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Original Research Paper

On Dimension of Some Finite Total Graphs

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Abstract. In this paper we decompose $\Gamma^{\alpha}(Z(\mathbb{Z}_{p_1p_2\cdots p_\alpha}))$, a graph with vertises $Z(\mathbb{Z}_{p_1p_2\cdots p_\alpha})$ and two distinct vertices x and y are adjacent if and only if there exists i, $1 \leq i \leq \alpha$, such that $p_i \mid x, y$. We obtain dimension, edge metric dimension, strong metric dimension, and fractional metric dimension for it.

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Keywords and Phrases: distance, resolving set,metric dimension, Total graph.

1 Introduction

The concept of *metric dimension* of a general metric space was introduced in 1953 by Blumenthal [2]. About twenty years later, it was applied by Slater [12] who introduced the concept of *locating set* of a

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graph. Independently Harary and Melter [6] introduced the same concept as the *resolving sets* and calculated the metric dimension of a tree graph. Since then, it has been frequently used in graph theory, chemistry, biology, robotics and many other disciplines.

Let $G = (V, E)$ be a simple, finite, undirected and connected graph. For graph theoretic terminology we refer to [5]. We say that a vertex $u \in V$ distinguishes (determines or recognizes) two vertices $x, y \in V$ if $d(u, x) \neq d(u, y)$, where $d(x, y)$ represents the length of a shortest $x-y$-path in G. A *metric generator* for G is a set $\mathtt{B} \subseteq V$ with the property that, for each pair of vertices $x, y \in V$, there exists a vertex $u \in \mathtt{B}$ that distinguishes x and y. If for some metric generator $\mathtt{A} \subseteq V$, we have that $|\mathtt{A}| = min\{|\mathtt{B}| : \mathtt{B}$ is a metric generator for $G\}$, we say that $\mathtt{A}$ is a *metric basis* for G and $dim(G) = |\mathtt{A}|$, is the *metric dimension* of G.

A set $\mathtt{L} \subseteq V$ is said to be a *local metric generator* for G if for each pair of vertices $x, y \in V$ such that $xy \in E$, there exists a vertex $u \in \mathtt{L}$ that distinguishes x and y. If for some local metric generator $\mathtt{M} \subseteq V$, we have that $|\mathtt{M}| = min\{|\mathtt{L}| : \mathtt{L}$ is a local metric generator for $G\}$, then we say that M is a *local metric basis* for G and $dim_\ell(G) = |\mathtt{M}|$, is the *local metric dimension* of G. The concept of adjacency generator was introduced by Jannesari and Omoomi [9] as a tool to study the metric dimension of lexicographic product of graphs. An *adjacency generator* for G is a set $\mathtt{B} \subseteq V$ such that for each $x, y \in V - \mathtt{B}$ there exists $b \in \mathtt{B}$ such that b is adjacent to exactly one of x and y. An adjancency which has the minimum cardinality among all adjacency generators of G is called an *adjacency basis* of G, and its cardinality is said to be the *adjacency dimension* of G, denoted by $dim_a(G)$. The concepts of *local adjacency generator*, *local adjacency basis* and *local adjacency dimension* are defined analogously, and the local adjacency dimension of a graph G is denoted by $dim_{a,\ell}(G)$.

The distance between the vertex v and the edge e is defined as $d(e, v) = min\{d(u, v), d(w, v)\}$, where $e = uw$. A vertex $w \in V$ distinguishes two edges $e_1, e_2 \in E$ if $d(w, e_1) \neq d(w, e_2)$. A nonempty set $\mathtt{S} \subseteq V$ is an *edge metric generator* for G if any two edges of G are distinguished by some vertex of S. An edge metric generator with the smallest possible cardinality is called an *edge metric basis* for G, and its cardinality is the *edge metric dimension*, which is denoted by $dim_e(G)$.

A set S of vertices of G is a *mixed metric generator* if any two elements (vertices or edges) of G are distinguished by some vertex of S. The smallest cardinality of a *mixed metric generator* for G is called the *mixed metric dimension* and is denoted by $dim_m(G)$. A mixed metric basis for G is a mixed metric generator for G of cardinality $dim_m(G)$. It immediately follows that $dim_m(G) \geq max\{dim(G), dim_e(G)\}$.

For any two vertices u and v of G, the interval $I[u, v]$ is defined as the collection of all vertices that belong to some shortest $u - v$-path. A vertex w strongly distinguishes u and v if $v \in I[u, w]$ or $u \in I[v, w]$. A set S of vertices in a connected graph G is a *strong metric generator* for G if every two vertices of G are strongly distinguished by some vertex of S. The smallest cardinality of a strong metric generator of G is called *strong metric dimension* and is denoted by $dim_s(G)$.

For any two vertices x and y of G, $R\{x, y\}$ denotes the set of vertices z such that $d(x, z) \neq d(y, z)$. In this view, a metric generating of G is a subset W of V such that $W \cap R\{x, y\} \neq \emptyset$ for any two distinct vertices x and y of G. Let $f : V(G) \longrightarrow [0, 1]$ be a real valued function. For $W \subseteq V$, denote $f(W) = \sum_{v \in W} f(v)$. We call f a *resolving function* of G if $f(R\{x, y\}) \geq 1$ for any two distinct vertices x and y of G. The *fractional metric dimension*, denoted by $dim_f(G)$, is given by $dim_f(G) = min\{|g| : g$ is a resolving function of $G\}$, where $|g| = g(V(G))$.

By definition of the different variants of generators, the following inequalities hold for any graph G. For more details see [9].

- $dim(G) \leq dim_a(G)$,
- $dim_l(G) \leq dim(G) \leq dim_l(G) + dim_{a,l}(G^c)$,
- $dim_l(G) \leq dim_{a,l}(G)$,
- $dim_{a,l}(G) \leq dim_a(G)$,

For a non-zero commutative ring R, let $Z(R)$ be the set of zero-divisors of R. Anderson and Badawi in [1] introduced the total graph of R, denoted by $\Gamma(R)$, as an undirected graph with all elements of R as vertices, and for distinct $x, y \in R$, the vertices x and y are adjacent if and only if $x + y \in Z(R)$. They also studied the induced subgraph $\Gamma(Z(R))$ of $\Gamma(R)$ with vertices $Z(R)$.

In this paper, we let $P = \{p_1, p_2, \dots, p_\alpha\}$ as a set of odd prime numbers and consider the total graph $\Gamma(\mathbb{Z}_{p_1p_2\dots p_\alpha})$ and its induced subgraph $\Gamma(Z(\mathbb{Z}_{p_1p_2\dots p_\alpha}))$. We define the graph $\Gamma^\alpha(Z(\mathbb{Z}_{p_1p_2\dots p_\alpha}))$ as a subgraph of $\Gamma(Z(\mathbb{Z}_{p_1p_2\dots p_\alpha}))$, with all elements of $Z(\mathbb{Z}_{p_1p_2\dots p_\alpha})$ as vertices, and two distinct vertices x and y are adjacent if and only if there exists i, $1 \leq i \leq \alpha$, such that $p_i \mid x, y$. We find a decomposition for this graph, and investigate the various dimensions.

2 Decomposition

In this section we first decompose the subgraph $\Gamma^2(Z(\mathbb{Z}_{pq}))$, where p, q are two distinct odd prime numbers and then generalize it into $\Gamma^\alpha(Z(\mathbb{Z}_{p_1p_2\dots p_\alpha}))$. We know that $Z(\mathbb{Z}_{p_1p_2\dots p_\alpha})$ is not an ideal of $\mathbb{Z}_{p_1p_2\dots p_\alpha}$, and $2 \notin Z(\mathbb{Z}_{p_1p_2\dots p_\alpha})$, so by [1], $\Gamma(Z(\mathbb{Z}_{p_1p_2\dots p_\alpha}))$ is a connected graph with diameter two.

Remark 2.1. *It is known by Euler's formula that* $|Z(\mathbb{Z}_n)| = n - \phi(n)$ *such that* $\phi(n) = n\Pi_{i=1}^{\alpha}(1 - \frac{1}{p_i})$. *So we have*

$$|Z(\mathbb{Z}_{p_1p_2\dots p_\alpha})| = p_1p_2\dots p_\alpha - \prod_{i=1}^{\alpha}(p_i - 1),$$

which is the number of vertices of $\Gamma^\alpha(Z(\mathbb{Z}_{p_1p_2\dots p_\alpha}))$.

Remark 2.2. *One can see that* $\Gamma^2(Z(\mathbb{Z}_{pq}))$ *is the graph* $\Gamma(Z(\mathbb{Z}_{pq}))$. *For* $\alpha \geq 3$, $\Gamma^\alpha(Z(\mathbb{Z}_{p_1p_2\dots p_\alpha}))$ *and* $\Gamma(Z(\mathbb{Z}_{p_1p_2\dots p_\alpha}))$ *are distinct. It is easy to see that* $\Gamma^\alpha(Z(\mathbb{Z}_{p_1p_2\dots p_\alpha}))$ *is a subgraph of* $\Gamma(Z(\mathbb{Z}_{p_1p_2\dots p_\alpha}))$ *because for any two adjacent vertices* x, y *in* $\Gamma^\alpha(Z(\mathbb{Z}_{p_1p_2\dots p_\alpha}))$, *there exists* i, $1 \leq i \leq \alpha$, *such that* $p_i \mid x, y$ *which leads to* $p_i \mid x + y$, *by [3]. So,* $x + y \in Z(\mathbb{Z}_{p_1p_2\dots p_\alpha})$, *and* $x \sim y$ *in* $\Gamma(Z(\mathbb{Z}_{p_1p_2\dots p_\alpha}))$. *Since the logical proposition* $p_i \mid x, y \Rightarrow p_i \mid x + y$, *is not reversible, there exist adjacent vertices in* $\Gamma(Z(\mathbb{Z}_{p_1p_2\dots p_\alpha}))$ *which are nonadjacent in* $\Gamma^\alpha(Z(\mathbb{Z}_{p_1p_2\dots p_\alpha}))$.

Example 2.3. Consider the graphs $\Gamma(Z(\mathbb{Z}_{105}))$ and $\Gamma^3(Z(\mathbb{Z}_{105}))$ with $P = \{3, 5, 7\}$. It is clear that the number of edges in $\Gamma(Z(\mathbb{Z}_{105}))$ is more than $\Gamma^3(Z(\mathbb{Z}_{105}))$. For example, $3 \sim 7$, $33 \sim 77$ in $\Gamma(Z(\mathbb{Z}_{105}))$ since $3+7, 33+77 \in Z(\mathbb{Z}_{105})$. But $3 \nsim 7$, $33 \nsim 77$ in $\Gamma^3(Z(\mathbb{Z}_{105}))$, since there isn't any common prime factor in P.

Lemma 2.4. *The following facts hold in* $\Gamma^2(Z(\mathbb{Z}_{pq}))$*:*

(i) If $u = kp$ *where* $1 \leq k \leq q-1$*, then* $deg(u) = q-1$*.*

(ii) If $v = kq$ *where* $1 \leq k \leq p-1$*, then* $deg(v) = p-1$*.*

(iii) $deg(0) = p+q-2$*.*

Proof. It is clear that $u = kp$ is adjacent to zero and all of the other multiples of p. Similarly, $v = kq$ is adjacent to zero and all of the other multiples of q. Obviously, u and v are not adjacent. □

Definition 2.5. *A decomposition of a graph* G *is a list of subgraphs* $G_1, G_2, \ldots, G_r$ *such that each edge appears in exactly one subgraph in the list. In this terminology, we say that* G *is decomposed by* $G_1, G_2, \ldots, G_r$ *and show it by* $G = G_1 + G_2 + \cdots + G_r$*.*

Theorem 2.6. $\Gamma^2(Z(\mathbb{Z}_{pq}))$ *has the following decomposition;*

$$\Gamma^2(Z(\mathbb{Z}_{pq})) = K_p + K_q.$$

Proof. In $\Gamma^2(Z(\mathbb{Z}_{pq}))$ we have q vertices of the form $v = kp$ which induce K_q and p vertices of the form $v = kq$ which induce K_p, by Lemma 2.4. It is clear that there is no other adjacency and zero is the only common vertex of them. □

Example 2.7. In Figure 1, we see the decomposition of $\Gamma^2(Z(\mathbb{Z}_{35}))$.

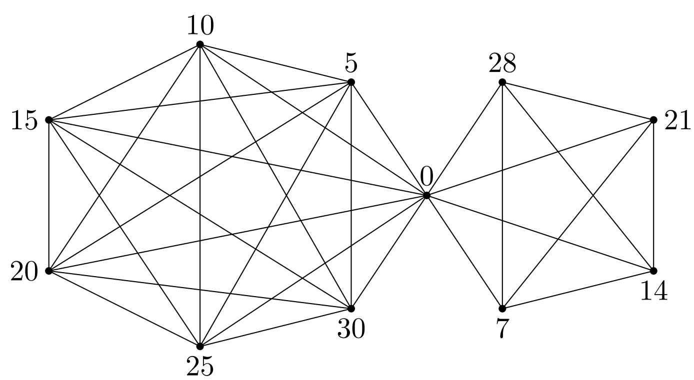

Figure 1: $\Gamma^2(Z(\mathbb{Z}_{35})) = K_7 + K_5$

Theorem 2.8. $\Gamma^3(Z(\mathbb{Z}_{pqr})) = K_{pq} + K_{pr} + K_{qr} - (K_p + K_q + K_r)$.

Proof. By Remark 2.1, $V|\Gamma^3(Z(\mathbb{Z}_{pqr}))| = pqr - (p-1)(q-1)(r-1)$. These vertices are classified in multiples of p, q and r. It is clear that qr elements of them are multiples of p, pr elements are multiples of q and pq elements are multiples of r. Hence, the edges of G can be decomposed into three complete graphs K_{qr}, K_{pr} and K_{pq}.

There are also some other edges formed by common multiples of p, q and r. Since there are r common multiples of p and q, p common multiples of q and r and q common multiples of p and r, which form K_r, K_p and K_q, respectively, in order to avoid considering repeated edges, we omit K_r, K_p and K_q from the decomposition. (See Figure 2.) □

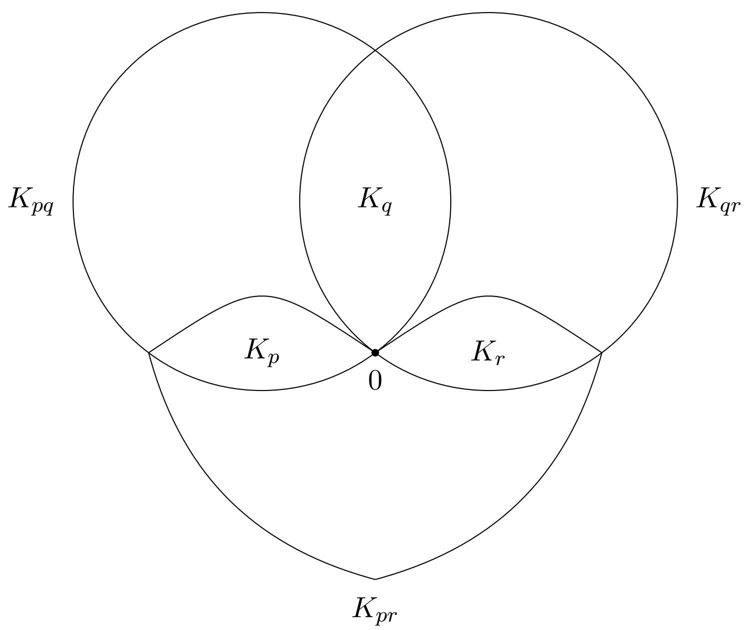

Figure 2: Decomposition of $\Gamma^3(Z(\mathbb{Z}_{pqr}))$

Theorem 2.9. $\Gamma^4(Z(\mathbb{Z}_{pqrs})) = K_{pqr} + K_{pqs} + K_{prs} + K_{qrs} - (K_{pq} + K_{pr} + K_{ps} + K_{qr} + K_{qs} + K_{rs}) + K_p + K_q + K_r + K_s$.

Proof. Similar to proof of theorem 2.8, multiples of p, q, r and s form the complete graphs $K_{qrs}, K_{prs}, K_{pqs}$ and K_{pqr}, respectively. Notice that there are $\binom{4}{2}$ common multiples of every two factors of p, q, r and s which are counted once in above complete graphs. So, we should omit the edges of $K_{pq} + K_{pr} + K_{ps} + K_{qr} + K_{qs} + K_{rs}$. Since the edges formed by common multiples of every three factors of p, q, r and s are missed by the procedure of deletion, we have to add the complete graphs $K_p + K_q + K_r + K_s$. (See Figure 3.) □

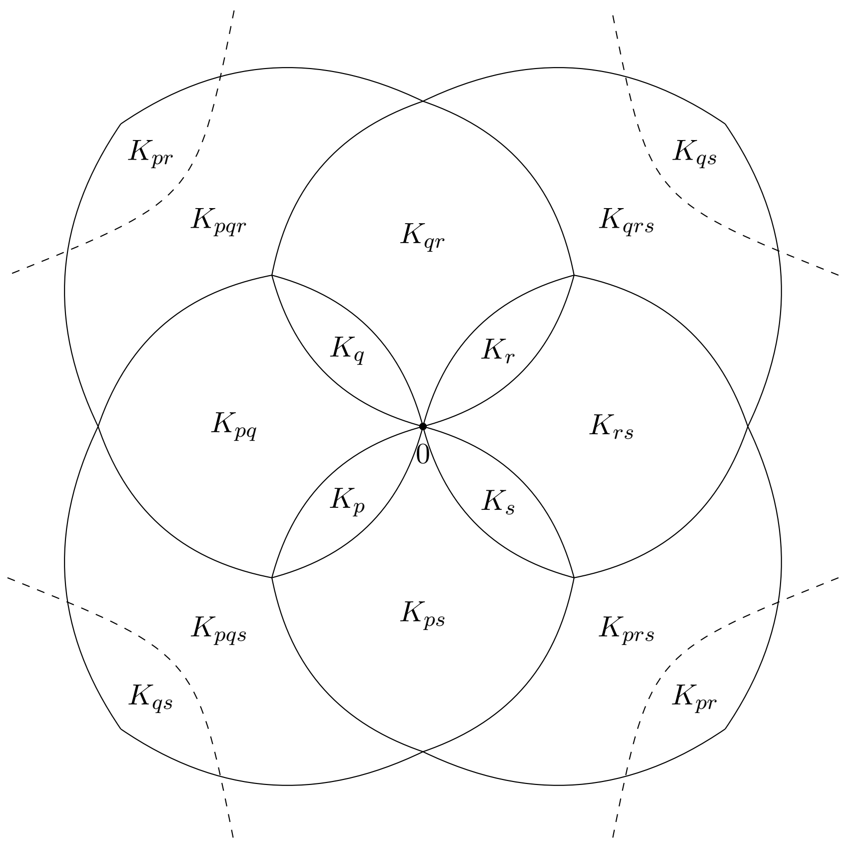

Figure 3: Decomposition of $\Gamma^4(Z(\mathbb{Z}_{pqrs}))$

In general, we have the following theorem.

Theorem 2.10.

$$\Gamma^{\alpha}(Z(\mathbb{Z}_{p_1p_2\ldots p_\alpha})) = \sum_{i=1}^{\alpha} K_{\prod\limits_{\substack{j=1\\ j\neq i}}^{\alpha} p_j} - \sum_{\substack{i,j=1\\ i<j}}^{\alpha} K_{\prod\limits_{\substack{k=1\\ k\neq i,j}}^{\alpha} p_k} + \sum_{\substack{i,j,k=1\\ i<j<k}}^{\alpha} K_{\prod\limits_{\substack{l=1\\ l\neq i,j,k}}^{\alpha} p_l} - \ldots + (-1)^{\alpha} \sum_{i=1}^{\alpha} K_{p_i}.$$

Proof. The proof follows by inclusion-exclusion principle and induction on α due to the proofs of Theorems 2.8, 2.9. □

Corollary 2.11. *The clique number of* $\Gamma^{\alpha}(Z(\mathbb{Z}_{p_1p_2\ldots p_\alpha}))$ *is* $\max\{\prod\limits_{\substack{j=1\\ j\neq i}}^{\alpha} p_j \mid 1 \leq i \leq \alpha\}$. *In the case* $p_1 < p_2 < \cdots < p_\alpha$, *the clique number is* $p_2p_3\ldots p_\alpha$.

Proof. According to Theorem 2.10, it is clear. □

Corollary 2.12. $diam(\Gamma^{\alpha}(Z(\mathbb{Z}_{p_1p_2\ldots p_\alpha}))) = 2$.

Proof. The decomposition in Theorem 2.10 shows that $\Gamma^{\alpha}(Z(\mathbb{Z}_{p_1p_2\ldots p_\alpha}))$ is not a complete graph. Since $\Gamma^{\alpha}(Z(\mathbb{Z}_{p_1p_2\ldots p_\alpha}))$ is a subgaph of $\Gamma(Z(\mathbb{Z}_{p_1p_2\ldots p_\alpha}))$, $diam(\Gamma^{\alpha}(Z(\mathbb{Z}_{p_1p_2\ldots p_\alpha}))) = 2$. □

3 Twin Equivalence Classes of $\Gamma(Z(\mathbb{Z}_{p_1p_2\ldots p_\alpha}))$

For a vertex u, the *open neighborhood* of u in G is $N(u) = \{v \in V \mid uv \in E\}$ and the *closed neighborhood* of u is $N[u] = N(u) \cup \{u\}$. Two vertices u, v are *true twins* of G if $N[u] = N[v]$. They are *false twins* if $N(u) = N(v)$; and *twins* if they are any of the previous. Define a relation $\equiv$ on $V(G)$ by $u \equiv v$ if and only if $u = v$ or u, v are twins. By Lemma 2.6 in [7], $\equiv$ is an equivalence relation. It is not difficult to see that the equivalence classes of the true-twin relations are cliques and those of the false-twin relations are independent sets. There are three possibilities for each twin equivalence class U:

(a) U is a singleton set, or

(b) $|U| > 1$ and $N(u) = N(v)$ for any $u, v \in U$, or

(c) $|U| > 1$ and $N[u] = N[v]$ for any $u, v \in U$.

We will refer to the type (c) as the true twin equivalence classes.

Consider the equivalence relation $\equiv$. For each vertex $v \in V(G)$, let v^* be the set of vertices of G that are equivalent to v under $\equiv$. Let $\{v_1^*, ..., v_k^*\}$ be the partition of $V(G)$ induced by $\equiv$, where each v_i is a representative of the set v_i^*. The *twin graph* of G, denoted by G^*, is the graph with vertex set $V(G^*) := \{v_1^*, ..., v_k^*\}$, where $v_i^* v_j^* \in E(G^*)$ if and only if $v_i v_j \in E(G)$. By Lemma 2.10 in [7], one can see that this definition is independent of the choice of representatives.

Note that in $\Gamma^\alpha(Z(\mathbb{Z}_{p_1 p_2 \dots p_\alpha}))$ the vertices can be classified in multiples and common multiples of p_i's. In the next, we show that this partition forms equivalence classes. Also, we obtain the number of the equivalence classes by counting the ways of selecting common multiples of p_i's.

Notation 3.1. For any nonempty subset $S \subseteq P$, let $A_S = \{x \in Z(\mathbb{Z}_{p_1 p_2 \dots p_\alpha}); \quad p \mid x \iff p \in S\}$. Set $\mathcal{A} = \{A_S; S \subseteq P\}$ and for all $1 \leq i \leq \alpha$, $\mathcal{A}_i = \{A_S; \quad S \subseteq P, |S| = i\}$.

One sees that $|\mathcal{A}_i| = \binom{\alpha}{i}$. In the next theorem we show that for all $\emptyset \neq S \subseteq P$, A_S is an equivalence class.

Theorem 3.2. *The number of twin equivalence classes of* $\Gamma^\alpha(Z(\mathbb{Z}_{p_1 p_2 \dots p_\alpha}))$ *is* $2^\alpha - 1$.

Proof. Let $S = \{p_{i_1}, \dots, p_{i_s}\} \subseteq P$. We show that $N[x] = N[y]$ for every $x, y \in A_S \in \mathcal{A}_s$. Suppose that $z \in N[x]$. There exists p_i such that $p_i \mid x$, $p_i \mid y$. If $p_i \in S$, then $p_i = p_{i_l}$, for some $1 \leq l \leq s$. So, $p_{i_l} \mid x$ and $p_{i_l} \mid z$. Since $p_{i_l} \mid y$, it leads to $z \sim y$. Therefore, $z \in N[y]$. If $p_i \notin S$, then $p_{i_l} \nmid y$. So, $y \notin A_S$, which contradicts to the assumption. Thus, A_S is an equivalence class. Note that $p \mid 0$ for all $p \in P$. So, A_P is the zero singleton class.

By assumption, we have $\binom{\alpha}{s}$ sets of A_S's. Therefore, the number of the equivalence classes is $\sum_{i=1}^{\alpha} \binom{\alpha}{i}$ which equals to $2^\alpha - 1$ by [3]. □

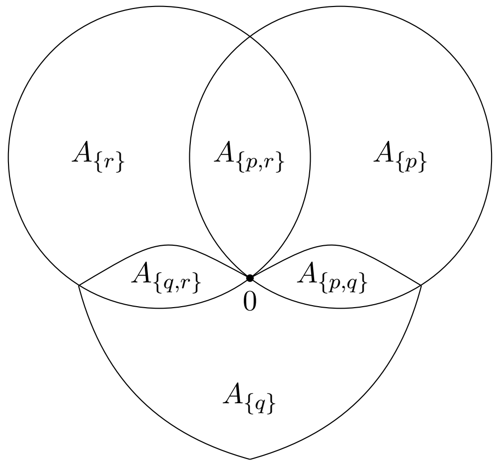

Figure 4: Equivalence classes of $\Gamma^3(Z(\mathbb{Z}_{pqr}))$

Lemma 3.3. *Let S and T be two nonempty subsets of $P = \{p_1, \ldots, p_\alpha\}$. For any $v \in A_S$ and $w \in A_T$;*

(i) If $S \cap T \neq \emptyset$, then $d(v, w) = 1$;

(ii) If $S \cap T = \emptyset$, then $d(v, w) = 2$.

Proof. (i) If $p_l \in S \cap T$, then $p_l \mid v$ and $p_l \mid w$. So, $v \sim w$. (ii) Let $S = \{p_{i_1}, \ldots, p_{i_s}\}$ and $T = \{p_{j_1}, \ldots, p_{j_t}\}$. Since the intersection is empty, $p_{i_l} \nmid w$ for any $1 \leq l \leq s$, and $p_{j_r} \nmid v$ for any $1 \leq r \leq t$. If $d(v, w) = 1$, then there exists $p_k \in P$ such that $p_k \mid v$ and $p_k \mid w$. By the assumption, $p_k \notin S \cup T$. It means that there is no prime factor in P which aliquots both v and w. So, $v \nsim w$. Hence, $d(v, w) = 2$. □

We know that a subset $S \subseteq V$ is an *independent set* in G if no two vertices in S are adjacent. *The independence number* of G is the maximum size of all independent sets of vertices, denoted by $\alpha(G)$.

Theorem 3.4. *The independence number of $\Gamma^\alpha(Z(\mathbb{Z}_{p_1 p_2 \cdots p_\alpha}))$ is α.*

Proof. Consider the equivalence classes $\mathcal{A}_1 = \{A_{\{p_i\}}; \quad 1 \leq i \leq \alpha\}$. Let $I = \{v_1, v_2, \ldots, v_\alpha\}$ such that $v_i \in A_{\{p_i\}}$. For any two distinct vertices $v_i, v_j \in I$, by part (ii) of Lemma 3.3, v_i and v_j are not adjacent.

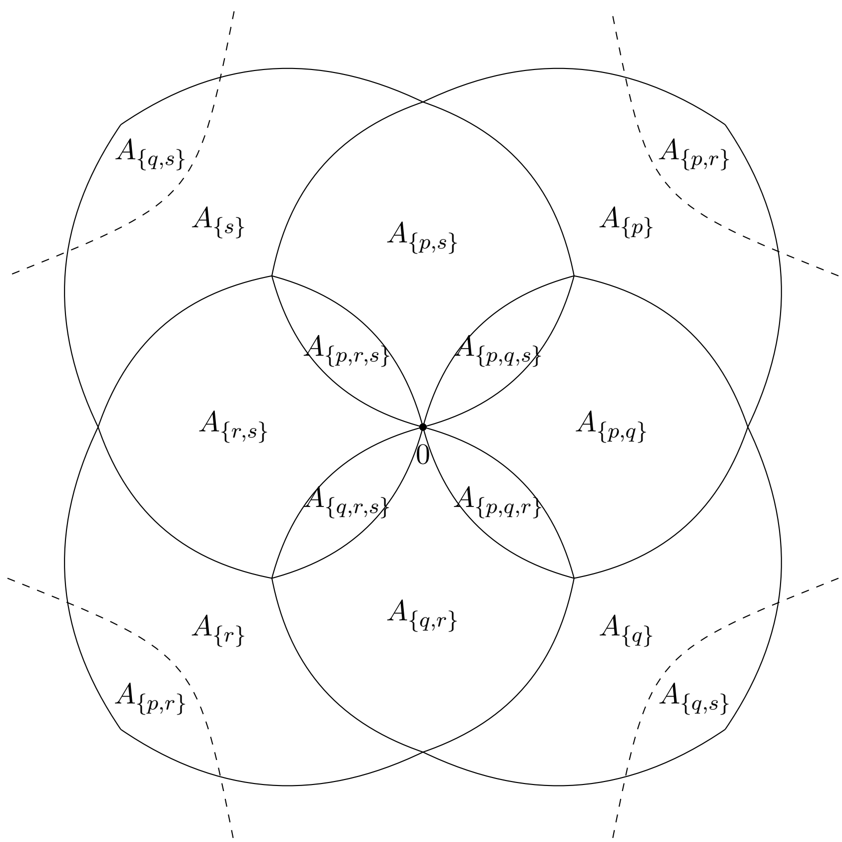

Figure 5: Equivalence classes of $\Gamma^4(Z(\mathbb{Z}_{pqrs}))$

Therefore, I is an independent set. Suppose that $u \notin I$. If $u \in \mathcal{A}_1$, then $u \in A_{\{p_t\}}$ for some $1 \leq t \leq \alpha$, so $p_t \mid u$. Since $p_t \mid v_t$, it leads to u is adjacent to v_t. Let $u \notin \mathcal{A}_1$, then there is an equivalence class A_S such that $u \in A_S$. Clearly, $A_S \notin \mathcal{A}_1$, $|S| \geq 2$, and $\{p_t\} \cap S \neq \emptyset$ for some $1 \leq t \leq \alpha$. Let $p_r \in S$ such that $p_r \neq p_t$, then $p_r \mid u$ and $p_t \mid u$. So, $u \sim v_r$ and $u \sim v_t$. Hence, I is the maximal independent set. $\square$

4 Dimension

In this section we obtain some types of dimensions for the graph $G = \Gamma^{\alpha}(Z(\mathbb{Z}_{p_1p_2\cdots p_\alpha}))$. First, we note that determining whether a given set $\mathtt{B}$ of vertices of G is a metric generating set of G, one need only investigate the pairs of vertices in $V(G) - \mathtt{B}$, since $u \in \mathtt{B}$ is the only vertex of G whose distance from u is 0.

Theorem 4.1. *[8] If G^* is the twin graph of G, then $dim(G) \geq n(G) - n(G^*)$.*

Theorem 4.2. *Let $G = \Gamma^{\alpha}(Z(\mathbb{Z}_{p_1p_2\cdots p_\alpha}))$. Then $dim(G) = n(G) - 2^{\alpha} + 1$.*

Proof. By Theorems 4.1 and 3.2, $dim(G) \geq n(G) - 2^{\alpha} + 1$. Set R as a set of representative vertices of equivalence classes. By Theorem 3.2, $|R| = 2^{\alpha} - 1$. We show that $\mathtt{M} = V(G) - R$ is a metric basis for G. For any two distinct vertices $x, y \in \mathtt{M}$, $d(x, x) = 0$ and $d(x, y) = 1$ or $d(x, y) = 2$. So, $d(x, x) \neq d(x, y)$.

Let $u, v \in R$ such that $u, v \neq 0$, and suppose S and T be two subsets of P such that $u \in A_S$ and $v \in A_T$. There exists $L \subseteq P$ such that $L \cap S \neq \emptyset$ and $L \cap T = \emptyset$. So, for every vertex $x \in A_L$, $x \notin R$; $d(x, u) = 1$ and $d(x, v) = 2$, by Lemma 3.3.

Again, let $u, v \in R$ and $u = 0$. There exists $L \subseteq P$ such that $L \cap S \neq \emptyset$ and $L \cap T = \emptyset$. Then for all $x \in A_L$, $d(x, v) = 2$ and $d(x, u) = 1$; since zero is adjacent to all vertices. Hence, $\mathtt{M}$ is a metric basis and $dim(G) \leq n(G) - 2^{\alpha} + 1$. □

Corollary 4.3. *Let $G = \Gamma^{\alpha}(Z(\mathbb{Z}_{p_1p_2\cdots p_\alpha}))$, Then $dim_a(G) = n(G) - 2^{\alpha} + 1$.*

Proof. Since $diam(G) = 2$, it is easy to see that $dim_a(G) = dim(G) = n(G) - 2^{\alpha} + 1$. □

Lemma 4.4. *Let y and z be true twins. If $e, f \in E(G)$ such that $e = xy$ and $f = xz$, then at least one of y and z belongs to an edge metric basis of G.*

Proof. Let $x \in A_S$ and S be a subset of P. Consider two edges $e = xy$ and $f = xz$. It is clear that $d(e, x) = d(f, x) = 0$. So, x does not distinguish e and f. Now for $v \neq x, y, z$, consider the following cases.

Case 1: Let $y, z \in A_S$, if $v \in A_S$, then $d(e, v) = d(f, v) = 1$. If $v \notin A_S$, then $v \in A_T$ for some $T \subseteq P$ and by Lemma 3.3, $d(e, v) = d(f, v) = 1$ or $d(e, v) = d(f, v) = 2$.

$$d(e,v) = d(f,v) = \begin{cases} 1 & S \cap T \neq \emptyset \\ 2 & S \cap T = \emptyset \end{cases}$$

Case 2: If $y, z \notin A_S$, since y and z are twin vertices, then by Lemma 3.3, $y, z \in A_T$ for some $T \subseteq P$ such that $S \cap T \neq \emptyset$. If $v \in A_{S \cap T}$, then v is adjacent to x, y and z. So, $d(e, v) = d(f, v) = 1$. If $v \in A_S$ or $v \in A_T$, then $v \in A_T$ is adjacent to x or y, z, respectively. Therefore, $d(e, v) = d(f, v) = 1$. If $v \in A_{P \setminus \{S \cup T\}}$, then $d(e, v) = d(f, v) = 2$, by Lemma 3.3.

$$d(e,v) = d(f,v) = \begin{cases} 1 & v \in A_{S \cup T} \\ 2 & v \in A_{P \setminus (S \cup T)} \end{cases}$$

So, no vertex $v \neq x, y, z$ distinguishes e and f. Also, x does not distinguish e and f, by the first part of the proof. Thus, at least one of y and z must be in an edge metric basis $\mathtt{E}$. $\square$

Lemma 4.5. *Let $G = \Gamma^{\alpha}(Z(\mathbb{Z}_{p_1 p_2 \cdots p_\alpha}))$ and $S \subseteq P$. Then every edge metric basis $\mathtt{E}$ of G contains at least $|A_S| - 1$ vertices of the equivalence class A_S.*

Proof. Let $S \subseteq P$ and consider the true twin equivalence class A_S of G. Let x, y, z be three vertices of A_S and consider two edges $e = xy$ and $f = xz$. Then by Lemma 4.4, at least one of y and z must be in an edge metric basis $\mathtt{E}$. Since A_S is a clique, this argument can be repeated for any pair of such edges. $\square$

Theorem 4.6. *Let $G = \Gamma^{\alpha}(Z(\mathbb{Z}_{p_1 p_2 \cdots p_\alpha}))$. Then $dim_e(G) = n(G) - 2$.*

Proof. For any edge metric basis $\mathtt{E}$ of G, by Theorem 3.2 and Lemma 4.5, $|\mathtt{E}| \geq n(G) - 2^{\alpha} + 1$. If $\mathtt{E} = V(G) - \{u, v, w\}$ such that $u \in A_U$, $v \in A_V$, and $w \in A_W$ where A_U, A_V, A_W are three distinct equivalence classes associated with $U, V, W \subseteq P$.

Let $u, v, w \neq 0$, then for two edges $e = 0u$, $f = 0v$, $d(e, 0) = d(f, 0) = 0$, and $d(e, x) = d(f, x) = 1$ for all $x \in \mathtt{E}$. Thus, $\mathtt{E}$ does not distinguish

e and f. If $u = 0$, $\mathtt{E} = V(G) - \{0, v, w\}$, then the edges $e = 0w$, $f = 0v$ have distance one to all vertices in $\mathtt{E}$, which is a contradiction to edge metric basis of $\mathtt{E}$. By this argument, there is no edge metric basis of size $n-3$. Hence, $dim_e(G) \geq n-2$.

Let $\mathtt{E}^{'}$ contains at least one of the vertices v and w. Suppose that $w \in \mathtt{E}^{'}$ and $\mathtt{E}^{'} = V(G) - \{0, v\}$. We show that $\mathtt{E}^{'}$ is an edge metric basis. According to the structure of the graph, for any pair edges e and f we have the following cases. In each case, we show that there is an $x \in \mathtt{E}^{'}$ which distinguishes e and f.

Case 1: Let $e = xy$, $f = zt$, such that $x, y, z, t \notin \{0, v\}$. If e, f belong to an equivalence class A_S, then $d(e, x) = 0$ and $d(f, x) = 1$. If e, f belong to distinct equivalence classes A_S , A_T, respectively, then $d(e, x) = 0$. Also, for $T \cap S \neq \emptyset$, $d(f, x) = 1$ and $T \cap S = \emptyset$, $d(f, x) = 2$.

Case 2: Let $e = 0v$, $f = 0x$, then $d(f, x) = 0$ and $d(e, x) = 1$; since zero is adjacent to all vertices.

Case 3: Let $e = 0x$, $f = 0y$, then $d(e, x) = 0$ and $d(f, x) = 1$ or 2; by Lemma 3.3.

Case 4: Let $e = 0v$, $f = xy$, then $d(f, x) = 0$ and $d(e, x) = 1$; since zero is adjacent to all vertices.

Case 5: Let $e = 0v$, $f = vx$, then $d(f, x) = 0$, $d(e, x) = 1$; since $x \in A_V$, or $x \in A_S$ for some $S \subseteq P$ such that $V \cap S \neq \emptyset$.

Case 6: Let $e = vx$, $f = yz$, then $d(e, x) = 0$, $d(f, x) = 1$ or 2; by Lemma 3.3.

Case 7: Let $e = vx$, $f = vy$, then $d(e, x) = 0$, $d(f, x) = 1$; since $y \in A_V$, or $y \in A_T$ for some $T \subseteq P$ such that $V \cap T \neq \emptyset$.

Case 8: Let $e = 0x$, $f = yz$, then $d(e, x) = 0$, $d(f, x) = 1$ or 2; by Lemma 3.3.

In each case, $\mathtt{E}^{'}$ is an edge metric basis of G. Thus, $dim_e(G) \leq n-2$.
$\square$

Definition 4.7. *A vertex $u \in N(v)$ is said to be a maximal neighbour of v if $N[v] \subseteq N[u]$.*

Theorem 4.8. *[10] Let G be a connected graph of order n. Then $dim_m(G) = n$ if and only if every vertex of the graph G has a maximal neighbour.*

Theorem 4.9. *Let $G = \Gamma^{\alpha}(Z(\mathbb{Z}_{p_1 p_2 \ldots p_\alpha}))$. Then $dim_m(G) = n(G)$.*

Proof. By the structure of the graph G, every vertex v belongs to a twin equivalence class of A_S for some $S \subseteq P$. So, each vertex has a maximal neighbour, and the result follows by Theorem 4.8. □

Definition 4.10. *$X \subseteq V$ is called a twin-free clique in G if the subgraph induced by X is a clique and for every $u, v \in X$ it follows $N[u] \neq N[v]$. We say that the twin-free clique number of G, denoted by $\bar{\omega}(G)$, is the maximum cardinality among all twin-free cliques in G.*

Theorem 4.11. *[11] Let H be a connected graph of order $n \geq 2$. Then $dim_s(H) \leq n - \bar{\omega}(H)$. Moreover, if H has diameter two, then $dim_s(H) = n - \bar{\omega}(H)$.*

Remark 4.12. *Let G^* be the twin graph of G and χ be a maximal clique in G^*. Then it induces a clique in G which is twin-free, i.e. $\omega(G^*) \leq \bar{\omega}(G)$. Let Θ be a maximal twin-free clique in G, i.e. every vertex is belong to a twin equivalence class A_S for some $S \subseteq P$. Then it induces a clique in G^*. So, $\bar{\omega}(G) \leq \omega(G^*)$. Hence, $\bar{\omega}(G) = \omega(G^*)$.*

Theorem 4.13. *Let $G = \Gamma^{\alpha}(Z(\mathbb{Z}_{p_1 p_2 \ldots p_\alpha}))$, then $dim_s(G) = n - 2^{\alpha-1}$.*

Proof. By remark 4.12, $\bar{\omega}(G) = \omega(G^*)$. If $\alpha = 2$, according to decomposition in Theorem 2.6, G^* is isomorphic to P_3. So, $\omega(G^*) = 2$.

For $\alpha \geq 3$, we claim that $\omega(G^*) = 2^{\alpha-1}$. Without lose of generality, let $T = \{p_1\} \subseteq P$ and $\mathtt{S}' = \{S \subseteq P; \quad p_1 \in S, |S| \geq 2\}$. Consider the equivalence classes $\mathcal{B} = \{A_S; \quad S \in \mathtt{S}'\} \cup \{A_{T^c}\}$. Let X be the set of representative vertices of the equivalence classes $\mathcal{B}$. It is clear that $|\mathtt{S}'| = 2^{\alpha-1} - 1$. So, $|X| = 2^{\alpha-1}$.For all $S \in \mathtt{S}'$, $S \cap T^c \neq \emptyset$. So, by Lemma 3.3, all the vertices of X are adjacent. It means that the graph induced by X is the complete graph $K_{2^{\alpha-1}}$.

Now, suppose $R \subseteq P$ such that $R \notin \mathtt{S}'$. If $R = T$, then the representative vertex of A_R is not adjacent to the representative vertex of A_{T^c}. If $R \neq T$, then $R^c = S$, for some $S \in \mathtt{S}'$. So, by Lemma 3.3, the representative vertex of A_R is not adjacent to at least one vertex of the clique $K_{2^{\alpha-1}}$. Hence, $K_{2^{\alpha-1}}$ is a maximal clique of G^*. □

Lemma 4.14. *[4] For any twin vertices x, y of a connected graph G, $R\{x, y\} = \{x, y\}$.*

Theorem 4.15. *[4] Let G be a connected graph of order at least two. Then $dim_f(G) = \frac{|V(G)|}{2}$ if and only if there exists a bijection $\alpha : V(G) \longrightarrow V(G)$ such that $\alpha(v) \neq v$ and $|R\{v, \alpha(v)\}| = 2$ for all $v \in V(G)$.*

Proposition 4.16. *Let $G = \Gamma^{\alpha}(Z(\mathbb{Z}_{p_1 p_2 \cdots p_\alpha}))$. Then $dim_f(G) = \frac{n(G)}{2}$.*

Proof. Let $\alpha : V(G) \longrightarrow V(G)$ such that $\alpha(x) = x + \prod_{p_i | x} p_i$. Then it is easy to check that α is a bijection which takes any vertex x to it's twin and $\alpha(x) \neq x$. Moreover, by Lemma 4.14, $R\{x, \alpha(x)\} = \{x, \alpha(x)\}$, and the result follows by Theorem 4.15. $\square$

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