis for various types of fuzzy-logic-based control systems [21],[11],[4],[17],[14],[22],[15],[8],[1]. In the majority of papers stability conditions of fuzzy control systems employ quadratic Lyapunov functions. Among various types of stability for control systems, BIBO (Bounded Input Bounded Output) stability has received less attention in fuzzy sense.

In this paper we will investigate fuzzy BIBO stability of linear continuous-time systems from a mathematical point of view through a theorem. In order to achieve this goal, we will use the model employed in [19] which considers state-space representation of fuzzy linear continuous-time systems under generalized H-differentiability.

The organization of the paper is as follows: In Section 2, some basic concepts of fuzzy theory as well as system theory are given. In Section 3, a theorem is proved which states necessary and sufficient conditions for fuzzy BIBO stability, subsequently the result is illustrated via an example. The conclusion is provided in section 4.

2.   Priliminaries

In this section for the sake of clarity some related definitions and theorems regarding the subject is represented.

2.1   Fuzzy theory

Denoting the set of all real numbers by $\mathbb{R}$, the set of all fuzzy numbers on $\mathbb{R}$, is denoted by E. A fuzzy number in parametric form is defined as the pair $(\underline{u}, \overline{u})$ of functions $\underline{u}(\alpha), \overline{u}(\alpha), 0 \leqslant \alpha \leqslant 1$, which satisfies the following properties [7],[5]:

1. $\underline{u}(\alpha)$ is a bounded non-decreasing left continuous function in (0,1], and right continuous at 0,

2. $\overline{u}(\alpha)$ is a bounded non-increasing left continuous function in (0,1], and right continuous at 0,

3. $\underline{u}(\alpha) \leqslant \overline{u}(\alpha), 0 \leqslant \alpha \leqslant 1$.

An equivalent definition that defines fuzzy number as a mapping is proposed in [20].

For every fuzzy number $u \in E$ the $\alpha - level$ set, $[u]_\alpha$, for $0 \leqslant \alpha \leqslant 1$ is defined as:

$$[u]_\alpha) = \begin{cases} \{x \in \mathbb{R} | u(x) \geqslant \alpha\} & if \quad \alpha > 0 \\ cl(suppu) & if \quad \alpha = 0 \end{cases} \tag{1}$$

So the α -level set of a fuzzy number defines a bounded and closed interval $[\underline{u}(\alpha), \overline{u}(\alpha)]$, where $\underline{u}(\alpha)$ and $\overline{u}(\alpha)$ denote the left-hand endpoint and the right-hand endpoint of $[u]_\alpha$, respectively.

Every real number $y \in \mathbb{R}$ can be considered as a fuzzy number by defining:

$$\tilde{y}(t)) = \begin{cases} 1 & if \quad t = y \\ 0 & if \quad t \neq y \end{cases} \tag{2}$$

which is called a singleton. For example $\tilde{0} = [0]_\alpha = (\underline{0}, \overline{0})$.

The Hausdorff distance [3] between two fuzzy numbers $u = (\underline{u}(\alpha), \overline{u}(\alpha))$ and $v = (\underline{v}(\alpha), \overline{v}(\alpha))$ is a mapping $d : E \times E \to \mathbb{R}_+ \cup \{0\}$ where

$$d(u, v) = sup \; max\{|\underline{u}(\alpha) - \underline{v}(\alpha)|, |\overline{u}(\alpha) - \overline{v}(\alpha)|\} \tag{3}$$

It is easy to verify that d defines a metric on E and satisfies the following properties [18]:

1. $d(u + w, v + w) = d(u, v), \forall u, v, w \in E$

2. $d(ku, kv) = |k| d(u, v), \forall k \in \mathbb{R}, u, v \in E$

3. $d(u + v, w + e) \leqslant d(u, w) + d(v, e), \forall u, v, w, e \in E$

4. (d, E) is a complete metric space.

In order to define differentiability of a fuzzy function we need the concept of H-difference which is given as bellow:

Definition 2.1.1. ([19]) *Let $x, y \in E$. If there exists $z \in E$ such that $x = y + z$, then z is called the H-difference of x and y, and it is denoted by $x \ominus y$.*

The sign " $\ominus$ " always stands for H-difference. We also mention that:

$$x \ominus y \neq x + (-1)y$$

Definition 2.1.2. ([3]) *Let $f : (a, b) \rightarrow E$ and $x_0 \in (a, b)$. The fuzzy function f is strongly generalized differentiable at x_0 , if there exists an element $f'(x_0) \in E$ such that:*

(i) *for all $h > 0$ sufficiently small $\exists f(x_0 + h) \ominus f(x_0)$, $\exists f(x_0) \ominus f(x_0 - h)$ and the limits (in the metric d):*

$$lim_{h\downarrow 0}\frac{f(x_0 + h) \ominus f(x_0)}{h} = lim_{h\downarrow 0}\frac{f(x_0) \ominus f(x_0 - h)}{h} = f'(x_0) \quad (4)$$

or

(ii) *for all $h > 0$ sufficiently small $\exists f(x_0) \ominus f(x_0 + h)$, $\exists f(x_0 - h) \ominus f(x_0)$ and the limits (in the metric d):*

$$lim_{h\downarrow 0}\frac{f(x_0) \ominus f(x_0 + h)}{-h} = lim_{h\downarrow 0}\frac{f(x_0 - h) \ominus f(x_0)}{-h} = f'(x_0) \quad (5)$$

or

(iii) *for all $h > 0$ sufficiently small $\exists f(x_0 + h) \ominus f(x_0)$, $\exists f(x_0 - h) \ominus f(x_0)$ and the limits (in the metric d):*

$$lim_{h\downarrow 0}\frac{f(x_0 + h) \ominus f(x_0)}{h} = lim_{h\downarrow 0}\frac{f(x_0 - h) \ominus f(x_0)}{-h} = f'(x_0) \quad (6)$$

or

(iv) *for all $h > 0$ sufficiently small $\exists f(x_0) \ominus f(x_0 + h)$, $\exists f(x_0) \ominus f(x_0 - h)$ and the limits (in the metric d):*

$$lim_{h\downarrow 0}\frac{f(x_0) \ominus f(x_0 + h)}{-h} = lim_{h\downarrow 0}\frac{f(x_0) \ominus f(x_0 - h)}{h} = f'(x_0) \quad (7)$$

Theorem 2.1.3. *Let $f(t)$ be a fuzzy valued function on $[a, \infty)$ represented by $[f(t)]_\alpha = (\underline{f}(t, \alpha), \overline{f}(t, \alpha))$. For any fixed $\alpha \in [0, 1]$ assume that $\underline{f}(t, \alpha)$ and $\overline{f}(t, \alpha)$ are Riemann-integrable on $[a, b]$ for every $b \geqslant a$, and assume there are two positive functions $\underline{M}(\alpha)$ and $\overline{M}(\alpha)$ such that $\int_a^b |\underline{f}(t, \alpha)| \mathrm{d}t \leqslant \underline{M}(\alpha)$ and $\int_a^b |\overline{f}(t, \alpha)| \mathrm{d}t \leqslant \overline{M}(\alpha)$ for every $b \geqslant a$. Then $f(t)$ is improper fuzzy Riemann-integrable on $[a, \infty)$ and the improper fuzzy Riemann-integral is a fuzzy number. Furthermore, we have:*

$$[\int_a^\infty f(t)\mathrm{d}t]_\alpha = \left(\int_a^\infty \underline{f}(t, \alpha)\mathrm{d}t, \int_a^\infty \overline{f}(t, \alpha)\mathrm{d}t \right) \quad (8)$$

Proof. See [24]. □

Definition 2.1.4. ([19]) Consider fuzzy valued function $f(t)$ as $[f(t)]_\alpha = (\underline{f}(t,\alpha), \overline{f}(t,\alpha))$. Then, $f(t)$ is (i)-absolute function if for all $\alpha \in [I_1, I_2] \subseteq [0,1]$:

$$|[f(t)]_\alpha| = (|\underline{f}(t,\alpha)|, |\overline{f}(t,\alpha)|) \tag{9}$$

and $f(t)$ is (ii)-absolute function if for all $\alpha \in [I_1, I_2] \subseteq [0,1]$:

$$|[f(t)]_\alpha| = (|\overline{f}(t,\alpha)|, |\underline{f}(t,\alpha)|) \tag{10}$$

Theorem 2.1.5. *Consider the fuzzy valued function $f(t)$ as $[f(t)]_\alpha = (\underline{f}(t,\alpha), \overline{f}(t,\alpha))$ where $\underline{f}(t,\alpha)$ and $\overline{f}(t,\alpha)$ are lower and upper functions of $f(t)$ for all $\alpha \in [0,1]$ respectively. Then:*

1. *If $\underline{f}(t,\alpha) \geqslant 0$ for all α then f is (i)-absolute fuzzy-valued function.*

2. *If $\overline{f}(t,\alpha) \leqslant 0$ for all α then f is (ii)-absolute fuzzy-valued function.*

Proof. See [19]. □

Theorem 2.1.6. *The absolute-value of the fuzzy valued function $f(t)$ is always a positive fuzzy-valued function.*

Proof. See [19]. □

2.2   System theory

Consider the following system of equations:

$$\begin{cases} \dot{x}(t) = f(x,t,u) \\ y(t) = g(x,t,u) \end{cases} \tag{11}$$

where $x \in E^n$, $y \in E^p$, $u \in E^m$, $f : \mathbb{R} \times E^n \times E^m \to E^n$ and $g : \mathbb{R} \times E^n \times E^m \to E^p$. Here, t denotes time and u, y denote fuzzy system input and output, respectively [19]. Note that $\dot{x} = \frac{\partial x}{\partial t}$ is computed under strongly generalized differentiability.

The equation (11) is called the state space description of fuzzy continuous-time system. In this form of representation the first equation is called the fuzzy state equation and the second one is called the fuzzy output equation.

In linear case, $f(t,x,u) = Ax + Bu$ and $g(t,x,u) = Cx + Du$. Therefore, (11) will appear as:

$$\begin{cases} \dot{x}(t) = Ax + Bu \\ y(t) = Cx + Du \end{cases} \tag{12}$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, $D \in \mathbb{R}^{p \times m}$ and $u : \mathbb{R} \to E^m$, $x : \mathbb{R} \to E^n$, $y : \mathbb{R} \to E^p$ denote the fuzzy system input, state and output vectors, respectively.

If $m = p = 1$ then (12) is called a fuzzy SISO system (Single-Input Single-Output system).

Although one can consider (11) as a special case of a hybrid fuzzy differential system and use various mehtods to solve it [2], here we only deal with it's stability and propose some relevant definitions.

Definition 2.2.1. *If $u(t)$ is a fuzzy impulse input, then the fuzzy output excited by $u(t)$ will be called a fuzzy impulse response which is denoted by $g(t)$.*
Using fuzzy impulse response, the fuzzy output will be obtained from:

$$y(t) = \int_0^t g(t - \tau)u(\tau)\mathrm{d}\tau = \int_0^t g(t)u(t - \tau)\mathrm{d}\tau \tag{13}$$

Definition 2.2.2. *In the fuzzy system (12), the output excited by the initial state $x(t_0) = \tilde{0}$, is called the fuzzy zero-state response.*

Definition 2.2.3. *An input $u(t)$ is called fuzzy bounded if there exists $u_m = (\underline{u_m}(\alpha), \overline{u_m}(\alpha))$ such that:*

$$d(|u(t)|, \tilde{0}) \leqslant d(u_m, 0) \tag{14}$$

Definition 2.2.4. *Assume fuzzy system (12) in zero-state response. Then it is called fuzzy BIBO stable (Bounded-Input Bounded-Output stable) if every fuzzy bounded input $u(t)$ excites a fuzzy bounded output.*

3.   Stability of Fuzzy Control Systems

In this section, we are going to state and prove a theorem which gives necessary and sufficient conditions for a fuzzy SISO control system to be BIBO stable.

Theorem 3.1. *The fuzzy SISO system (12) is BIBO stable if and only if the fuzzy impulse response $g(t)$, is fuzzy absolutely integrable in $[0, \infty)$ or*

$$d\Big(\int_0^\infty |g(t)|\mathrm{d}t, \tilde{0}\Big) \leqslant d(M, \tilde{0}) < \infty$$

for some $M \in E$, where $M = (\underline{M}(\alpha), \overline{M}(\alpha))$.

Proof. Let $g(t)$ be fuzzy absolutely integrable and $u(t)$ be an arbitrary fuzzy bounded input i.e. there exists $u_m = (\underline{u_m}(\alpha), \overline{u_m}(\alpha))$ such that:

$$d(|u(t)|, \tilde{0}) \leqslant d(u_m, \tilde{0}), \forall t \geqslant 0.$$

By (13) we have:

$$d(|y(t)|, \tilde{0}) = d(|\int_0^t g(t-\tau)u(\tau)\mathrm{d}\tau|, \tilde{0}).$$

On the other hand:

$$|\int_0^t \underline{g}(\tau, \alpha)\underline{u}(t-\tau, \alpha)\mathrm{d}\tau| \leqslant \int_0^t |\underline{g}(\tau, \alpha)||\underline{u}(t-\tau, \alpha)|\mathrm{d}\tau \leqslant \underline{u_m}(\alpha)\int_0^t \underline{g}(\tau, \alpha)\mathrm{d}\tau \leqslant \underline{u_m}(\alpha)\underline{M}(\alpha),$$

and

$$|\int_0^t \overline{g}(\tau, \alpha)\overline{u}(t-\tau, \alpha)\mathrm{d}\tau| \leqslant \int_0^t |\overline{g}(\tau, \alpha)||\overline{u}(t-\tau, \alpha)|\mathrm{d}\tau \leqslant \overline{u_m}(\alpha)\int_0^t \overline{g}(\tau, \alpha)\mathrm{d}\tau \leqslant \overline{u_m}(\alpha)\overline{M}(\alpha).$$

So, by definition of Hausdorf distance we have:

$$d\left(\int_0^\infty |g(t)|\mathrm{d}t, \tilde{0}\right) \leqslant d(M, \tilde{0}).$$

Thus the fuzzy output is bounded.

Now suppose that the fuzzy system (12) is BIBO stable but g(t) is not fuzzy absolutely integrable, then there exists t_1 such that:

$$d\left(\int_0^{t_1} |g(\tau)|\mathrm{d}\tau, \tilde{0}\right) = \infty.$$

Or, for all fuzzy numbers $N = (\underline{N}(\alpha), \overline{N}(\alpha))$ we have:

$$d\left(\int_0^{t_1} |g(\tau)|\mathrm{d}\tau, \tilde{0}\right) \geqslant d(N, \tilde{0}).$$

To show that the fuzzy system is not BIBO stable we choose the fuzzy bounded input as:

$$u(t_1 - \tau, \alpha) = \begin{cases} \tilde{1} & g(\tau, \alpha) \geqslant \tilde{0} \\ \widetilde{-1} & g(\tau, \alpha) < \tilde{0} \, ; \end{cases}$$

then:

$$\underline{y}(t,\alpha) = \int_0^{t_1} \underline{g}(\tau,\alpha)\underline{u}(t_1 - \tau,\alpha)\mathrm{d}\tau = \int_0^{t_1} |\underline{g}(\tau,\alpha)|\mathrm{d}\tau \geqslant \underline{N}(\alpha).$$

and:

$$\overline{y}(t,\alpha) = \int_0^{t_1} \overline{g}(\tau,\alpha)\overline{u}(t_1 - \tau,\alpha)\mathrm{d}\tau = \int_0^{t_1} |\overline{g}(\tau,\alpha)|\mathrm{d}\tau \geqslant \overline{N}(\alpha)$$

Therefore based on the definition of Hausdorf distance one can reach the following relation:

$$d(y(t),\tilde{0}) \geqslant d(N,\tilde{0})$$

which shows that the output is not bounded. Thus the system is not fuzzy BIBO stable which contradicts the hypothesis. □

Example 3.2. Consider a BIBO stable linear system with fuzzy impulse response $g(t,\tau,\alpha) = a(\alpha)e^{-2t-\tau}$ where $a(\alpha) = [2+\alpha, 4-\alpha]$, for $t \geqslant \tau$. Or:

$$g(t,\tau,\alpha) = [(2+\alpha)e^{-2t-\tau}, (4-\alpha)e^{-2t-\tau}].$$

This function is integrable and we have:

$$\int_0^t |g(t,\alpha)|\mathrm{d}\tau = \int_0^t [(2+\alpha)e^{-2t-\tau}, (4-\alpha)e^{-2t-\tau}]\mathrm{d}\tau = [-(2+\alpha)(e^{-3t}-e^{-2t}), (\alpha-4)(e^{-3t}-e^{-2t})],$$

so $g(t,\tau,\alpha)$ is fuzzy absolutely integrable.
Conversly, to show BIBO stability of fuzzy system with fuzzy impulse response:

$$g(t,\tau,\alpha) = [(2+\alpha)e^{-2t-\tau}, (4-\alpha)e^{-2t-\tau}],$$

one can apply an arbitrary fuzzy bounded input $u(t)$. According to (13) fuzzy integralibility of $g(t,\tau)$ and boundedness of $u(t)$ yields boundedness of $y(t)$. So for every bounded input there exists a bounded output.

4. Conclusion

In this paper, we discussed fuzzy BIBO stability for single-input single-output control systems. In order to prove a theorem about the necessary and sufficient conditions for BIBO stability, some new concepts such as fuzzy impulse input, fuzzy zero state response and other related concepts were proposed. Here, only BIBO stability is considered. Other types of stability such as exponential stability can be verified as a future work.

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