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Original Research Paper

A Common Fixed Point Theorem via Two and Three Mapping in Banach Algebra

S. Verma
Shri Mata Vaishno Devi University

K. Raj
Shri Mata Vaishno Devi University

S.K. Sharma*
Shri Mata Vaishno Devi University

Abstract. In this paper we make an attempt to prove common fixed point theorem for two and three mappings in Banach algebra by using measure of non-compactness. In the second section of the paper we prove the common fixed point theorem for two continuous linear operators on Banach algebra and also an effort is made to prove common fixed point theorem for three continuous commuting operators in Banach algebra.

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1 Introduction and Preliminaries

Schauder [23] used Compactness in fixed point theory and G. Darbo in 1955 [18] used non compactness and non compact operators in fixed

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*Corresponding Author

point theory. The foremost aim of study of non compact operators is to create a new family of operators which transform the bounded set to compact set.

Definition 1.1.[20] *Let M be a metric space and N be its subset then Kuratowski's measure of non compactness is defined as*

$$\gamma(N) = \inf \Big\{ \theta > 0 : N = \cup_{i=1}^{n} N_i; \\ \textit{for some } N_i \textit{ with diam}(N_i) \leq \theta, 1 \leq i \leq n < \infty \Big\}.$$

Here diam $(N_i) = \sup\{d(a,b) : a, b \in N_i, i = 1, 2, \cdots, n\}$.

Definition 1.2. [13] *Let S be a Banach space and N be a non empty bounded subset of N_S, where N_S denotes the family of non empty subsets of S. Then the mapping $\delta : N_S \to [0, \infty)$ is called Hausdorff measure of non compactness of $N \subseteq N_S$ and is defined as*

$$\delta(N) = \inf \Big\{ \alpha > 0 : N \textit{ has a finite } \alpha - \textit{net in } S \Big\}.$$

Definition 1.3.[13] *Let (M, d) be a complete metric space and N be a non empty bounded subset of N_M, where N_M denotes the family of non empty subsets of M. Then the mapping $\delta : N_M \to [o, \infty)$ is called Hausdorff measure of non compactness of $N \subseteq N_M$ and is defined as*

$$\delta(N) = \inf \Big\{ \alpha > 0 : N \subseteq \cup_{i=1}^{n} B(x_i, r_i), x_i \in M, r_i < \alpha \;\; i = 1, 2, 3, ..., n \Big\}.$$

Definition 1.4. [13] *Let (M, d) be a complete metric space and N be a non empty bounded subset of N_M then the mapping $\epsilon : N_M \to [0, \infty)$ is called Istratescu measure of non compactness of $N \subseteq N_M$ s.t*

$$\epsilon(N) = \inf \Big\{ \beta > 0 : N \textit{ has no infinite } \beta - \textit{ discrete subsets} \Big\}.$$

Theorem 1.5.[13] *Let (M, d) be a complete metric space. If $\{\phi_n\}$ be a decreasing sequence of non-empty, closed and bounded subsets of M s.t $\lim_{n\to\infty} \gamma(\phi_n) = 0$ then the intersection $\phi_\infty = \bigcap_{i=1}^{\infty} \phi_n$ is non empty and compact subset of M.*

Proposition 1.6. *Let N, N_1 and N_2 be non empty and bounded subsets of a complete metric space (M, d) then*
1. $\gamma(N) = 0 \Longleftrightarrow \overline{N}$ *is compact,* $\overline{N}$ *is the closure of* N
2. $\gamma(N) = \gamma(\overline{N})$
3. $N_1 \subseteq N_2 \rightarrow \gamma(N_1) \leq \gamma(N_2)$
4. $\gamma(N_1 \bigcup N_2) = max\{\gamma(N_1), \gamma(N_2)\}$
5. $\gamma(N_1 \bigcap N_2) \leq \min\{\gamma(N_1), \gamma(N_2)\}$.

Proposition 1.7. *Let N, N_1 and N_2 be non empty and bounded subsets of a complete metric space (M, d) then*
1. $\delta(N) = 0 \Longleftrightarrow \overline{N}$ *is compact,* $\overline{N}$ *is the closure of* N
2. $\delta(N) = \delta(\overline{N})$
3. If $N_1 \subseteq N_2 \rightarrow \delta(N_1) \leq \delta(N_2)$
4. $\delta(N_1 \bigcup N_2) = \max\{\delta(N_1), \delta(N_2)\}$
5. $\delta(N_1 \bigcap N_2) \leq \min\{\delta(N_1), \delta(N_2)\}$

Proposition 1.8. *Let N, N_1 and N_2 be non empty and bounded subsets of a Banach space $(S, \|.\|)$ over F then*
1. $\delta(N_1 + N_2) \leq \delta(N_1) + \delta(N_2)$
2. $\delta(N + x) = \delta(N), \forall x \in S$
3. $\delta(\lambda(N) = |\lambda|\delta(N), \forall \lambda \in F$
4. $\delta(N) = \delta(Conv(N))$, *where $Conv(N)$ is the convex hull of N*

Note: Similar Properties hold for Istratescu measure of Non Compactness

Theorem 1.9.[13] **Schauder's fixed point theorem:** *Let S be a Banach space and $\aleph$ be a non-empty bounded subsets of S then every continuous mapping $P : \aleph \rightarrow \aleph$ has atleast one fixed point. We use abbreviation S. f. t for Schauder's fixed point theorem throughout the paper.*

Definition 1.10.[13] *A function $m* : \mu_s \longrightarrow \mathbf{R}_+$ (where S is the Banach space and μ_s is the family of all non empty bounded subsets of S) is said to be measure of non compactness in the space S if it satisfies the following conditions.*

1. The family Ker $m*=\{E\in\mu_s : m^*(E)=0\}$ is non empty and Ker $m^*\subseteq\nu_s$, where ν_s is the family of non empty relatively compact subsets of S.
2. $E_1\subseteq E_2\Rightarrow m^*(E_1)\le m^*(E_2)$
3. $m^*(E)=m^*(\overline{E})$
4. $m^*(E)=m^*(conv(E))$
5. $m^*\{\lambda E_1+(1-\lambda)(E_2)\}\le\lambda m^*(E_1)+(1-\lambda)m^*(E_2),\forall\in[0,1]$
6. If $\{E_n\}$ is a sequence of closed sets from μ_s such that $E_{n+1}\subseteq E_n$ for $n=1,2,3,....$ and $\lim\limits_{n\to\infty}m^*(E_n)=0$, then the intersection $E_\infty=\cap_{i=1}^{\infty}E_n$ is non empty.

Definition 1.11. *Let $\aleph$ be a non empty bounded, closed and convex subset of a Banach space S. A self mapping $P:\aleph\to\aleph$ is said to be a m^* contractive mapping if there exists some constant $k\in(0,1)$ s.t $m^*(P(E))\le km^*(E)$, for non empty subset E of $\aleph$.*

Theorem 1.12.[5] Darbo's fixed point theorem: *Let S be a Banach space and $\aleph$ be a non-empty, closed and bounded subsets of S and let mapping $P:\aleph\to\aleph$ be continuous and if P is a $m*$ contraction then P has atleast one fixed point.*

Definition 1.13.[14] *Let T be a non empty set then T is said to be an algebra if*
1. $((T,+,.)$ is a vector space over F,
2. $(T,+,\circ)$ is a ring and
3. $(\alpha a)\circ b=\alpha(a\circ b)=a\circ(\alpha b)$, $\forall\alpha\in F and\ a,b\in T$.

Definition 1.14.[14] *If T is an algebra and $\|.\|$ a norm on T satisfying $\|ab\|\le\|a\|\|b\|$, $\forall a,b\in T$ then the pair $(T,\|.\|)$ is called normed algebra. A complete norm algebra T is called Banach algebra.*
For example, let K be a compact Hausdorff space and $T=C(K)$ then with respect to point wise multiplication of functions, T is a commutative unital algebra with norm $\|f\|_\infty=\sup\limits_{t\in K}|f(t)|$ is a Banach algebra.
For more details about Banach algebra and measure of non compactness one may refer to ([1]-[4], [6]-[7], [9]-[12], [15]-[17], [19], [21]-[22], [24]) and references therein.

Note: $m*$ is the measure of non-compactness satisfying the conditions of definition 1.5. Now if the Banach space S has the structure of Banach algebra T and for given subsets E_1, E_2 of μ_T, let $E_1E_2 = \{ab : a \in E_1, b \in E_2\}$ then the following inequality holds

$$m^*(E_1E_2) \leq (||E_1||)m*(E_2) + (||E_2||)m*(E_1).$$

The foremost goal of this paper is to prove common fixed point theorem for two and three commuting mapping in Banach algebra by using measure of non compactness.

2 Common Fixed Point Theorem For Two Continuous Linear Operators On Banach Algebra T

Theorem 2.1. *Let $\aleph$ be a non empty bounded, closed and convex subset of the Banach algebra T and operators P_1, P_2, Q_1, Q_2 transforms $\aleph$ into T are continuous such that $P_1(\aleph), P_2(\aleph), Q_1(\aleph)$, $Q_2(\aleph)$ are bounded. Suppose $P = P_1Q_1$ and $Q = P_2Q_2$ be the operators from $\aleph$ into $\aleph$ are continuous and Q is linear such that*

$$Q(P(E)) \subseteq P(E), \quad E \subseteq \aleph$$

. If the operators P_1, P_2, Q_1, Q_2 on set $\aleph$ satisfies the following conditions

$$m^*(P_1(E)) \leq \xi_1(m^*(E)),$$
$$m^*(Q_1(E)) \leq \xi_2(m^*(E)),$$
$$m^*(P_2(E)) \leq \xi_1'(m^*(E)),$$
$$m^*(Q_2(E)) \leq \xi_2'(m^*(E)),$$

where E is the non empty subset of $\aleph$, m^* is an arbitrary measure of non compactness defined on μ_T and $\xi_1, \xi_2, \xi_1', \xi_2'$ are non decreasing functions from R_+ to R_+ such that

$$\lim_{n\to\infty} \xi_1^n(x) = 0,$$

$$\lim_{n\to\infty} \xi_2^n(x) = 0,$$
$$\lim_{n\to\infty} \xi_1'^n(x) = 0,$$
$$\lim_{n\to\infty} \xi_2'^n(x) = 0$$

and

$$\lim_{n\to\infty} (||P_1(\aleph)||\xi_2 + ||Q_1(\aleph)||\xi_1)^n(x) = 0,$$
$$\lim_{n\to\infty} (||P_2(\aleph)||\xi_2' + ||Q_2(\aleph)||\xi_1')^n(x) = 0,$$

for any $x \geq 0$, *then* P *and* Q *have a common fixed point.*

Proof. Let E be a non empty subset of $\aleph$. Then in view of the assumption that m^* is a measure of non compactness defined on μ_T, we have

$$\begin{aligned} m^*(P(E)) &= m^*(P_1Q_1(E)) \\ &\leq m^*(P_1(E)Q_1(E)) \\ &\leq m^*(Q_1(E))||P_1(E)|| + m^*(P_1(E))||Q_1(E)|| \\ &\leq m^*(Q_1(E))||P_1(\aleph)|| + m^*(P_1(E)||Q_1(\aleph))|| \\ &\leq \xi_2(m^*(E))||P_1(\aleph)|| + \xi_1(m^*(E))||Q_1(\aleph)|| \\ &= (\xi_2)||P_1(\aleph)|| + (\xi_1)||Q_1(\aleph)||)(m^*(E) \qquad (2.1) \end{aligned}$$

Let

$$\psi_1(x) = ((\xi_2)||P_1(\aleph)|| + (\xi_1)||Q_1(\aleph)||)(x).$$

So from equation 2.1, we have

$$m^*(P(E)) \leq \psi_1(m^*(E)).$$

Similarly,

$$m^*(Q(E)) \leq \psi_2(m^*(E)),$$

where $\psi_2(x) = ((\xi_2')||P_2(\aleph)|| + (\xi_1')||Q_2(\aleph)||)(x)$.
Since $\xi_1, \xi_2, \xi_1', \xi_2'$ are non decreasing functions from $R_+$to R_+ so ψ_1 and ψ_2 are also non decreasing thus because of given condition

$$\lim_{n\to\infty} \psi_1^n(x) = 0$$

and

$$\lim_{n\to\infty} \psi_2^n(x) = 0.$$

Now we can define a sequence of subsets $\{\aleph_n\}$ of T as $\aleph = \aleph_0, \aleph_n = convP\aleph_{n-1}, n \geq 1$.
Then $\aleph_n \subseteq \aleph_{n-1}$ and $Q(\aleph_n) \subset \aleph_n$ (2.2)
Clearly

$$\aleph_1 \subset \aleph_0$$

and

$$Q(\aleph_1) \subseteq Conv(QP(\aleph_0)) \subseteq Conv(P(\aleph_0)) = \aleph_1.$$

Therefore $Q(\aleph_1) \subseteq \aleph_1$ and so equation 2.2 holds for n=1.
Suppose it is true for $n \geq 1$, then

$$(\aleph_{n+1}) = Conv(P(\aleph_n)) \subseteq Conv(P(\aleph_{n-1})) = \aleph_n,$$

as

$$\aleph_n \subseteq \aleph_{n-1},$$

thus

$$\aleph_{n+1} \subseteq \aleph_n$$

and

$$Q(\aleph_{n+1}) = Q(Conv(P(\aleph_n))) \subseteq Conv(QP(\aleph_n)) \subseteq ConvP(\aleph_n) = \aleph_{n+1}.$$

Hence

$$Q(\aleph_{n+1}) \subseteq \aleph_{n+1}$$

and this implies that

$$\aleph_0 \supset \aleph_1 \supset \aleph_2 ... .$$

If there exists $n \geq 0$ such that $m^*(\aleph_n) = 0$, then $\aleph_n$ is relatively compact and since

$$P(\aleph_n) \subseteq ConvP(\aleph_n) = \aleph_{n+1} \subseteq \aleph_n,$$

so by S. f. t P has a fixed point.
Now, we assume that $m^*(\aleph_n) \neq 0, n \geq 0$. Then by assumption,

$$\begin{aligned} m^*(\aleph_{n+1}) &= m^*(Conv(P(\aleph_n))) \\ &= m^*(P(\aleph_n)) \\ &\leq \psi_1(m^*(\aleph_n)) \\ &= \psi_1(m^*(ConvP(\aleph_{n-1}))) \\ &\leq \psi_1^2(m^*(\aleph_{n-1})) \\ &\quad \cdots\cdots\cdots \\ &\leq \psi_1^n(m^*(\aleph_0)). \end{aligned}$$

Since $\psi_1 : \mathbb{R}_+ \to \mathbb{R}_+$ is a non decreasing and

$$\lim_{n\to\infty} \psi_1^n(x) = 0.$$

If we consider

$$r = \lim_{n\to\infty} m^*(\aleph_{n+1}) \leq \lim_{n\to\infty} m^*\psi_1^n(m^*(\aleph_0)) = \lim_{n\to\infty} \psi_1^n(x) = 0,$$

for $x \geq 0, x = m^*(\aleph_0)$.
Then, we have

$$m^*(\aleph_n) \to 0 \quad \text{as} \quad n \to \infty.$$

Since

$$\aleph_{n+1} \subseteq \aleph_n$$

and

$$P(\aleph_n) \subseteq \aleph_n, n \geq 1.$$

Thus, by condition 6 in the definition of m^*, we have

$$\aleph_\infty = \cap_{n=1}^\infty \aleph_n$$

is non empty, closed and convex subset in $\aleph$. Moreover $\aleph_\infty$ is invariant under the operator P and belongs to Kerm^* is relatively compact. Hence, by S. f. t P has a fixed point in $\aleph$.
Now, Suppose

$$G_P = \{t \in \aleph : P(t) = t\}.$$

Then clearly G_P is closed by continuity of P and by assumption Q is linear, we have

$$Q(G_P) \subseteq G_P.$$

So $Q(t)$ is a fixed point of P for any $t \in G_P$ and

$$m^*(G_P) = m^*(P(G_P)) \leq \psi_1(m^*(G_P)) \leq m^*(G_P).$$

Therefore,

$$m^*(G_P) = 0$$

and G_P is compact. Then by S. f. t, Q has a fixed point and set

$$G_Q = \{t \in \aleph : Q(t) = t\}$$

is closed by continuity of Q.
Also

$$Q(G_P) \subseteq G_P$$

thus by S. f. t, $P(t)$ is a fixed point of Q, for all $t \in G_Q$.
Since

$$G_P \cap G_Q \subseteq G_P \subseteq \aleph$$

is compact subset and

$$P, Q : G_P \cap G_Q \to G_P \cap G_Q$$

are continuous self maps so by S. f. t, P & Q have a common fixed point in $\aleph$. □

In next theorem we prove common fixed point theorem for two continuous Commutative operators on Banach Algebra T.

Theorem 2.2. *Let $\aleph$ be a non empty bounded, closed and convex subset of the Banach algebra T and operators P_1, P_2, Q_1, Q_2 transforms $\aleph$ into T are continuous and mutually commutative s.t*

$$P_1(\aleph), P_2(\aleph), Q_1(\aleph), Q_2(\aleph)$$

are bounded. Suppose $P = P_1Q_1$ and $Q = P_2Q_2$ be the commutative operators from $\aleph$ into $\aleph$ are continuous. If the operators P_1, P_2, Q_1, Q_2 on set $\aleph$ satisfies the following conditions

$$m^*(P_1(E)) \leq \xi_1(m^*(E)), m^*(Q_1(E)) \leq \xi_2(m^*(E)),$$

$$m^*(P_2(E)) \leq \xi_1'(m^*(E)), m^*(Q_2(E)) \leq \xi_2'(m^*(E)),$$

where E is the non empty subset of $\aleph$ and m^* is an arbitrary measure of non compactness defined on μ_T. Also $\xi_1, \xi_2, \xi_1', \xi_2'$ are non decreasing functions from R_+ to R_+ s.t

$$\lim_{n\to\infty} \xi_1^n(x) = 0,$$

$$\lim_{n\to\infty} \xi_2^n(x) = 0,$$

$$\lim_{n\to\infty} \xi_1'^n(x) = 0,$$

$$\lim_{n\to\infty} \xi_2'^n(x) = 0$$

and

$$\lim_{n\to\infty} (||P_1(\aleph)||\xi_2 + ||Q_1(\aleph)||\xi_1)^n(x) = 0,$$

$$\lim_{n\to\infty} (||P_2(\aleph)||\xi_2' + ||Q_2(\aleph)||\xi_1')^n(x) = 0,$$

for any $x \geq 0$, then P and Q have a common fixed point.

Proof. Let E be a non empty subset of $\aleph$ then by definition of m^* on μ_T, we have

$$\begin{aligned} m^*(P(E)) &= m^*(P_1Q_1(E)) \\ &\leq m^*(P_1(E)Q_1(E)) \\ &\leq m^*(Q_1(E))||P_1(E)|| + m^*(P_1(E))||Q_1(E)|| \\ &\leq m^*(Q_1(E))||P_1(\aleph)|| + m^*(P_1(E)||Q_1(\aleph))|| \\ &\leq \xi_2(m^*(E))||P_1(\aleph)|| + \xi_1(m^*(E))||Q_1(\aleph)|| \\ &= (\xi_2)||P_1(\aleph)|| + (\xi_1)||Q_1(\aleph)||)(m^*(E) \quad ....(2.3) \end{aligned}$$

Let

$$\psi_1(x) = ((\xi_2)||P_1(\aleph)|| + (\xi_1)||Q_1(\aleph)||)(x).$$

So from equation 2.3, we have

$$m^*(P(E)) \leq \psi_1(m^*(E)).$$

Similarly

$$m^*(Q(E)) \leq \psi_2(m^*(E)),$$

where

$$\psi_2(x) = ((\xi_2')||P_2(\aleph)|| + (\xi_1')||Q_2(\aleph)||)(x).$$

Since

$$\xi_1, \xi_2, \xi_1', \xi_2'$$

are non decreasing functions from R_+ into R_+ so ψ_1, ψ_2 are also non decreasing functions. Now because of given condition, we have

$$\lim_{n\to\infty} \psi_1^n(x) = 0$$

and

$$\lim_{n\to\infty} \psi_2^n(x) = 0.$$

Now we can define a sequence of subsets $\{\aleph_n\}$ of T as

$$\aleph = \aleph_0, \aleph_n = convP\aleph_{n-1}, n \geq 1.$$

Then $\aleph_n \subseteq \aleph_{n-1}$ and $Q(\aleph_n) \subset \aleph_n$ ...(2.4).
Clearly

$$\aleph_1 \subset \aleph_0$$

and

$$Q(\aleph_1) \subseteq Conv(QP(\aleph_0)) = Conv(PQ(\aleph_0)) \subseteq Conv.(P(\aleph_0)) = \aleph_1(as \quad PQ = QP).$$

Therefore,

$$Q(\aleph_1) \subseteq \aleph_1$$

and so 2.4 holds for $n = 1$. Suppose it is true for $n \geq 1$ then

$$(\aleph_{n+1}) = Conv(P(\aleph_n)) \subseteq Conv(P(\aleph_{n-1})) = \aleph_n,$$

as $\aleph_n \subseteq \aleph_{n-1}$.
Thus

$$\aleph_{n+1} \subseteq \aleph_n$$

and

$$Q(\aleph_{n+1}) = Q(Conv(P(\aleph_n))) \subseteq Conv(QP(\aleph_n)) \subseteq ConvP(\aleph_n) = \aleph_{n+1}.$$

Thus

$$Q(\aleph_{n+1}) \subseteq \aleph_{n+1}$$

and this implies that

$$\aleph_0 \supset \aleph_1 \supset \aleph_2 \ldots \quad .$$

If there exists $n \geq 0$ s.t

$$m^*(\aleph_n) = 0$$

then $\aleph_n$ is relatively compact. Since $P(\aleph_n) \subseteq ConvP(\aleph_n) = \aleph_{n+1} \subseteq \aleph_n$, so by S. f. t, P has a fixed point.
Now, we assume that $m^*(\aleph_n) \neq 0, n \geq 0$.
Then by assumption,

$$\begin{aligned} m^*(\aleph_{n+1}) &= m^*(Conv(P(\aleph_n))) \\ &= m^*(P(\aleph_n)) \\ &\leq \psi_1(m^*(\aleph_n)) \\ &= \psi_1(m^*(ConvP(\aleph_{n-1}))) \\ &\leq \psi_1^2(m^*(\aleph_{n-1})) \\ &\quad \cdots\cdots \\ &\leq \psi_1^n(m^*(\aleph_0)). \end{aligned}$$

Since $\psi_1 : \mathbb{R}_+ \to \mathbb{R}_+$ is a non decreasing and $\lim_{n\to\infty} \psi_1^n(x) = 0$.
If we consider

$$r = \lim_{n\to\infty} m^*(\aleph_{n+1}) \leq \lim_{n\to\infty} m^*\psi_1^n(m^*(\aleph_0)) = \lim_{n\to\infty} \psi_1^n(x) = 0,$$

for

$$x \geq 0, x = m^*(\aleph_0),$$

then,

$$m^*(\aleph_n) \to 0$$

as $n \to \infty$.
Since $\aleph_{n+1} \subseteq \aleph_n$ and

$$P(\aleph_n) \subseteq \aleph_n, n \geq 1,$$

so by condition 6 in the definition of m^*, we have $\aleph_\infty = \cap_{n=1}^{\infty} \aleph_n$ is a non empty closed and convex subset in $\aleph$. Moreover $\aleph_\infty$ is invariant under the operator P and belongs to $Kerm^*$ is relatively compact. So, by S. f. t, P has a fixed point in $\aleph$.
Now, Suppose

$$G_P = \{t \in \aleph : P(t) = t\}$$

then clearly G_P is closed by continuity of P and we have

$$Q(G_P) \subseteq G_P$$

then $Q(t)$ is a fixed point of P for any $t \in G_P$ and

$$m^*(G_P) = m^*(P(G_P)) \leq \psi_1(m^*(G_P)) \leq m^*(G_P).$$

Therefore, $m^*(G_P) = 0$ and G_P is compact then by S. f. t, Q has a fixed point and set

$$G_Q = \{t \in \aleph : Q(t) = t\}$$

is closed by continuity of Q.
Also $Q(G_P) \subseteq G_P$ then by S. f. t, $P(t)$ is a fixed point of Q, for all $t \in G_Q$.
Since

$$G_P \cap G_Q \subseteq G_P \subseteq \aleph$$

is compact subset and

$$P, Q : G_P \cap G_Q \to G_P \cap G_Q$$

are continuous self maps so by S. f. t, P & Q have a common fixed point in $\aleph$. $\square$

3 Common Fixed Point Theorem For Three Continuous Commutative Operators On Banach Algebra T

Theorem 3.1. *Let $\aleph$ be a non empty bounded, closed and convex subset of the Banach algebra T and operators $P_1, P_2, P_3, Q_1, Q_2, Q_3$ transforms $\aleph$ into T are continuous and mutually commutative such that*

$$P_1(\aleph), P_2(\aleph), P_3(\aleph), Q_1(\aleph), Q_2(\aleph), Q_3(\aleph)$$

are bounded. Suppose $P = P_1Q_1$, $Q = P_2Q_2$ and $R = P_3Q_3$ be three continuous and commuting mappings from $\aleph$ into $\aleph$ such that Q and R are linear and

$$Q(P(E)) \subseteq P(E), R(P(E)) \subseteq P(E), E \subseteq \aleph.$$

If the operators $P_1, P_2, P_3, Q_1, Q_2, Q_3$ on $\aleph$ satisfies the following conditions

$$m^*(P_1(E)) \leq \xi_1(m^*(E)),$$

$$m^*(Q_1(E)) \leq \xi_2(m^*(E)),$$

$$m^*(P_2(E)) \leq \xi_1'(m^*(E)),$$

$$m^*(Q_2(E)) \leq \xi_2'(m^*(E)),$$

$$m^*(P_3(E)) \leq \xi_1''(m^*(E)),$$

and

$$m^*(Q_3(E)) \leq \xi_2''(m^*(E)),$$

where E is the non empty subset of $\aleph$ and m^* is an arbitrary measure of non compactness defined on μ_T and $\xi_1, \xi_2, \xi_1', \xi_2', \xi_1'', \xi_2''$ are non decreasing functions from R_+ to R_+ s.t

$$\lim_{n\to\infty} \xi_1^n(x) = 0,$$

$$\lim_{n\to\infty} \xi_2^n(x) = 0,$$

$$\lim_{n\to\infty} \xi_1'^n(x) = 0,$$

$$\lim_{n\to\infty} \xi_2'^n(x) = 0,$$

$$\lim_{n\to\infty} \xi_1''^n(x) = 0,$$

$$\lim_{n\to\infty} \xi_2''^n(x) = 0$$

and

$$\lim_{n\to\infty} (||P_1(\aleph)||\xi_2 + ||Q_1(\aleph)||\xi_1)^n(x) = 0,$$

$$\lim_{n\to\infty} (||P_2(\aleph)||\xi_2' + ||Q_2(\aleph)||\xi_1')^n(x) = 0,$$

$$\lim_{n\to\infty} (||P_3(\aleph)||\xi_2'' + ||Q_3(\aleph)||\xi_1'')^n(x) = 0,$$

for any $x \geq 0$, then we have
1. P, Q and R have a common fixed point in $\aleph$.
2. If $Q(Conv(\aleph)) \subseteq Conv(Q(\aleph))$ then PQ, Q and R have a fixed point in $\aleph$.

Proof. Let E be a non empty subset of $\aleph$ and m^* be a measure of non compactness defined on μ_T, T is a Banach algebra, then we have

$$\begin{aligned} m^*(P(E)) &= m^*(P_1Q_1(E)) \\ &\leq m^*(P_1(E)Q_1(E)) \\ &\leq m^*(Q_1(E))||P_1(E)|| + m^*(P_1(E))||Q_1(E)|| \\ &\leq m^*(Q_1(E))||P_1(\aleph)|| + m^*(P_1(E)||Q_1(\aleph))|| \\ &\leq \xi_2(m^*(E))||P_1(\aleph)|| + \xi_1(m^*(E))||Q_1(\aleph)|| \\ &= (\xi_2)||P_1(\aleph)|| + (\xi_1)||Q_1(\aleph)||)(m^*(E)... \quad .(3.1) \end{aligned}$$

Let

$$\psi_1(x) = ((\xi_2)||P_1(\aleph)|| + (\xi_1)||Q_1(\aleph)||)(x).$$

So from equation 3.1, we have

$$m^*(P(E)) \leq \psi_1(m^*(E)).$$

Similarly $m^*(Q(E)) \leq \psi_2(m^*(E))$ and $m^*(R(E)) \leq \psi_3(m^*(E))$, where

$$\psi_2(x) = ((\xi_2')||P_2(\aleph)|| + (\xi_1')||Q_2(\aleph)||)(x)$$

and

$$\psi_3(x) = ((\xi_2'')||P_3(\aleph)|| + (\xi_1'')||Q_3(\aleph)||)(x).$$

Since $\xi_1, \xi_2, \xi_1', \xi_2', \xi_1'', \xi_2''$ are non decreasing functions from R_+ to R_+, so ψ_1, ψ_2 and ψ_3 are also non decreasing because of given conditions, so we have

$$\lim_{n\to\infty} \psi_1^n(x) = 0, \lim_{n\to\infty} \psi_2^n(x) = 0 \text{ and } \lim_{n\to\infty} \psi_3^n(x) = 0.$$

Now we can define a sequence of subsets $\{\aleph_n\}$ of T as $\aleph = \aleph_0$, $\aleph_n = convP\aleph_{n-1}, n \geq 1$, then $\aleph_n \subseteq \aleph_{n-1}$ and $Q(\aleph_n) \subset \aleph_n$ ....(3.2)
Clearly $\aleph_1 \subset \aleph_0$ and $Q(\aleph_1) \subseteq Conv(QP(\aleph_0)) = Conv(PQ(\aleph_0)) \subseteq Conv(P(\aleph_0)) = \aleph_1$ as $PQ = QP$.
Therefore, $Q(\aleph_1) \subseteq \aleph_1$ and so 3.2 holds for $n = 1$. Suppose it is true for $n \geq 1$, then $(\aleph_{n+1}) = Conv(P(\aleph_n)) \subseteq Conv(P(\aleph_{n-1})) = \aleph_n$, because $\aleph_n \subseteq \aleph_{n-1}$.
Thus $\aleph_{n+1} \subseteq \aleph_n$ and

$$Q(\aleph_{n+1}) = Q(Conv(P(\aleph_n))) \subseteq Conv(QP(\aleph_n)) \subseteq ConvP(\aleph_n) = \aleph_{n+1}.$$

So $Q(\aleph_{n+1}) \subseteq \aleph_{n+1}$ and this implies that

$$\aleph_0 \supset \aleph_1 \supset \aleph_2 ... \quad .$$

If there exists $n \geq 0$ s.t $m^*(\aleph_n) = 0$ then $\aleph_n$ is relatively compact.
Since

$$P(\aleph_n) \subseteq ConvP(\aleph_n) = \aleph_{n+1} \subseteq \aleph_n,$$

so by S. f. t, P has a fixed point.
Now, we assume that $m^*(\aleph_n) \neq 0, n \geq 0$. Then by supposition

$$\begin{aligned}
m^*(\aleph_{n+1}) &= m^*(Conv(P(\aleph_n))) \\
&= m^*(P(\aleph_n)) \\
&\leq \psi_1(m^*(\aleph_n)) \\
&= \psi_1(m^*(ConvP(\aleph_{n-1}))) \\
&\leq \psi_1^2(m^*(\aleph_{n-1})) \\
&\quad \cdots\cdots \\
&\leq \psi_1^n(m^*(\aleph_0)).
\end{aligned}$$

Since $\psi_1 : \mathbb{R}_+ \to \mathbb{R}_+$ is a non decreasing function and $\lim_{n\to\infty} \psi_1^n(x) = 0$.
If we consider

$$r = \lim_{n\to\infty} m^*(\aleph_{n+1}) \leq \lim_{n\to\infty} \psi_1^n(m^*(\aleph_0)) = \lim_{n\to\infty} \psi_1^n(x) = 0,$$

for $x \geq 0$ and $x = m^*(\aleph_0)$, then

$$m^*(\aleph_n) \to 0 \text{ as } n \to \infty.$$

Since $\aleph_{n+1} \subseteq \aleph_n$ and $P(\aleph_n) \subseteq \aleph_n, n \geq 1$, so by condition 6 in the definition of m^*, we have $\aleph_\infty = \cap_{n=1}^{\infty} \aleph_n$ is a non empty closed and convex subset of $\aleph$. Moreover, $\aleph_\infty$ is invariant under the operator P and belongs to Ker m^* is relatively compact. Thus, by S. f. t, P has a fixed point in $\aleph$. Similarly, we have

$$Q(\aleph_n) \subseteq \aleph_n,$$

$$R(\aleph_n) \subseteq \aleph_n$$

and

$$Q(\aleph_\infty) \subseteq \aleph_\infty,$$

$$R(\aleph_\infty) \subseteq \aleph_\infty.$$

Thus, by S. f. t, Q, R have fixed point in $\aleph$. Now consider $G_R = \{t \in \aleph : R(t) = t\}$. Then clearly G_R is closed, convex and bounded by continuity of R is a subset of $\aleph$ such that $Q(G_R) \subseteq G_R$ and $R(G_R) \subseteq G_R$, we have

$$m^*(G_R) = m^*(R(G_R)) \leq \psi_3(m^*(G_R)) \leq m^*(G_R).$$

Therefore, $m^*(G_R) = 0$ and G_R is compact. Thus by S. f. t Q has a fixed point in G_R and so Q and R have a common fixed point.
Next consider $G = \{t \in \aleph : Q(t) = R(t) = t\}$ is closed and convex subset of $\aleph$ and $P(G) \subseteq G$, so P have fixed point in G. Hence P, Q, R have a common fixed point in $\aleph$.
(ii) Given $Q(Conv(\aleph)) \subseteq Conv(Q(\aleph))$, Consider the sequence $\{\aleph_n\}$ such that

$$\aleph = \aleph_0, \aleph_n = conv(QP(\aleph_{n-1})), n \geq 1.$$

We have clearly

$$\aleph_{n+1} = (Conv(QP(\aleph_n))) \subseteq Conv(QP(\aleph_{n-1})) \subseteq Conv PQ(\aleph_{n-1}) = \aleph_n$$

and so

$$\aleph_n \subseteq \aleph_{n-1}, \forall n \geq 1.$$

Thus it follows that

$$R(\aleph_n) \subseteq R(\aleph_{n-1}), \forall n \geq 1.$$

So $\{m^*(R(\aleph_n))\}$ is a positive non increasing sequence of real numbers such that

$$\lim_{n\to\infty} m^*(R(\aleph_n)) = 0.$$

Now choose $\aleph'_n = \overline{R(\aleph_n)}$ such that

$$m^*(\aleph'_n) = m^*(R(\aleph_n)) = m^*(\overline{R(\aleph_n)}) \text{ by definition of } m^*.$$

So

$$\lim_{n\to\infty} m^*(\aleph'_n) = 0.$$

Since $\{\aleph'_n\}$ is a nested sequence and $\aleph'_{n+1} \subseteq \aleph'_n, \forall n \geq 1$, we have $\aleph'_\infty = \cap_{n=1}^{\infty}\aleph'_n$ is non empty and $m^*(\aleph'_\infty) \leq m^*(\aleph'_n), \forall n \geq 1$. Thus $\lim_{n\to\infty} m^*(\aleph'_\infty) = 0$ and $\overline{\aleph'_\infty} = \aleph'_\infty$ is compact and convex as R is continuous linear map. Now

$$(PQ = QP)(\aleph_n) \subseteq \aleph_n$$

and

$$(QP)(\aleph_n) \subseteq Conv(QP)\aleph_n \subseteq Conv(QP)(\aleph_{n-1}) = \aleph_n, \quad n = 1, 2, 3... \ .$$

Thus, $(QP)(\aleph_n) \subseteq \aleph_n$, $\forall n \in \mathbb{N}$ and

$$(Q)(\aleph_n) \subseteq \aleph_n,$$

$$(R)(\aleph_n) \subseteq \aleph_n,$$

$$(QP)(\aleph'_n) = QP(\overline{R(\aleph_n)}) \subseteq \overline{QP(R\aleph_n}) \subseteq \overline{R(QP(\aleph_n))} \subseteq \overline{R(\aleph_n)} = \aleph'_n.$$

Hence,

$$(QP)(\aleph'_n) \subseteq \aleph'_n (Q)(\aleph'_n) \subseteq \aleph'_n (R)(\aleph'_n) \subseteq \aleph'_n$$

and

$$(QP)(\aleph'_\infty) \subseteq \aleph'_\infty, (Q)(\aleph'_\infty) \subseteq \aleph'_\infty, (R)(\aleph'_\infty) \subseteq \aleph'_\infty.$$

So, by S. f. t, QP, Q, R have a fixed point in $\aleph$. $\square$

Theorem 3.2. *Let $\aleph$ be a non empty bounded, closed and convex subset of the Banach algebra T and operators $P_1, P_2, P_3, Q_1, Q_2, Q_3$ transforms $\aleph$ into T are continuous such that*

$$P_1(\aleph), P_2(\aleph), P_3(\aleph), Q_1(\aleph), Q_2(\aleph), Q_3(\aleph)$$

are bounded. Suppose $P = P_1Q_1$, $Q = P_2Q_2$ and $R = P_3Q_3$ be three continuous mappings from $\aleph$ into $\aleph$ such that Q and R are linear with

$$Q(P(E)) \subseteq P(E)$$

and

$$R(P(E)) \subseteq P(E), \quad \text{for} \quad E \subseteq \aleph.$$

If the operators P_1, P_2, P_3, Q_1, Q_2, Q_3 on $\aleph$ satisfies the following conditions

$$m^*(P_1(E)) \leq \xi_1(m^*(E)),$$
$$m^*(Q_1(E)) \leq \xi_2(m^*(E)),$$
$$m^*(P_2(E)) \leq \xi'_1(m^*(E)),$$
$$m^*(Q_2(E)) \leq \xi'_2(m^*(E)),$$
$$m^*(P_3(E)) \leq \xi''_1(m^*(E)),$$
$$m^*(Q_3(E)) \leq \xi''_2(m^*(E)),$$

where m^* is an arbitrary measure of non compactness defined on μ_T and $\xi_1, \xi_2, \xi'_1, \xi'_2, \xi''_1, \xi''_2$ are non decreasing functions from R_+ to R_+ s.t

$$\lim_{n\to\infty} \xi_1^n(x) = 0,$$
$$\lim_{n\to\infty} \xi_2^n(x) = 0,$$
$$\lim_{n\to\infty} \xi_1'^n(x) = 0,$$
$$\lim_{n\to\infty} \xi_2'^n(x) = 0,$$

$$\lim_{n\to\infty} \xi_1''^{n}(x) = 0,$$
$$\lim_{n\to\infty} \xi_2''^{n}(x) = 0$$

and

$$\lim_{n\to\infty} (||P_1(\aleph)||\xi_2 + ||Q_1(\aleph)||\xi_1)^n(x) = 0,$$
$$\lim_{n\to\infty} (||P_2(\aleph)||\xi_2' + ||Q_2(\aleph)||\xi_1')^n(x) = 0,$$
$$\lim_{n\to\infty} (||P_3(\aleph)||\xi_2'' + ||Q_3(\aleph)||\xi_1'')^n(x) = 0,$$

for any $x \geq 0$*, then we have*
1. P*,* Q *and* R *have a common fixed point in* $\aleph$*.*
2. If $Q(Conv(\aleph)) \subseteq Conv(Q(\aleph))$ *then* PQ*,* Q *and* R *have a fixed point in* $\aleph$*.*

Proof. The proof is straightforward, so we omit the details. □

Conclusion: In this paper we proved fixed point theorem for two and three commuting and non commuting mapping in Banach algebra using measure of non compactness. In future we will use the results for Meir Keller two and three Operators to prove common fixed point theorem.

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Sanjeev Verma
School of Mathematics
Research Scholar
Shri Mata Vaishno Devi University
Katra, J&K, India.
E-mail: vsanjev28@gmail.com

Kuldip Raj
School of Mathematics
Assistant Professor of Mathematics
Shri Mata Vaishno Devi University
Katra, J&K, India.
E-mail: kuldipraj68@gmail.com

Sunil Kumar Sharma
School of Mathematics
Assistant Professor of Mathematics
Shri Mata Vaishno Devi University
Katra, J&K, India.
E-mail: sunilksharma42@gmail.com