Journal of Mathematical Extension
Vol. 17, No. 10, (2023) (2)1-17
URL: https://doi.org/10.30495/JME.2023.2546
ISSN: 1735-8299
Original Research Paper

Generalized Weakly Symmetric Sasakian Manifolds

V. Pirhadi*
University of Kashan

G. Fasihi Ramandi
Imam Khomeini International University

S. Azami
Imam Khomeini International University

Abstract. In this parer, we give a necessary condition for Sasakian manifolds to be generalized weakly symmetric. We prove the odd-dimensional spheres are the only generalized weakly symmetric Sasakian manifolds. Then, we show that generalized weakly Ricci-symmetric Sasakian manifolds are Einstein. Thereafter, we define the sense of weakly parallel Riemannian submanifolds and show that every weakly parallel invariant submanifold of a Sasakian manifold is totally geodesic. Finally, we provide some examples which verify our main results.

AMS Subject Classification: 53D15; 53C25
Keywords and Phrases: Sasakian manifolds, generalized weakly symmetric manifolds, generalized weakly Ricci-symmetric manifolds, weakly parallel invariant submanifolds

Received: October 2022; Accepted: December 2023
*Corresponding Author

1 Introduction

In [1], E. Cartan defined the sense of locally symmetric and semi symmetric manifolds. The Riemannian manifold (M, g) is said to be locally symmetric if it satisfies $\nabla R = 0$ and is called semi-symmetric if $R.R = 0$ in which R is the curvature tensor of (M, g). Also, a Riemannian manifold with $\nabla S = 0$ is called Ricci-symmetric where S denotes the Ricci tensor of g. Thereafter, Z. I. Szabó [2] considered the semi-symmetric Riemannian manifolds and showed that locally symmetric Riemannian manifolds are semi-symmetric. Note that the converse fails in general. Moreover, M. C. Chaki extended the sense of locally symmetric Riemannian manifolds to pseudo symmetric manifolds. In [3], R. N. Sen and M. C. Chaki proved that the curvature tensor of a conformally flat Riemannian manifold with some additional conditions satisfies

$$\begin{aligned}(\nabla_W R)(X, Y)Z &= 2A(W)R(X, Y)Z + A(X)R(W, Y)Z \\ &+ A(Y)R(X, W)Z + A(Z)R(X, Y)W + g(R(X, Y)Z, W)\rho, \quad (1)\end{aligned}$$

where ∇ denotes the Levi-Civita connection of g and $\rho = \sharp A$ is a non-zero vector field. A non-flat Riemannian manifold with (1) is called pseudo-symmetric. M. C. Chaki [4] also defined the sense of pseudo Ricci-symmetric manifolds. A non-flat Riemannian manifold whose Ricci tensor satisfies the equation

$$(\nabla_X S)(Y, Z) = 2\alpha(X)S(Y, Z) + \alpha(Y)S(X, Z) + \alpha(Z)S(X, Y),$$

is called pseudo Ricci-symmetric. In [5], M. Tarafdar proved that pseudo symmetric and pseudo-Ricci symmetric Sasakian manifolds are locally symmetric and Ricci-symmetric, respectively. Thus, there exist no proper pseudo symmetric and pseudo-Ricci symmetric Sasakian manifolds.
Thereafter, L. Tamássy and T. Q. Binh [6] introduced the notions of weakly symmetric and weakly Ricci-symmetric manifolds and showed that weakly symmetric and weakly Ricci-symmetric Sasakian manifolds must satisfy $A + B + D = 0$. They [6] also defined the notion of weakly φ-symmetric manifolds and proved a similar result for weakly φ-symmetric Sasakian manifolds. The non-flat almost contact metric

manifold $(M, \varphi, \xi, \eta, g)$ whose the curvature tensor satisfies the equation

$$\begin{aligned}(\nabla_W R)(X,Y)Z &= A(W)R(X,Y)Z + B(X)R(W,Y)Z \\ &+ B(Y)R(X,W)Z + D(Z)R(X,Y)W + g(R(X,Y)Z,W)\rho, \quad (2)\end{aligned}$$

is called weakly symmetric and is called weakly φ-symmetric if

$$\begin{aligned}\varphi^2(\nabla_W R)(X,Y)Z &= A(W)R(X,Y)Z + B(X)R(W,Y)Z \\ &+ B(Y)R(X,W)Z + D(Z)R(X,Y)W + g(R(X,Y)Z,W)\rho, \quad (3)\end{aligned}$$

in which A, B and D are smooth 1-forms and $\rho = \sharp D$. Further, a non-flat Riemannian manifold which satisfies:

$$(\nabla_X S)(Y,Z) = A(X)S(Y,Z) + B(Y)S(X,Z) + D(Z)S(X,Y),$$

is called weakly Ricci-symmetric. Inspired by this, R. S. D. Dubey [7] defined the sense of generalized weakly symmetric manifolds. A non-flat Riemannian manifold which admits non-zero 1-forms A_i, B_i and D_i such that

$$\begin{aligned}(\nabla_W R)(X,Y)Z &= A_1(W)R(X,Y)Z + B_1(X)R(W,Y)Z \\ &+ B_1(Y)R(X,W)Z + D_1(Z)R(X,Y)W + g(R(X,Y)Z,W)\rho_1 \\ &+ A_2(W)G(X,Y)Z + B_2(X)G(W,Y)Z + B_2(Y)G(X,W)Z \\ &+ D_2(Z)G(X,Y)W + g(G(X,Y)Z,W)\rho_2, \quad (4)\end{aligned}$$

is called generalized weakly symmetric in which $\rho_i := \sharp D_i$ for $i = 1, 2$ and $G(X,Y)Z$ is given by

$$G(X,Y)Z := g(Y,Z)X - g(X,Z)Y.$$

Similarly, a generalized weakly Ricci-symmetric manifold defines as follows

$$\begin{aligned}(\nabla_X S)(Y,Z) &= A_1(X)S(Y,Z) + B_1(Y)S(X,Z) + D_1(Z)S(X,Y) \\ &+ A_2(X)g(Y,Z) + B_2(Y)g(X,Z) + D_2(Z)g(X,Y). \quad (5)\end{aligned}$$

In [8], the first author presented an equation for the covariant derivative of the curvature tensor of Kenmotsu manifolds. We also gave a necessary condition for Kenmotsu manifolds to be generalized weakly symmetric [9]. Further, weakly ϕ-symmetric and weakly ϕ-Ricci symmetric Kenmotsu manifolds have been studied in [15] by S. K. Hui. He showed that theses manifolds are η-Einstein. He also studied ϕ-pseudo symmetric and ϕ-pseudo Ricci symmetric on Kenmotsu and para-Sasakian manifolds and obtained some interesting result [16]-[18].

Motivated by these works, we prove the odd dimensional spheres are the only generalized weakly symmetric Sasakian manifolds. Next, we show that generalized weakly Ricci-symmetric Sasakian manifolds are Ricci symmetric. Hence, generalized weakly Ricci-symmetric Sasakian manifolds are Einstein. Then, we introduce the notion of weakly parallel invariant submanifolds of Riemannian manifolds and show that every weakly parallel invariant submanifold of a Sasakian manifold is totally geodesic.

This paper is organized as follows: In Section 2, we prepare some definitions and basic formulas on Sasakian manifolds and invariant submanifolds of Sasakian manifolds. In Section 3, we prove that every generalized weakly symmetric Sasakian manifold is locally symmetric. In Section 4, we show that generalized weakly Ricci-symmetric Sasakian manifolds are Ricci-symmetric. In Section 5, we illustrate that every weakly parallel invariant submanifold of a Sasakian manifold is totally geodesic. Finally, In Section 6, we give some examples which verifies our results in previous sections.

2 Some Preliminaries on Sasakian Manifolds

In this section, we give some definitions and basic formulas concerning Sasakian manifolds and Riemannian submanifolds. The Riemannian manifold (M^n, g) with a $(1,1)$-tensor field φ, a vector field ξ, and a 1-form η such that

$$\varphi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1, \qquad \varphi(\xi) = 0, \qquad \eta o \varphi = 0,$$

$$g(X,Y) = g(\varphi(X), \varphi(Y)) + \eta(X)\eta(Y),$$

$$g(\varphi(X), Y) = -g(X, \varphi(Y)), \qquad g(\xi, X) = \eta(X),$$

is called an almost contact metric manifold. The almost contact metric manifold $(M^n, \varphi, \xi, \eta, g)$ which satisfies the equation

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X.$$

is called a Sasakian manifold. Sasakian manifolds are normal ($N_\varphi + 2d\eta \otimes \xi = 0$) and satisfy the following equations [10]:

$$\nabla_X \xi = -\varphi X, \quad R(X, Y)\xi = \eta(Y)X - \eta(X)Y,$$
$$S(X, \xi) = (n-1)\eta(X), \quad d\eta(X, Y) = g(X, \varphi Y) \tag{6}$$
$$S(\varphi X, \varphi Y) = S(X, Y) - (n-1)\eta(X)\eta(Y),$$
$$(\nabla_W R)(X, Y)\xi = g(X, \varphi W)Y - g(Y, \varphi W)X + R(X, Y)\varphi W, \tag{7}$$
$$(\nabla_X S)(Y, \xi) = -(n-1)g(\varphi X, Y) + S(\varphi X, Y). \tag{8}$$

The Reeb vector field ξ in Sasakian manifolds is a Killing vector field. Hence, Sasakian manifolds are K-contact. The converse is true only in dimension 3. Suppose that π is a 2-plane of T_pM which is spanned by u and v, then the sectional curvature of (M, g) defines as follows:

$$K(\pi) = \frac{g(R(u, v)v, u)}{g(u, u)g(v, v) - (g(u, v))^2}.$$

Assume that $X \in \ker(\eta)$. It is well-known that Sasakian manifolds satisfy the equation $K(\langle \xi, X \rangle) = 1$. Moreover, $K(\langle X, \varphi X \rangle)$ is called the φ-holomorphic sectional curvature of M. In [11], K. Ogiue proved that the Sasakian manifold $(M^n, \varphi, \xi, \eta, g)$ is of constant point-wise φ-holomorphic sectional curvature $H \in C^\infty(M)$ if and only if the curvature tensor of M is of the following form

$$\begin{aligned} R(X, Y)Z = \frac{H+3}{4}\{g(Y, Z)X - g(X, Z)Y\} + \frac{H-1}{4}\{\eta(X)\eta(Z)Y \\ - \eta(Y)\eta(Z)X + \eta(Y)g(X, Z)\xi - \eta(X)g(Y, Z)\xi \\ + g(X, \varphi Z)\varphi Y - g(Y, \varphi Z)\varphi X + 2g(X, \varphi Y)\varphi Z\}. \end{aligned}$$

He also showed that H is a constant function if $n > 3$. Let $M \subseteq \overline{M}$ be an isometrically immersed submanifold of $(\overline{M}, g)$. The tangent bundle $T\overline{M}$ as a vector bundle on the base manifold M decomposes as follows

$$T\overline{M} = TM \oplus T^\perp M,$$

where $T^{\perp}M$ is called the normal vector bundle and is the complementary of TM in $T\overline{M}$. According to the above decomposition, the Gauss-Weingarten formulas are written as follows:

$$\overline{\nabla}_X Y = h(X,Y) + \nabla_X Y,$$

$$\overline{\nabla}_X N = -A_N(X) + \nabla^{\perp}_X Y,$$

in which $X, Y \in \Gamma(TM)$, $N \in \Gamma(T^{\perp}M)$ and $\nabla^{\perp}$ is the induced connection on the normal bundle $T^{\perp}M$. In the above equations h and A are called the second fundamental form and the shape operator of the submanifold M, respectively and are related by $g(h(X,Y),N) = g(A_N(X),Y)$. The submanifold M of the Riemannian manifold $(\overline{M}, g)$ with $h = 0$ is called totally geodesic. Further, the submanifold M^n of $(\overline{M}, g)$ whose the second fundamental form satisfies the equation:

$$h(X,Y) = g(X,Y)H,$$

is called totally umbilical where H is the mean curvature vector field of M^n and is given as follows:

$$H := \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i).$$

In the case where $H = 0$, the submanifold M is called minimal. Also, the submanifold M of the almost contact metric manifold $(\overline{M}, \varphi, \xi, \eta, g)$ is called invariant if $\xi \in \Gamma(TM)$ and $\varphi(TM) \subseteq \Gamma(TM)$. It is well-known that invariant submanifolds of Sasakian manifolds are minimal and satisfy the following equations (cf. [12]):

$$h(\varphi X, Y) = \varphi h(X,Y), \qquad A_{\varphi(N)}(X) = \varphi A_N(X) = -A_N(\varphi X), \quad (9)$$

$$h(\xi, X) = A_N(\xi) = 0, \quad (10)$$

3 Generalized Weakly Symmetric Sasakian Manifolds

It is well-known that the odd-dimensional spheres are Sasakian. In this section, we show that the odd-dimensional spheres are the only generalized weakly symmetric Sasakian manifolds. Hence, generalized weakly symmetric Sasakian manifolds are Einstein and have constant scalar curvature.

Theorem 3.1. *The odd-dimensional spheres are the only generalized weakly symmetric Sasakian manifolds. Moreover, the associated* 1*-forms* A_i, B_i *and* D_i *satisfy* $A_1 + A_2 = 0$*,* $B_1 + B_2 = 0$ *and* $D_1 + D_2 = 0$*.*

Proof. Let $(\overline{M}, \varphi, \xi, \eta, g)$ be a generalized weakly symmetric Sasakian manifold. Setting $Z = \xi$ in (4) and using (7), we get

$$\begin{aligned}
&g(Y, -\varphi W)X - g(X, -\varphi W)Y + R(X, Y)\varphi W = [A_1(W) + A_2(W)]\\
&\{\eta(Y)X - \eta(X)Y\} + [B_1(X) + B_2(X)]\{\eta(Y)W - \eta(W)Y\}\\
&+ [B_1(Y) + B_2(Y)]\{\eta(W)X - \eta(X)W\} + D_1(\xi)R(X, Y)W\\
&+ D_2(\xi)\{g(Y, W)X - g(X, W)Y\} + g(\eta(Y)X - \eta(X)Y, W)\rho_1\\
&+ g(\eta(Y)X - \eta(X)Y, W)\rho_2.
\end{aligned}$$

Let $X \in \ker(\eta)$ and putting $Y = \varphi X$ and $W = X$ in the above equation, we find that

$$\begin{aligned}
-g(\varphi X, \varphi X)X + R(X, \varphi X)\varphi X &= D_1(\xi)R(X, \varphi X)X\\
&\quad - D_2(\xi)g(X, X)\varphi X.
\end{aligned}$$

Taking inner product with X in the above equation, we observe that

$$g(R(X, \varphi X)\varphi X, X) = g(\varphi X, \varphi X)g(X, X).$$

Thus, M is of constant point-wise φ-holomorphic sectional curvature 1. Hence, the curvature tensor of M is of the following form

$$R(X, Y)Z = g(Y, Z)X - g(X, Z)Y.$$

Thus, M is necessarily an odd-dimensional sphere. Further

$$S(X, Y) = (n - 1)g(X, Y), \qquad \nabla R = 0.$$

Now, looking once again at (4) we see

$$\begin{aligned}
0 = {}&[A_1(W) + A_2(W)]\{g(Y, Z)X - g(X, Z)Y\} + [B_1(X) + B_2(X)]\\
&\{g(Y, Z)W - g(W, Z)Y\} + [B_1(Y) + B_2(Y)]\{g(Z, W)X - g(Z, X)W\}\\
&+ [D_1(Z) + D_2(Z)]\{g(W, Y)X - g(W, X)Y\}\\
&+ \{g(Z, Y)g(X, W) - g(Z, X)g(Y, W)\}[\rho_1 + \rho_2]. \qquad (11)
\end{aligned}$$

Suppose that $\{\xi, e_1, ..., e_m, \varphi e_1, ..., \varphi e_m\}$ is a φ-basis for T_pM where $\dim M = (2m+1)$. Taking in (11) $X = \varphi e_i$, $Y = Z = e_i$ and $W = \xi$ we obtain

$$[A_1(\xi) + A_2(\xi)]\,\varphi e_i + [B_1(\varphi e_i) + B_2(\varphi e_i)]\,\xi = 0,$$

which implies $(A_1 + A_2)(\xi) = (B_1 + B_2)(\varphi e_i) = 0$. Next, setting $X = \xi$, $Y = Z = e_i$ and $W = \varphi e_i$ we get

$$[A_1(\varphi e_i) + A_2(\varphi e_i)]\,\xi + [B_1(\xi) + B_2(\xi)]\,\varphi e_i = 0,$$

which proves that $(A_1 + A_2)(\varphi e_i) = (B_1 + B_2)(\xi) = 0$. This together with $(A_1+A_2)(\xi) = (B_1+B_2)(\varphi e_i) = 0$ assert that 1-forms A_1+A_2 and $B_1 + B_2$ are both identically zero. Finally, putting $\{X = Z = \varphi e_i, W = Y = e_i\}$ and $\{X = Z = e_i, W = Y = \varphi e_i\}$ in (11), respectively, we conclude that

$$\rho_1 + \rho_2 = [D_1(\varphi e_i) + D_2(\varphi e_i)]\,\varphi e_i,$$
$$\rho_1 + \rho_2 = -\,[D_1(e_i) + D_2(e_i)]\,e_i,$$

which results the 1-form $D_1 + D_2$ is also identically zero and completes the proof. □

In [6], L. Tamassy and T. Q. Binh showed that weakly symmetric and weakly φ-symmetric Sasakian manifolds must satisfy $A+B+D = 0$. Applying the above theorem, we extend the Tamassy and Binh's results [6] as follows.

Corollary 3.2. *Weakly symmetric Sasakian manifolds are Einstein and locally symmetric. Also, 1-forms A, B and D satisfy $A = B = D = 0$.*

Corollary 3.3. *Every weakly φ-symmetric Sasakian manifold is Einstein and locally symmetric. Meanwhile, the associated 1-forms A, B and D satisfy $A = B = D = 0$.*

Proof. Let $(M, \varphi, \xi, \eta, g)$ be a weakly φ-symmetric Sasakian manifold. Setting $Z = \xi$ in (3), it follows

$$\begin{aligned}-(\nabla_W R)(X,Y)\xi + g((\nabla_W R)(X,Y)\xi, \xi)\xi &= A(W)R(X,Y)\xi \\ &+ B(X)R(W,Y)\xi + B(Y)R(X,W)\xi \\ &+ D(\xi)R(X,Y)W + g(R(X,Y)\xi, W)\rho.\end{aligned}$$

Using $g((\nabla_W R)(X,Y)\xi,\xi) = 0$. A similar argument as mentioned in the proof of Theorem 1 proves that M is Einstein and 1-forms A, B and D satisfy $A = B = D = 0$. $\square$

4 Generalized Weakly Ricci Symmetric Sasakian Manifolds

In this section, we show that the Einstein Sasakian manifolds are the only Sasakian manifolds whose Ricci tensors satisfy (5). It is well-known that the covariant derivative of the Ricci tensor of the Sasakian manifold $(M^n, \varphi, \xi, \eta, g)$ is as follows (see [10] page 284):

$$\begin{aligned}(\nabla_Z S)(X,Y) = &(\nabla_X S)(Y,Z) + (\nabla_{\varphi Y} S)(\varphi X, Z) - \eta(X)S(\varphi Y, Z)\\ &- 2\eta(Y)S(\varphi X, Z) + (n-1)\eta(X)g(\varphi Y, Z)\\ &+ 2(n-1)\eta(Y)g(\varphi X, Z). \end{aligned} \tag{12}$$

This enable us to state the following proposition.

Proposition 4.1. *Let $(M^n, \varphi, \xi, \eta, g)$ be a Sasakian manifold. Then $(\nabla_\xi S)(X,Y) = 0$ for all $X, Y \in \Gamma(TM)$.*

Proof. Setting $Z = \xi$ in (12) and applying (6) and (8), we conclude that

$$\begin{aligned}(\nabla_\xi S)(X,Y) = &(n-1)g(Y, -\varphi X) + S(Y, \varphi X) - S(\nabla_{\varphi Y}\varphi X, \xi)\\ &+ S(\varphi X, \varphi^2 Y) = -(n-1)g(Y, \varphi X) + S(Y, \varphi X)\\ &- (n-1)g(\nabla_{\varphi Y}\varphi X, \xi) - S(\varphi X, Y) = -(n-1)g(Y, \varphi X)\\ &- (n-1)g((\nabla_{\varphi Y}\varphi)X, \xi) = -(n-1)g(Y, \varphi X)\\ &- (n-1)g(\varphi Y, X) = 0,\end{aligned}$$

which is the desired result. $\square$

Using the above proposition, we have the following theorem.

Theorem 4.2. *Every generalized weakly Ricci-symmetric Sasakian manifold is Ricci-symetric. Further, the 1-forms A_i, B_i and D_i satisfy $(n-1)A_1 + A_2 = 0$, $(n-1)B_1 + B_2 = 0$ and $(n-1)D_1 + D_2 = 0$.*

Proof. Let $(M^{2m+1}, \varphi, \xi, \eta, g)$ be a generalized weakly Ricci-symmetric Sasakian manifold. Suppose $\{\xi, e_1, ..., e_m, e_{m+1} := \varphi e_1, ..., e_{2m} := \varphi e_m\}$ is a φ-basis for T_pM. Taking $\{X = \xi, Y = e_i, Z = \xi\}$, $\{X = \xi, Y = \xi, Z = e_i\}$ and $\{X = e_i, Y = \xi, Z = \xi\}$ in (5), respectively, we obtain

$$(n-1)A_1(e_i) + A_2(e_i) = 0, \;\; (n-1)B_1(e_i) + B_2(e_i) = 0,$$
$$(n-1)D_1(e_i) + D_2(e_i) = 0. \tag{13}$$

Next, setting $\{X = e_i, Y = \varphi e_i, Z = \xi\}$ and $\{X = \varphi e_i, Y = e_i, Z = \xi\}$ in (5), respectively, and taking account of (8), we find that

$$\begin{cases} -(n-1) + S(e_i, e_i) = D_1(\xi)S(e_i, \varphi e_i), \\ (n-1) - S(e_i, e_i) = D_1(\xi)S(e_i, \varphi e_i), \end{cases}$$

which demonstrate that $S(e_i, e_i) = (n-1)$. On the other hand, taking $\{X = Y = e_i, Z = \xi\}$ in (5) results

$$S(\varphi e_i, e_i) = (n-1)D_1(\xi) + D_2(\xi). \tag{14}$$

Similar computations as above show that

$$S(\varphi e_i, e_i) = (n-1)B_1(\xi) + B_2(\xi). \tag{15}$$

Further, inserting $\{X = \xi, Y = Z = e_i\}$, we derive

$$(n-1)A_1(\xi) + A_2(\xi) = 0. \tag{16}$$

Putting $X = Y = Z = \xi$ in (5) yields

$$0 = [(n-1)A_1(\xi) + A_2(\xi)] + [(n-1)B_1(\xi) + B_2(\xi)] + [(n-1)D_1(\xi) + D_2(\xi)]. \tag{17}$$

In view of (14)-(17), we get $(n-1)A_1(\xi)+A_2(\xi) = (n-1)B_1(\xi)+B_2(\xi) = (n-1)D_1(\xi) + D_2(\xi) = 0$. This together with (13) asserts that 1-forms $(n-1)A_1 + A_2$, $(n-1)B_1 + B_2$ and $(n-1)D_1 + D_2$ are all identically zero and prove the theorem. □

It is well known that Ricci symmetric Sasakian manifolds are Einstein [10]. Thus, we can gave the following.

Corollary 4.3. *Generalized weakly Ricci-symmetric Sasakian manifolds are Einstein and have constant scalar curvature.*

5 Weakly Parallel Invariant Submanifolds

In this section, we first define the sense of weakly parallel submanifolds of Riemannian manifolds and prove that every weakly parallel invariant submanifold of a Sasakian manifold is totally geodesic.

Definition 5.1. The submanifold M of the Riemannian manifold $(\overline{M}, g)$ whose the second fundamental form h satisfies:

$$(\overline{\nabla}_X h)(Y,Z) = A(X)h(Y,Z) + B(Y)h(X,Z) + D(Z)h(X,Y), \quad (18)$$

is called weakly parallel in which A, B and D are smooth 1-forms on M.

Suppose that M is an invariant submanifold of the Sasakian manifold $(\overline{M}, \varphi, \xi, \eta, g)$. It immediately follows from (9) and (10) that

$$\begin{aligned}(\overline{\nabla}_X \varphi)(N) &= \overline{\nabla}_X \varphi(N) - \varphi(\overline{\nabla}_X N) \\ &= \nabla^{\perp}_X \varphi(N) - A_{\varphi(N)}(X) - \varphi(\nabla^{\perp}_X N - A_N X) \\ &= \nabla^{\perp}_X \varphi(N) - \varphi(\nabla^{\perp}_X N) = (\nabla^{\perp}_X \varphi)(N), \end{aligned} \quad (19)$$

which leads to the following theorem.

Theorem 5.2. *Weakly parallel invariant submanifolds of Sasakian manifolds are totally geodesic.*

Proof. Let M be a weakly parallel invariant submanifold of the Sasakian manifold $(\overline{M}, \varphi, \xi, \eta, g)$. Taking $N = h(Y,Z)$ and using (19), we get

$$\begin{aligned}(\nabla^{\perp}_X \varphi)h(Y,Z) = (\overline{\nabla}_X \varphi)h(Y,Z) = g(X, h(Y,Z))\xi \\ - \eta(h(Y,Z))X = 0.\end{aligned}$$

This together with the fact that M is also a Sasakian manifold result

$$\begin{aligned}(\overline{\nabla}_X h)(\varphi Y, Z) &= \nabla^{\perp}_X h(\varphi Y, Z) - h(\nabla_X \varphi Y, Z) \\ &- h(\varphi Y, \nabla_X Z) = \nabla^{\perp}_X \varphi h(Y,Z) - h((\nabla_X \varphi)Y \\ &+ \varphi(\nabla_X Y), Z) - h(\varphi Y, \nabla_X Z) = (\nabla^{\perp}_X \varphi)h(Y,Z) \\ &+ \varphi(\nabla^{\perp}_X h(Y,Z)) - h(g(X,Y)\xi - \eta(Y)X + \varphi(\nabla_X Y), Z) \\ &- h(\varphi Y, \nabla_X Z) = \varphi(\overline{\nabla}_X h)(Y,Z) + \eta(Y)h(X,Z). \end{aligned} \quad (20)$$

From this we conclude that

$$A(X)h(\varphi Y, Z) + B(\varphi Y)h(X, Z) + D(Z)h(X, \varphi Y) = A(X)\varphi h(Y, Z) \\ + B(Y)\varphi h(X, Z) + D(Z)\varphi h(X, Y) + \eta(Y)h(X, Z),$$

which gives

$$B(\varphi Y)h(X, Z) = B(Y)\varphi h(X, Z) + \eta(Y)h(X, Z). \tag{21}$$

Putting $Y = \xi$ in the above equation, we find that $B(\xi) = 0$. Next, substituting Y with φY in (21) we obtain $B(\varphi Y) = 0$ which proves the 1-form B is identically zero. Replacing Y with Z in (20). Similar computations illustrate that the 1-form D is also identically zero. Therefore, the Equation (18) can be written as follows:

$$(\overline{\nabla}_X h)(Y, Z) = A(X)h(Y, Z). \tag{22}$$

Finally, setting $Y = \xi$ in the equation above yields

$$h(\varphi X, Z) = 0, \tag{23}$$

which completes the proof. $\square$

6 Examples

In previous sections, we proved that generalized weakly symmetric and generalized weakly Ricci-symmetric Sasakian manufolds are Einstein and have constant scalar curvature. Now, we give an example of a non-Einstein Sasakian manifold and show that it is neither weakly symmetric nor generalized weakly symmetric.

Example 6.1. Let $M = \mathcal{R}^3$. Setting [14]

$$e_1 := 2\frac{\partial}{\partial y}, \;\; e_2 := 2(\frac{\partial}{\partial x} - y^2\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}), \;\; e_3 := 2\frac{\partial}{\partial z}. \tag{24}$$

Let g be the Riemannian metric that is given by

$$g(e_i, e_j) = \delta_{ij}, \quad i, j = 1, 2, 3,$$

where δ_{ij} denotes Kronecker's delta. Taking $\xi := e_3$ and assume that η is the 1-form dual to ξ. Suppose that φ is the $(1,1)$ tensor field which is defined by

$$\varphi(e_1) = e_2, \quad \varphi(e_2) = -e_1, \quad \varphi(e_3) = 0,$$

Clearly, $(M, \varphi, \xi, \eta, g)$ is an almost contact metric manifold. Applying (24), we compute

$$[e_1, e_2] = -4ye_1 + 2e_3, \quad [e_1, e_3] = [e_2, e_3] = 0. \tag{25}$$

These equations together with Koszul's formula yield

$$\begin{aligned}
&\nabla_{e_1} e_1 = 4ye_2, && \nabla_{e_1} e_2 = -4ye_1 + e_3, && \nabla_{e_1} e_3 = -e_2,\\
&\nabla_{e_2} e_1 = -e_3, && \nabla_{e_2} e_2 = 0, && \nabla_{e_2} e_3 = e_1,\\
&\nabla_{e_3} e_1 = -e_2, && \nabla_{e_3} e_2 = e_1, && \nabla_{e_3} e_3 = 0,
\end{aligned}$$

Using the above equations, we observe that

$$(\nabla_{e_j}\varphi)e_i = g(e_i, e_j)\xi - \eta(e_i)e_j,$$

which demonstrates $(M, \varphi, \xi, \eta, g)$ is a Sasakian manifold. Moreover, the components of the curvature tensor of M can be written as follows

$$\begin{aligned}
&R(e_1, e_2)e_1 = (3 + 24y^2)e_2, && R(e_1, e_2)e_2 = (-3 - 24y^2)e_1,\\
&R(e_1, e_2)e_3 = 0, && R(e_1, e_3)e_1 = -e_3,\\
&R(e_1, e_3)e_2 = 0, && R(e_1, e_3)e_3 = e_1\\
&R(e_2, e_3)e_1 = 0, && R(e_2, e_3)e_2 = -e_3,\\
&R(e_2, e_3)e_3 = e_2.
\end{aligned}$$

From the above equations, it follows

$$S(X, Y) = (-2 - 24y^2)g(X, Y) + (4 - 24y^2)\eta(X)\eta(Y),$$

which asserts that M is not Einstein. Next, suppose that M is weakly symmetric. Applying the above equations into (2), we get

$$\begin{aligned}
0 = (\nabla_{e_3} R)(e_2, e_3)e_1 = D(e_1)e_2, \;\; 0 &= (\nabla_{e_2} R)(e_1, e_3)e_3\\
&= A(e_2)e_1 + B(e_1)e_2,
\end{aligned}$$

$$0 = (\nabla_{e_3} R)(e_2, e_1)e_1 = A(e_3)\left[-3 - 24y^2\right] e_2 + B(e_2)e_3,$$
$$0 = (\nabla_{e_3} R)(e_1, e_3)e_2 = D(e_2)e_1,$$

which implies $A(e_2) = B(e_2) = D(e_1) = D(e_2) = 0$. Computing $(\nabla_{e_2} R)(e_2, e_1)e_1$ and using the above equations, we find that

$$\begin{aligned} e_2(-3 - 24y^2)e_2 = (\nabla_{e_2} R)(e_2, e_1)e_1 &= A(e_2)\left[-3 - 24y^2\right] e_2 \\ &+ B(e_2)\left[-3 - 24y^2\right] e_2 + D(e_1)\left[3 + 24y^2\right] e_1 \\ &+ \left[-3 - 24y^2\right] \rho = \left[-3 - 24y^2\right] \rho, \end{aligned}$$

which reduces to $e_2(-3 - 24y^2)e_2 = \left[-3 - 24y^2\right] \rho$. This together with $D(e_2) = 0$ shows that $e_2(-3 - 24y^2) = 0$ which is a contradiction and proves that M is not weakly symmetric. Moreover, similar computations as mentioned in the proof of Theorem 1 assert that $(\nabla_W R)(X, Y)Z = 0$ for all $W, X, Y, Z \in \Gamma(TM)$. On the other side, we compute

$$(\nabla_{e_2} R)(e_2, e_1)e_1 = e_2(-3 - 24y^2)e_2,$$

which is a contradiction. Thus, M is not also a generalized weakly symmetric Sasakian manifold.

To illustrate the existence of totally geodesic invariant submanifolds of Sasakian manifolds, we use the following example of 5-dimensional Sasakian manifolds as follows [13].

Example 6.2. Taking $\overline{M} = \{(x, y, z, u, v) \in \mathbf{R}^5\}$ in which (x, y, z, u, v) are the standard coordinates in $\mathbf{R}^5$. Putting

$$\begin{aligned} e_1 := \frac{\partial}{\partial x} - 2y\frac{\partial}{\partial z}, \quad e_2 := \frac{\partial}{\partial y}, \quad e_3 := \frac{\partial}{\partial z}, \\ e_4 := \frac{\partial}{\partial u} - 2v\frac{\partial}{\partial z}, \quad e_5 := \frac{\partial}{\partial v}. \end{aligned} \tag{26}$$

Suppose g is the Riemannian metric which is defined by

$$g(e_i, e_j) = \delta_{ij}, \quad i, j = 1, ..., 5.$$

where δ_{ij} denotes Kronecker's delta. Taking $\xi := e_3$ and $\eta(X) := g(X, e_3)$. Assume that φ is the $(1, 1)$ tensor field which is given by

$$\begin{aligned} &\varphi(e_1) = e_2, \quad \varphi(e_2) = -e_1, \quad \varphi(e_3) = 0, \\ &\varphi(e_4) = e_5, \quad \varphi(e_5) = -e_4. \end{aligned}$$

Obviously, $(\overline{M}, \varphi, \xi, \eta, g)$ is an almost contact metric manifold. Applying (26), we compute

$$[e_1, e_2] = 2e_3, \quad [e_4, e_5] = 2e_3,$$

and $[e_i, e_j] = 0$ for the other pairs (e_i, e_j). These equations together with Koszul's formula, yield

$$\begin{array}{lllll}
\overline{\nabla}_{e_1} e_1 = 0, & \overline{\nabla}_{e_1} e_2 = e_3, & \overline{\nabla}_{e_1} e_3 = -e_2, & \overline{\nabla}_{e_1} e_4 = 0, & \overline{\nabla}_{e_1} e_5 = 0, \\
\overline{\nabla}_{e_2} e_1 = -e_3, & \overline{\nabla}_{e_2} e_2 = 0, & \overline{\nabla}_{e_2} e_3 = e_1, & \overline{\nabla}_{e_2} e_4 = 0, & \overline{\nabla}_{e_2} e_5 = 0, \\
\overline{\nabla}_{e_3} e_1 = -e_2, & \overline{\nabla}_{e_3} e_2 = e_1, & \overline{\nabla}_{e_3} e_3 = 0, & \overline{\nabla}_{e_3} e_4 = -e_5, & \overline{\nabla}_{e_3} e_5 = e_4, \\
\overline{\nabla}_{e_4} e_1 = 0, & \overline{\nabla}_{e_4} e_2 = 0, & \overline{\nabla}_{e_4} e_3 = -e_5, & \overline{\nabla}_{e_4} e_4 = 0, & \overline{\nabla}_{e_4} e_5 = e_3, \\
\overline{\nabla}_{e_5} e_1 = 0, & \overline{\nabla}_{e_5} e_2 = 0, & \overline{\nabla}_{e_5} e_3 = e_4, & \overline{\nabla}_{e_5} e_4 = -e_3, & \overline{\nabla}_{e_5} e_5 = 0.
\end{array} \tag{27}$$

From the equations above, we conclude that

$$(\overline{\nabla}_{e_i} \varphi) e_j = g(e_i, e_j)\xi - \eta(e_j) e_i,$$

for all $i, j = 1, ..., 5$ which demonstrates that $(\overline{M}, \varphi, \xi, \eta, g)$ is a Sasakian manifold. Suppose that $M = \langle e_1, e_2, e_3 \rangle$. In view of equation (27), we infer that

$$h(e_i, e_j) = 0,$$

for i, j= 1, 2, 3 which shows that M is a totally geodesic submanifold of $\overline{M}$.

References

[1] E. Cartan, Sur une classe remarquable d'espaces de Riemann, *Bull. Soc. Math. France* 54 (1926), 214–264.

[2] Z. I. Szaó, Structure theorems on Riemannian spaces satisfying $R(X, Y).R = 0$, *Geometriae Dedicata*, 19 (1985), 65–108.

[3] R. N. Sen and M. C. Chaki, On curvature restrictions of a certain kind of conformally flat Riemannian space of class one, *Proc. Nat. Inst. Sci. India*, 33 (1967), 100–102.

[4] M. C. Chaki, On pseudo Ricci symmetric manifolds, *Bulg. J. Phys* 15 (1988), 526-531.

[5] M. Tarafdar, On pseudo-symmetric and pseudo-Ricci-symmetric Sasakian manifolds, *Periodica Mathematica Hungarica*, 22 (1991), 125–128.

[6] L. Tamássy and T. Q. Binh, On weak symmetries of Einstein and Sasakian manifolds, *Tensor, N. S* 53 (1993), 140–148.

[7] R. S. D. Dubey, Generalized recurrent spaces, *Indian J. Pure Appl. Math* 10 (1979), 1508–1513.

[8] V. Pirhadi, Covariant derivative of the curvature tensor of Kenmotsu manifolds, *Bull. Iran.Math. Soc* 48 (2022), 1-18.

[9] V. Pirhadi, Gh.Fasihi-Ramandi and S. Azami, Some Characterization Results on Generalized Weakly Symmetric Kenmotsu Manifolds, *Bull. Iran. Math. Soc* 48 (2022), 3779—3794.

[10] K. Yano and M. Kon, Structure on manifold. *World scientific publishing, Series in pure mathematics*, 3 (1984).

[11] K. Ogiue, On almost contact manifolds admitting axiom of planes or axiom of free mobility, *Kodai Math* 16 (1964), 223–232.

[12] K.Yano and S. Ishihara, Invariant submanifotds of almost contact manifolds, *Kodai Math. Sere. Rep* 2 (1969), 350–364.

[13] S. Roy, S. Dey, A. Bhattacharyya and S. K. Hui, $\star$-Conformal η-Ricci soliton on Sasakian manifold, *Asian-European Journal of Math*(2021), 2250035.

[14] D. E. Blair, *Riemannian geometry of contact and symplectic manifolds*, 203, Second edition, Progress in Math, (2002)

[15] S. K. Hui, On weakly ϕ-symmetric Kenmotsu manifolds, *Acta Univ. Palac. Olom., Fac. Rer. Nat., Math.*, 51 (2012), 43–50.

[16] S. K. Hui, On ϕ-pseudo symmetric Kenmotsu manifolds, *Novi Sad J. Math.*, 43 (2013), 89–98.

[17] S. K. Hui, On ϕ-pseudo symmetric Para-Sasakian manifolds, *Acta Universitatis Apulensis*, 39 (2014), 161–178.

[18] D. G. Prakasha, S. K. Hui and K. Vikas, On weakly ϕ-Ricci symmetric Kenmotsu manifolds, *Int. J. Pure Appl. Math.*, 95 (2014), 515–521.

Vahid Pirhadi
Assistant Professor of Mathematics
Department of Mathematics
Faculty of Mathematics, University of Kashan
Kashan, Iran
E-mail: v.pirhadi@kashanu.ac.ir

Ghodratallah Fasihi-Ramandi
Assistant Professor of Mathematics
Department of Mathematics
Faculty of Science, Imam Khomeini International University
Qazvin, Iran
E-mail: fasihi@sci.ikiu.ac.ir

Shahroud Azami
Associate Professor of Mathematics
Department of Mathematics
Faculty of Science, Imam Khomeini International University
Qazvin, Iran
E-mail: azami@sci.ikiu.ac.ir