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The Frèchet Normal Cone of Optimization Problems with Switching Constraints

Z. Jafariani
Payame Noor University

N. Kanzi
Payame Noor University

M. Naderi Parizi *
Payame Noor University

Abstract. The paper deals with the mathematical programming problems with switching constraints that are defined with continuously differentiable functions. The main results are the upper approximations of the Frèchet normal cone of the feasible set for the problem. As applications of the main results, we present some stationary conditions of the considered problem.

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1 Introduction

Mathematical programming with switching constraints (MPSC, in brief), as a generalization of *mathematical programming with vanishing con-*

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*Corresponding Author

straints (briefly, MPVC) and *mathematical programming with equilibrium constraints* (MPEC, in short), is introduced in 2019 ([12]). Although MPSCs have always existed in topological optimization and optimal control, before reference [12], it was not a definite name for it and was not independently addressed by researchers. Theoretical aspects and a wide range of applications of MPSCs have been studied intensively by many researchers; see, e.g., [2, 3, 6, 9, 10, 11, 12, 15, 17].

The general form of a MPSC is as

$$(\Delta): \qquad \begin{array}{lll} \min & f(x) & \\ s.t. & g_j(x) \leq 0, & j \in J \\ & G_i(x)H_i(x) = 0, & i \in I, \end{array}$$

where the continuously differentiable functions $f, g_j, H_i, G_i : \mathbb{R}^n \to \mathbb{R}$ (for $j \in J$ and $i \in I$) are given and index sets I and J are finite. Throughout this article, we will suppose that the feasible set of (Δ), denoted by $\mathcal{S}$, is nonempty, i.e.,

$$\mathcal{S} = \{x \in \mathbb{R}^n \mid g_j(x) \leq 0, \ j \in J, \ \ G_i(x)H_i(x) = 0, \ i \in I\} \neq \emptyset.$$

As we know from [16, Theorem 6.12], if $x_0 \in \mathcal{S}$ is an optimal solution of (Δ), we have

$$-\nabla f(x_0) \in N_F(\mathcal{S}, x_0), \tag{1}$$

where $N_F(\mathcal{S}, x_0)$ denotes the Frèchet normal cone of $\mathcal{S}$ at x_0 (definitions will be described in the next section). So, the upper estimation of $N_F(\mathcal{S}, x_0)$ with respect to $\nabla g_j(x_0)$, $\nabla G_i(x_0)$, and $\nabla H_i(x_0)$ is required to express the first-order necessary optimality condition for (Δ). Since appropriate approximations for MPVCs and MPECs where made in [1, 4, 5, 7, 8, 13, 14], and so far this has not been done for MPSCs, we will address this issue for the first time in this paper.

It should be noted that the necessary optimality conditions introduced in [12] used the approximation of the problem for the smooth MPSCs. Further, this method is generalized for the nonsmooth case in [2, 3]. This article presents another method to reach the necessary optimality conditions for smooth MPSCs, which is based on the estimating of the Frèchet normal cone of $\mathcal{S}$, and its generalization to the nonsmooth case (if possible) requires independent research. As a reference, we note that the results of this article are based on Theorem 3.1 and this theorem does not work in the nonsmooth case.

We organize the paper as follows. In the next section, we provide the preliminary results to be used in the rest of the paper. In Section 3 we will seek to find some suitable upper approximations for Frèchet normal cone of the feasible set of MPSCs under three kinds of constraint qualifications in Guignard type, and we will use these approximations to reach the three kinds necessary optimality conditions for (Δ), named strongly stationary condition, weakly stationary condition, and M-stationary condition.

2 Notations and preliminaries

In this section, we overview some notations and preliminary results from [16] that will be used throughout this paper.

The set of all non-negative (resp. non-positive) real numbers is shown by $\mathbb{R}_+$ (resp. $\mathbb{R}_-$), and the zero vector in $\mathbb{R}^n$ is denoted by 0_n. For a non-empty subset Ω of $\mathbb{R}^n$, its polar cone is defined as

$$\Omega^- := \{x \in \mathbb{R}^n \mid \langle x, y\rangle \leq 0, \quad \forall y \in \Omega\},$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner-product in $\mathbb{R}^n$. Also, $cone(\Omega)$, $\mathrm{cl}(\Omega)$, and $\overline{cone}(\Omega)$ denote the convex cone, the closure, and the closed convex cone of $M \subseteq \mathbb{R}^n$, respectively. With convention $\emptyset^- = \mathbb{R}^n$, it is easy to see ([16, Section 14]) that Ω^- is a closed convex cone for each $\Omega \subseteq \mathbb{R}^n$, and

$$\left(\overline{cone}(\Omega)\right)^- = \Omega^-. \tag{2}$$

We recall from[16, Theorem 3.3] that

$$cone(\Omega) = \left\{ \sum_{\kappa=1}^{s} \alpha_\kappa \omega_\kappa \mid s \in \mathbb{N},\ \alpha_\kappa \geq 0,\ \omega_\kappa \in \Omega \right\}. \tag{3}$$

The following theorems are recalled from [16].

Theorem 2.1. *If Ω is finite, then $cone(\Omega)$ is closed.*

Theorem 2.2. *Suppose that Ω_1 and Ω_2 are nonempty closed convex cones in $\mathbb{R}^n$. Then*

$$(\Omega_1 \cup \Omega_2)^- = \Omega_1^- \cap \Omega_2^-, \quad \text{and} \quad (\Omega_1 \cap \Omega_2)^- = \text{cl}\big(\Omega_1^- + \Omega_2^-\big).$$

Theorem 2.3. *If the linear function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is defined as $\varphi(x) = \langle a_0, x\rangle$ for a given $a_0 \in \mathbb{R}^n$, and $\Omega \subseteq \mathbb{R}$ is a given convex set, then*

$$\big(\varphi^{-1}\big(cl(\Omega)\big)\big)^- = \Omega^- a_0,$$

in which $\varphi^{-1}\big(cl(\Omega)\big) := \{x \in \mathbb{R}^n \mid \varphi(x) \in cl(\Omega)\}$.

The Bouligand tangent cone and the Fréchet normal cone of $\Omega \neq \emptyset$ at $x_0 \in \Omega$ are respectively defined as

$$\Gamma(\Omega, x_0) := \Big\{v \in \mathbb{R}^n \mid \exists t_s \downarrow 0,\ \exists v_s \to v \text{ such that } x_0 + t_s v_s \in \Omega\ \ \forall s \in \mathbb{N}\Big\},$$

$$N_F(\Omega, x_0) := \Big(\Gamma(\Omega, x_0)\Big)^-.$$

3 Main Results

As the starting point of this section, we state the following technical theorem that has a key rule in this paper.

Theorem 3.1. *If $K := \big\{y \in \mathbb{R}^q \mid \mathbf{A}y \leq 0_p,\ \mathbf{B}y = 0_r\big\}$ for some matrices $\mathbf{A} \in \mathbb{R}^{p\times q}$ and $\mathbf{B} \in \mathbb{R}^{r\times q}$, then*

$$K^- = \Big\{\mathbf{A}^\top \mu + \mathbf{B}^\top \eta \mid \eta \in \mathbb{R}^r,\ \mu \in \mathbb{R}^p_+\Big\}.$$

Proof. Let

$$\mathbf{A} := \begin{bmatrix} a_{11} & ... & a_{1q} \\ \vdots & & \vdots \\ a_{p1} & ... & a_{pq} \end{bmatrix}_{p\times q} \quad \text{and} \quad \mathbf{B} := \begin{bmatrix} b_{11} & ... & b_{1q} \\ \vdots & & \vdots \\ b_{r1} & ... & a_{rq} \end{bmatrix}_{r\times q}.$$

Also, for each $i = 1, ..., p$ and $j = 1, ..., r$ put

$$A_i := \begin{bmatrix} a_{i1} \\ \vdots \\ a_{iq} \end{bmatrix} \quad \text{and} \quad B_j := \begin{bmatrix} b_{j1} \\ \vdots \\ b_{jq} \end{bmatrix}.$$

So, we can write K as follows:

$$K = \left\{ y \in \mathbb{R}^q \;\middle|\; \begin{array}{ll} \langle A_i, y \rangle \leq 0, & \forall i = 1, ..., p \\ \\ \langle B_j, y \rangle = 0, & \forall j = 1, ..., r \end{array} \right\}.$$

We define the linear functions $\varphi_i, \psi_j : \mathbb{R}^q \to \mathbb{R}$, for $i \in \{1, ..., p\}$ and $j \in \{1, ..., r\}$, as

$$\varphi_i(y) := \langle A_i, y \rangle, \qquad \text{and} \qquad \psi_j := \langle B_j, y \rangle,$$

and rewrite K as follows:

$$\begin{aligned} K &= \Big\{ y \in \mathbb{R}^q \mid \varphi_i(y) \in \mathbb{R}_-, \ \ \forall i = 1, ..., p \quad \text{and} \quad \psi_j(y) = 0, \ \ \forall j = 1, ..., r \Big\} \\ &= \Big(\bigcap_{i=1}^{p} \varphi_i^{-1}(\mathbb{R}_-) \Big) \cap \Big(\bigcap_{j=1}^{r} \psi_j^{-1}(\{0\}) \Big). \end{aligned}$$

This equality and Theorem 2.2 conclude that

$$K^- = \mathrm{cl}\Big(\sum_{i=1}^{p} \big(\varphi_i^{-1}(\mathbb{R}_-) \big)^- \ + \ \sum_{j=1}^{r} \big(\psi_j^{-1}(\{0\}) \big)^- \Big). \tag{4}$$

Now, owing to Theorem 2.3, we get

$$\begin{aligned}
\sum_{i=1}^{p}\left(\varphi_i^{-1}(\mathbb{R}_-)\right)^- &= \sum_{i=1}^{p}(\mathbb{R}_-)^- A_i = \sum_{i=1}^{p}\mathbb{R}_+ A_i \\
&= \Big\{\sum_{i=1}^{p}\mu_i A_i \mid \mu_i \geq 0,\ i = 1, ..., p\Big\} \\
&= \Big\{\sum_{i=1}^{p}\mu_i \begin{bmatrix} a_{i1} \\ \vdots \\ a_{iq}\end{bmatrix} \mid \mu_i \geq 0,\ i = 1, ..., p\Big\} \\
&= \Big\{\begin{bmatrix} \mu_1 a_{11} + \mu_2 a_{21} + ... + \mu_p a_{p1} \\ \vdots \\ \mu_1 a_{1q} + \mu_2 a_{2q} + ... + \mu_p a_{pq}\end{bmatrix} \mid \mu_i \geq 0,\ i = 1, ..., p\Big\} \\
&= \Big\{\begin{bmatrix} a_{11} & ... & a_{p1} \\ \vdots & & \vdots \\ a_{1q} & ... & a_{qp}\end{bmatrix}\begin{bmatrix} \mu_1 \\ \vdots \\ \mu_p\end{bmatrix} \mid \mu_i \geq 0,\ i = 1, ..., p\Big\} \\
&= \Big\{\mathbf{A}^\top \mu \mid \mu \in \mathbb{R}^p_+\Big\}. \qquad (5)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\sum_{j=1}^{r}\left(\psi_i^{-1}(\{0\})\right)^- &= \sum_{j=1}^{r}(\{0\})^- B_i = \sum_{j=1}^{r}\mathbb{R} B_i \\
&= \Big\{\begin{bmatrix} \eta_1 b_{11} + \eta_2 b_{21} + ... + \eta_r b_{r1} \\ \vdots \\ \eta_1 b_{1q} + \eta_2 b_{2q} + ... + \eta_r b_{rq}\end{bmatrix} \mid \eta_j \in \mathbb{R},\ j = 1, ..., r\Big\} \\
&= \Big\{\begin{bmatrix} b_{11} & ... & b_{r1} \\ \vdots & & \vdots \\ b_{1q} & ... & b_{rq}\end{bmatrix}\begin{bmatrix} \eta_1 \\ \vdots \\ \eta_r\end{bmatrix} \mid \eta_j \in \mathbb{R},\ j = 1, ..., r\Big\} \\
&= \Big\{\mathbf{B}^\top \eta \mid \eta \in \mathbb{R}^r\Big\}.
\end{aligned}$$

The above equality, (4), and (5) imply that

$$K^- = \mathrm{cl}\bigg(\Big\{\mathbf{A}^\top \mu + \mathbf{B}^\top \eta \mid \eta \in \mathbb{R}^r,\ \mu \in \mathbb{R}^p_+\Big\}\bigg). \qquad (6)$$

On the other hand, considering (3), we deduce that

$$cone\Big(\big\{A_i, B_j, -B_j \mid i = 1, ..., p,\ j = 1, ..., r\big\}\Big) =$$
$$\Big\{\sum_{i=1}^{p}\mu_i A_i + \sum_{j=1}^{r}\hat{\eta}_j B_i + \sum_{j=1}^{r}\bar{\eta}_j(-B_i) \mid \mu_i, \hat{\eta}_j, \bar{\eta}_j \geq 0,\ i = 1, ..., p,\ j = 1, ..., r\Big\}$$
$$= \Big\{\mathbf{A}^\top\mu + \mathbf{B}^\top\eta \mid \eta \in \mathbb{R}^r,\ \mu \in \mathbb{R}^p_+\Big\},$$

in which $\eta_j := \hat{\eta}_j - \bar{\eta}_j$ for all $j = 1, ..., r$. The above equality, Theorem 2.1, and finiteness of set $\big\{A_i, B_j, -B_j \mid i = 1, ..., p,\ j = 1, ..., r\big\}$ conclude that the following set is closed:

$$\Big\{\mathbf{A}^\top\mu + \mathbf{B}^\top\eta \mid \eta \in \mathbb{R}^r,\ \mu \in \mathbb{R}^p_+\Big\}.$$

Consequently, (6) implies

$$K^- = \Big\{\mathbf{A}^\top\mu + \mathbf{B}^\top\eta \mid \eta \in \mathbb{R}^r,\ \mu \in \mathbb{R}^p_+\Big\},$$

as required. □

Considering a feasible point $\hat{x} \in \mathcal{S}$ (this point will be fixed throughout this paper), we define the following index sets:

$$\begin{aligned}
J_0 &:= \{j \in J \mid\ g_j(\hat{x}) = 0\},\\
I_G &:= \{i \in I \mid\ G_i(\hat{x}) = 0,\ H_i(\hat{x}) \neq 0\},\\
I_H &:= \{i \in I \mid\ G_i(\hat{x}) \neq 0,\ H_i(\hat{x}) = 0\},\\
I_{GH} &:= \{i \in I \mid G_i(\hat{x}) = 0,\ H_i(\hat{x}) = 0\}.
\end{aligned}$$

Suppose that the constraints of (Δ) have the following order:

$$\begin{aligned}
&g_1, g_2, ..., g_{|J_0|}, g_{|J_0|+1}, ..., g_{|J|},\\
&G_1H_1, G_2H_2, ..., G_{|I_G|}H_{|I_G|}, ..., G_{|I_G|+|I_H|}H_{|I_G|+|I_H|}, ..., G_{|I|}H_{|I|}.
\end{aligned}$$

Motivated by [2, 12], we define the following Guignard type constraint qualifications for MPSCs.

Definition 3.2. We say that (Δ) satisfies

- the Guignard constraint qualification (GCQ) at $\hat{x} \in \mathcal{S}$ if

$$L := \left\{ w \in \mathbb{R}^n \;\middle|\; \begin{array}{ll} \langle w, \nabla g_j(\hat{x})\rangle \leq 0, & j \in J_0 \\ \langle w, \nabla G_i(\hat{x})\rangle = 0, & i \in I_G \\ \langle w, \nabla H_i(\hat{x})\rangle = 0, & i \in I_H \end{array} \right\} \subseteq \overline{cone}\big(\Gamma(\mathcal{S}, \hat{x})\big).$$

- the weak-GCQ (WGCQ) at $\hat{x} \in \mathcal{S}$ if

$$L_1 := \left\{ w \in \mathbb{R}^n \;\middle|\; \begin{array}{ll} \langle w, \nabla g_j(\hat{x})\rangle \leq 0, & j \in J_0 \\ \langle w, \nabla G_i(\hat{x})\rangle = 0, & i \in I_G \cup I_{GH} \\ \langle w, \nabla H_i(\hat{x})\rangle = 0, & i \in I_H \cup I_{GH} \end{array} \right\} \subseteq \overline{cone}\big(\Gamma(\mathcal{S}, \hat{x})\big).$$

- the MPSC-GCQ at $\hat{x} \in \mathcal{S}$ if

$$L_2 := \left\{ w \in \mathbb{R}^n \;\middle|\; \begin{array}{ll} \langle w, \nabla g_j(\hat{x})\rangle \leq 0, & j \in J_0 \\ \langle w, \nabla G_i(\hat{x})\rangle = 0, & i \in I_G \\ \langle w, \nabla H_i(\hat{x})\rangle = 0, & i \in I_H \\ \langle w, \nabla G_i(\hat{x})\rangle \langle w, \nabla H_i(\hat{x})\rangle = 0, & i \in I_{GH} \end{array} \right\} \subseteq \overline{cone}\big(\Gamma(\mathcal{S}, \hat{x})\big).$$

It should be noted that the clear inclusions $L_1 \subseteq L_2 \subseteq L$ imply that the following implications are true at $\hat{x}$:

$$\text{GCQ} \implies \text{MPSC-GCQ} \implies \text{WGCQ}. \tag{7}$$

The following theorem introduces a broad and important class of MPSCs that satisfy MPSC-GCQ (and hence, WGCQ) at all of their feasible points.

Theorem 3.3. *Consider the following optimization problem with linear switching constraints:*

$$\begin{array}{lll} (\Theta): & \min & f(x) \\ & s.t. & \langle u_j, x\rangle \leq 0, \qquad j \in J, \\ & & \langle p_i, x\rangle\langle q_i, x\rangle = 0, \quad i \in I, \\ & & x \in \mathbb{R}^n, \end{array}$$

where, u_j, p_i, and q_i are nonzero vectors in $\mathbb{R}^n$ for all $j \in J$ and $i \in I$. This problem satisfies MPSC-GCQ at all of its feasible points.

Proof. At the first point, we recall that the considered problem can be written as (Δ) with the following data:

$$g_j(x) = \langle u_j, x\rangle, \quad G_i(x) = \langle p_i, x\rangle, \quad H_i(x) = \langle q_i, x\rangle.$$

Suppose that $\hat{x} \in \mathcal{S}$ and

$$w \in L_2 = \left\{ w \in \mathbb{R}^n \;\middle|\; \begin{array}{ll} \langle w, u_j\rangle \le 0, & j \in J_0 \\ \langle w, p_i\rangle = 0, & i \in I_G \\ \langle w, q_i\rangle = 0, & i \in I_H \\ \langle w, p_i\rangle\langle w, q_i\rangle = 0, & i \in I_{GH} \end{array} \right\}, \tag{8}$$

are arbitrarily given. Let $\mathcal{S}_1 := \{x \in \mathbb{R}^n \mid \langle x, u_j\rangle \le 0,\ j \in J\}$ and $\mathcal{S}_2 := \{x \in \mathbb{R}^n \mid \langle x, p_i\rangle\langle x, q_i\rangle = 0,\ i \in I\}$. If $t \ge 0$ and $j \in J_0$ are given, we have

$$\langle u_j, \hat{x} + tw\rangle = \underbrace{\langle u_j, \hat{x}\rangle}_{=0} + t \underbrace{\langle u_j, w\rangle}_{\le 0} \le 0. \tag{9}$$

If $j \in J \setminus J_0$, then $\langle u_j, \hat{x}\rangle < 0$, and so for some small non-negative $t \ge 0$ we have $\langle u_j, \hat{x}\rangle + t\langle u_j, w\rangle \le 0$, i.e., there exists a $\delta_j > 0$ such that

$$\langle u_j, \hat{x} + tw\rangle \le 0, \qquad \text{for } t \in [0, \delta_j).$$

This inequality and (9) imply that for all $0 \le t < \delta := \min\{\delta_j \mid j \in J \setminus J_0\} > 0$ and all $j \in J$, we have $\langle u_j, \hat{x} + tw\rangle \le 0$, and so

$$\hat{x} + tw \in \mathcal{S}_1, \qquad \text{for } t \in [0, \delta). \tag{10}$$

On the other hand, for all $t > 0$ we get

$$\begin{aligned} &\big(\langle p_i, \hat{x} + tw\rangle\big)\big(\langle q_i, \hat{x} + tw\rangle\big) = \\ &\quad \underbrace{\langle p_i, \hat{x}\rangle\langle q_i, \hat{x}\rangle}_{(a)} + t \underbrace{\langle p_i, \hat{x}\rangle\langle q_i, w\rangle}_{(b)} + t \underbrace{\langle p_i, w\rangle\langle q_i, \hat{x}\rangle}_{(c)} + t^2 \underbrace{\langle p_i, w\rangle\langle q_i, w\rangle}_{(d)}. \end{aligned}$$

For $i \in I_G$, we have $(a) = (b) = 0$, and $(c) = (d) = 0$ by $w \in L_2$. For $i \in I_H$, we have $(a) = (c) = 0$, and $(b) = (d) = 0$ by $w \in L_2$. For $i \in I_{GH}$, we have $(a) = (b) = (c) = 0$, and $(d) = 0$ by $w \in L_2$. Thus,

$$\big(\langle p_i, \hat{x} + tw\rangle\big)\big(\langle q_i, \hat{x} + tw\rangle\big) = 0, \quad \forall t \ge 0,\ i \in I,$$

and hence $\hat{x}+tw \in \mathcal{S}_2$. This inclusion with (10) and $\mathcal{S} = \mathcal{S}_1 \cap \mathcal{S}_2$ deduces that

$$\hat{x} + tw \in \mathcal{S}, \qquad \text{for } t \in [0, \delta),$$

and hence $w \in \Gamma(\mathcal{S}, \hat{x})$. Since w is an arbitrary element in L_2 and $\Gamma(\mathcal{S}, \hat{x}) \subseteq \overline{cone}\big(\Gamma(\mathcal{S}, \hat{x})\big)$, we get $L_2 \subseteq \overline{cone}\big(\Gamma(\mathcal{S}, \hat{x})\big)$, as required. □

The following example shows that the GCQ may not hold at the optimal solution of problem (Θ) in Theorem 3.3.

Example 3.4. Consider the following optimization problem:

$$\begin{aligned} \min \quad & -3x_1 - 4x_2 \\ s.t. \quad & x_1 + x_2 \leq 0, \\ & x_1 x_2 = 0. \end{aligned}$$

We can formalize this problem as (Θ) with the following data,

$$f(x_1, x_2) = \langle \begin{bmatrix} -3 \\ -4 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \rangle, \quad u_1(x_1, x_2) = \langle \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \rangle$$

$$p_1(x_1, x_2) = \langle \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \rangle, \quad q_1(x_1, x_2) = \langle \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \rangle.$$

Since $\mathcal{S} = \big(\mathbb{R}_- \times \{0\}\big) \cup \big(\{0\} \times \mathbb{R}_-\big)$, the optimal value of problem is attained at $\hat{x} = 0_2$. Clearly, $I = J = J_0 = I_{GH}$, and

$$L = \Big\{ \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \in \mathbb{R}^2 \mid w_1 + w_2 \leq 0 \Big\} \nsubseteq \mathbb{R}_- \times \mathbb{R}_- = \overline{cone}\big(\Gamma(\mathcal{S}, 0_2)\big).$$

Thus, GCQ does not hold at $\hat{x}$. Note that, since

$$L_2 = \Big\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix} \Big\}^- = \{0_2\},$$

the MPSC-GCQ (and hence, WGCQ) holds at $\hat{x}$.

Now, we can present our main results.

Theorem 3.5. *Suppose that GCQ holds at $\hat{x}$. Then*

$$N_F(\mathcal{S},\hat{x}) \subseteq \Bigg\{ \sum_{j=1}^{|J_0|} \lambda_j \nabla g_j(\hat{x}) + \sum_{i=1}^{|I|} \Big(\alpha_i \nabla G_i(\hat{x}) + \beta_i \nabla H_i(\hat{x})\Big) \;\Bigg|\; \begin{array}{ll} \lambda_j \geq 0, & j \in J_0 \\ \alpha_i = 0, & i \in I_H \cup I_{GH} \\ \beta_i = 0, & i \in I_G \cup I_{GH} \end{array} \Bigg\}.$$

Proof. According to GCQ and (2), we have

$$N_F(\mathcal{S},\hat{x}) = \big(\Gamma(\mathcal{S},\hat{x})\big)^- = \Big(\overline{cone}\big(\Gamma(\mathcal{S},\hat{x})\big)\Big)^- \subseteq L^-. \tag{11}$$

Let

$$\mathbf{g} := \begin{bmatrix} \frac{\partial g_1}{\partial x_1}(\hat{x}) & \dots & \frac{\partial g_1}{\partial x_n}(\hat{x}) \\ \vdots & & \vdots \\ \frac{\partial g_{|J_0|}}{\partial x_1}(\hat{x}) & \dots & \frac{\partial g_{|J_0|}}{\partial x_n}(\hat{x}) \end{bmatrix}_{|J_0|\times n},$$

$$\mathbf{G} := \begin{bmatrix} \frac{\partial G_1}{\partial x_1}(\hat{x}) & \dots & \frac{\partial G_1}{\partial x_n}(\hat{x}) \\ \vdots & & \vdots \\ \frac{\partial G_{|I_G|}}{\partial x_1}(\hat{x}) & \dots & \frac{\partial G_{|I_G|}}{\partial x_n}(\hat{x}) \end{bmatrix}_{|I_G|\times n}$$

$$\textit{and} \quad \mathbf{H} := \begin{bmatrix} \frac{\partial H_{|I_G|+1}}{\partial x_1}(\hat{x}) & \dots & \frac{\partial H_{|I_G|+1}}{\partial x_n}(\hat{x}) \\ \vdots & & \vdots \\ \frac{\partial H_{|I_G|+|I_H|}}{\partial x_1}(\hat{x}) & \dots & \frac{\partial H_{|I_G|+|I_H|}}{\partial x_n}(\hat{x}) \end{bmatrix}_{|I_H|\times n}.$$

Owing to

$$L = \Big\{ w \in \mathbb{R}^n \mid \mathbf{g}w \leq 0_{|J_0|},\ \mathbf{G}w = 0_{|I_G|},\ \mathbf{H}w = 0_{|I_H|} \Big\},$$

and Theorem 3.1, we deduce that

$$L^- = \Big\{ \mathbf{g}^\top \lambda + \mathbf{G}^\top \alpha + \mathbf{H}^\top \beta \mid \alpha \in \mathbb{R}^{|I_G|},\ \beta \in \mathbb{R}^{|I_H|},\ \lambda \in \mathbb{R}_+^{|J_0|} \Big\}. \tag{12}$$

Since

$$\mathbf{g}^\top\lambda+\mathbf{G}^\top\alpha+\mathbf{H}^\top\beta=$$

$$\begin{bmatrix} \lambda_1\frac{\partial g_1}{\partial x_1}(\hat{x})+...+\lambda_{|J_0|}\frac{\partial g_{|J_0|}}{\partial x_1}(\hat{x}) \\ \vdots \\ \lambda_1\frac{\partial g_1}{\partial x_n}(\hat{x})+...+\lambda_{|J_0|}\frac{\partial g_{|J_0|}}{\partial x_n}(\hat{x}) \end{bmatrix} + \begin{bmatrix} \alpha_1\frac{\partial G_1}{\partial x_1}(\hat{x})+...+\alpha_{|I_G|}\frac{\partial G_{|I_G|}}{\partial x_1}(\hat{x}) \\ \vdots \\ \alpha_1\frac{\partial G_1}{\partial x_n}(\hat{x})+...+\alpha_{|I_G|}\frac{\partial G_{|I_G|}}{\partial x_n}(\hat{x}) \end{bmatrix}$$

$$+\begin{bmatrix} \beta_{|I_G|+1}\frac{\partial H_{|I_G|+1}}{\partial x_1}(\hat{x})+...+\beta_{|I_G|+|I_H|}\frac{\partial H_{|I_G|+|I_H|}}{\partial x_1}(\hat{x}) \\ \vdots \\ \beta_{|I_G|+1}\frac{\partial H_{|I_G|+1}}{\partial x_n}(\hat{x})+...+\beta_{|I_G|+|I_H|}\frac{\partial H_{|I_G|+|I_H|}}{\partial x_n}(\hat{x}) \end{bmatrix}=$$

$$\sum_{j=1}^{|J_0|}\lambda_j\begin{bmatrix} \frac{\partial g_j}{\partial x_1}(\hat{x}) \\ \vdots \\ \frac{\partial g_j}{\partial x_n}(\hat{x}) \end{bmatrix}+\sum_{i=1}^{|I_G|}\alpha_i\begin{bmatrix} \frac{\partial G_i}{\partial x_1}(\hat{x}) \\ \vdots \\ \frac{\partial G_i}{\partial x_n}(\hat{x}) \end{bmatrix}+\sum_{i=|I_G|+1}^{|I_G|+|I_H|}\beta_i\begin{bmatrix} \frac{\partial H_i}{\partial x_1}(\hat{x}) \\ \vdots \\ \frac{\partial H_i}{\partial x_n}(\hat{x}) \end{bmatrix}=$$

$$\sum_{j=1}^{|J_0|}\lambda_j\nabla g_i(\hat{x})+\sum_{i=1}^{|I_G|}\alpha_i\nabla G_i(\hat{x})+\sum_{i=|I_G|+1}^{|I_G|+|I_H|}\beta_i\nabla H_i(\hat{x}),$$

by taking $\alpha_i=0$ for $i\in I_H\cup I_{GH}$ and $\beta_i=0$ for $i\in I_G\cup I_{GH}$, we have

$$\mathbf{g}^\top\lambda+\mathbf{G}^\top\alpha+\mathbf{H}^\top\beta=\sum_{j=1}^{|J_0|}\lambda_j\nabla g_i(\hat{x})+\sum_{i=1}^{|I|}\Big(\alpha_i\nabla G_i(\hat{x})+\beta_i\nabla H_i(\hat{x})\Big).$$

This equality, (11), and (12) imply that

$$N_F(\mathcal{S},\hat{x})\subseteq\Bigg\{\sum_{j=1}^{|J_0|}\lambda_j\nabla g_j(\hat{x})$$
$$+\sum_{i=1}^{|I|}\Big(\alpha_i\nabla G_i(\hat{x})+\beta_i\nabla H_i(\hat{x})\Big)\ \Bigg|\ \begin{array}{ll} \lambda_j\geq 0,\&j\in J_0 & \\ \alpha_i=0, & i\in I_H\cup I_{GH} \\ \beta_i=0, & i\in I_G\cup I_{GH} \end{array}\Bigg\}.$$

□

Since the proof of the following theorem is exactly the same as Theorem 3.5, we do not repeat it.

Theorem 3.6. *Suppose that WGCQ holds at $\hat{x}$. Then*

$$N_F(\mathcal{S},\hat{x}) \subseteq \left\{ \sum_{j=1}^{|J_0|} \lambda_j \nabla g_j(\hat{x}) + \sum_{i=1}^{|I|} \Big(\alpha_i \nabla G_i(\hat{x}) + \beta_i \nabla H_i(\hat{x}) \Big) \;\middle|\; \begin{array}{ll} \lambda_j \geq 0, & j \in J_0 \\ \alpha_i = 0, & i \in I_H \\ \beta_i = 0, & i \in I_G \end{array} \right\}.$$

Because L_2 does not have a representation like set K in Theorem 3.1, the proof of the following theorem is not exactly the same as Theorems 3.5 and 3.6.

Theorem 3.7. *Suppose that MPSC-GCQ holds at $\hat{x}$. Then*

$$N_F(\mathcal{S},\hat{x}) \subseteq \left\{ \sum_{j=1}^{|J_0|} \lambda_j \nabla g_j(\hat{x}) + \sum_{i=1}^{|I|} \Big(\alpha_i \nabla G_i(\hat{x}) + \beta_i \nabla H_i(\hat{x}) \Big) \;\middle|\; \begin{array}{ll} \lambda_j \geq 0, & j \in J_0 \\ \alpha_i = 0, & i \in I_H \\ \beta_i = 0, & i \in I_G \\ \alpha_i \beta_i = 0, & i \in I_{GH} \end{array} \right\}.$$

Proof. At starting of the proof, we observe that

$$\big\{ w \in \mathbb{R}^n \mid \langle w, \nabla G_i(\hat{x}) \rangle \langle w, \nabla H_i(\hat{x}) \rangle = 0,\ i \in I_{GH} \big\} =$$

$$\left\{ w \in \mathbb{R}^n \mid \exists \tilde{I} \subseteq I_{GH} \;\middle|\; \begin{array}{ll} \langle w, \nabla G_i(\hat{x}) \rangle = 0, & i \in \tilde{I}, \\ \langle w, \nabla H_i(\hat{x}) \rangle = 0, & i \in \big(I_{GH} \setminus \tilde{I} \big) \end{array} \right\},$$

and thus,

$$L_2 = \left\{ w \in \mathbb{R}^n \mid \exists \tilde{I} \subseteq I_{GH} \;\middle|\; \begin{array}{ll} \langle w, \nabla g_j(\hat{x}) \rangle \leq 0, & j \in J_0 \\ \langle w, \nabla G_i(\hat{x}) \rangle = 0, & i \in I_G \cup \tilde{I} \\ \langle w, \nabla H_i(\hat{x}) \rangle = 0, & i \in I_H \cup \big(I_{GH} \setminus \tilde{I} \big) \end{array} \right\}.$$

From the above equality and Theorem 3.1 we find some non-negative numbers $\lambda_j \geq 0$ for $j \in J_0$, as well as some real numbers α_i for $i \in\in I_G \cup \tilde{I}$ and β_i for $i \in I_H \cup \big(I_{GH} \setminus \tilde{I} \big)$ such that

$$L_2^- = \Big\{ \sum_{j=1}^{|J_0|} \lambda_j \nabla g_j(\hat{x}) + \sum_{i \in I_G \cup \tilde{I}} \alpha_i \nabla G_i(\hat{x}) + \sum_{i \in I_H \cup \big(I_{GH} \setminus \tilde{I} \big)} \beta_i \nabla H_i(\hat{x}) \mid \lambda_j \geq 0 \Big\}.$$

Taking $\alpha_i = 0$ as $i \in I_H \cup \big(I_{GH} \setminus \tilde{I}\big)$ and $\beta_i = 0$ as $i \in I_G \cup \tilde{I}$, we conclude that

$$L_2^- = \Big\{ \sum_{j=1}^{|J_0|} \lambda_j \nabla g_j(\hat{x}) + \sum_{i=1}^{|I|} \Big(\alpha_i \nabla G_i(\hat{x}) + \beta_i \nabla H_i(\hat{x})\Big) \mid \exists \tilde{I} \subseteq I_{GH} \left| \begin{array}{ll} \lambda_j \geq 0, & j \in J_0 \\ \alpha_i = 0, & i \in I_H \cup \big(I_{GH} \setminus \tilde{I}\big) \\ \beta_i = 0, & i \in I_G \cup \tilde{I} \end{array} \right. \Big\}$$

$$\subseteq \Big\{ \sum_{j=1}^{|J_0|} \lambda_j \nabla g_j(\hat{x}) + \sum_{i=1}^{|I|} \Big(\alpha_i \nabla G_i(\hat{x}) + \beta_i \nabla H_i(\hat{x})\Big) \left| \begin{array}{ll} \lambda_j \geq 0, & j \in J_0 \\ \alpha_i = 0, & i \in I_H \\ \beta_i = 0, & i \in I_G \\ \alpha_i\beta_i = 0, & i \in I_{GH} \end{array} \right. \Big\}. \tag{13}$$

Now, MPSC-GCQ, (2), and (13) conclude that

$$N_F(\mathcal{S}, \hat{x}) = \big(\Gamma(\mathcal{S}, \hat{x})\big)^- = \Big(\overline{conv}\big(\Gamma(\mathcal{S}, \hat{x})\big)\Big)^- \subseteq L_2^- \subseteq$$

$$\Big\{ \sum_{j=1}^{|J_0|} \lambda_j \nabla g_j(\hat{x}) + \sum_{i=1}^{|I|} \Big(\alpha_i \nabla G_i(\hat{x}) + \beta_i \nabla H_i(\hat{x})\Big) \left| \begin{array}{ll} \lambda_j \geq 0, & j \in J_0 \\ \alpha_i = 0, & i \in I_H \\ \beta_i = 0, & i \in I_G \\ \alpha_i\beta_i = 0, & i \in I_{GH} \end{array} \right. \Big\}$$

and the proof is complete. □

As applications of the above theorems, we state the KKT type necessary optimality condition for MPSCs as follows. Note that this optimality conditions are presented in [9, 10, 12] for the smooth case and in [2, 3] for the nonsmooth case, using other methods.

Theorem 3.8. *Let $\hat{x}$ be an optimal solution of (Δ).*

1. *If GCQ holds at $\hat{x}$, then we can find some coefficients λ_j, α_i, and*

β_i as $j \in J_0$ and $i \in I$, such that:

$$\begin{cases} \nabla f(\hat{x}) + \sum_{j=1}^{|J_0|} \lambda_j \nabla g_j(\hat{x}) + \sum_{i=1}^{|I|} \Big(\alpha_i \nabla G_i(\hat{x}) + \beta_i \nabla H_i(\hat{x})\Big) = 0_n, \\ \lambda_j \geq 0, \quad \text{for } j \in J_0, \\ \alpha_i = 0, \quad \text{for } i \in I_H \cup I_{GH}, \\ \beta_i = 0, \quad \text{for } i \in I_G \cup I_{GH}. \end{cases} \tag{14}$$

2. *If WGCQ holds at $\hat{x}$, then we can find some coefficients λ_j, α_i, and β_i as $j \in J_0$ and $i \in I$, such that:*

$$\begin{cases} \nabla f(\hat{x}) + \sum_{j=1}^{|J_0|} \lambda_j \nabla g_j(\hat{x}) + \sum_{i=1}^{|I|} \Big(\alpha_i \nabla G_i(\hat{x}) + \beta_i \nabla H_i(\hat{x})\Big) = 0_n, \\ \lambda_j \geq 0, \quad \text{for } j \in J_0, \\ \alpha_i = 0, \quad \text{for } i \in I_H, \\ \beta_i = 0, \quad \text{for } i \in I_G. \end{cases} \tag{15}$$

3. *If MPSC-GCQ holds at $\hat{x}$, then we can find some coefficients λ_j, α_i, and β_i as $j \in J_0$ and $i \in I$, such that:*

$$\begin{cases} \nabla f(\hat{x}) + \sum_{j=1}^{|J_0|} \lambda_j \nabla g_j(\hat{x}) + \sum_{i=1}^{|I|} \Big(\alpha_i \nabla G_i(\hat{x}) + \beta_i \nabla H_i(\hat{x})\Big) = 0_n, \\ \lambda_j \geq 0, \quad \text{for } j \in J_0, \\ \alpha_i = 0, \quad \text{for } i \in I_H, \\ \beta_i = 0, \quad \text{for } i \in I_G, \\ \alpha_i \beta_i = 0, \quad \text{for } i \in I_{GH}. \end{cases} \tag{16}$$

Proof. It is enough that we prove (14), and the proofs of (15) & (16) are similar. According to (1) and Theorem 5, we deduce that

$$-\nabla f(\hat{x}) \in N_F(\mathcal{S},\hat{x}) \subseteq \Bigg\{ \sum_{j=1}^{|J_0|} \lambda_j \nabla g_j(\hat{x}) + \sum_{i=1}^{|I|} \Big(\alpha_i \nabla G_i(\hat{x}) + \beta_i \nabla H_i(\hat{x})\Big) \;\Bigg|\; \begin{array}{ll} \lambda_j \geq 0, & j \in J_0 \\ \alpha_i = 0, & i \in I_H \cup I_{GH} \\ \beta_i = 0, & i \in I_G \cup I_{GH} \end{array} \Bigg\}.$$

Hence, there exist some scalars λ_i, α_i, and β_i as $i \in I$ and $j \in J$, satisfying (14). □

It is worth mentioning that condition (14) (res: (15) and (16)) is referred in [2, 3, 9, 10, 12] by "strongly stationarity condition" (res: "weakly stationarity condition" and "M-stationarity condition") at $\hat{x}$. The difference between these three stationary conditions is that the multipliers α_i and β_i as $i \in I_{GH}$ in M-stationarity are freer than in strong stationarity, and in weakly stationarity are freer than M-stationarity ("M" stands for Mordukhovich). Several examples that show the comparison between these three kinds of stationary conditions can be seen in [2, 12].

4 Conclusion

In this paper, we derived three kinds of Guignard type constraint qualifications as well as optimality conditions, named (weakly, strongly, M-) stationary conditions, for the mathematical programming with switching constraints involving continuously differentiable functions. The main results were focused on upper estimating the Frèchet normal cone of the feasible set of the considered problem.

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References

[1] A. Ansari Ardali, N. Movahedian, S. Nobakhtian, Optimality conditions for nonsmooth mathematical programs with equilibrium constraints using convexificators, *Optimization*, 65 (2014), 67-85 .

[2] F. Gorgini Shabankareh, N. Kanzi, K. Fallahi, J. Izadi, Stationarity in nonsmooth optimization with switching constraints. *Iranian Journal of Science and Technology*, Transactions A: Science, (2022), 1-9.

[3] F. Gorgini Shabankareh, N. Kanzi , J. Izadi, K. Fallahi Guignard Qualifications and Stationary Conditions for Mathematical Programming with Nonsmooth Switching Constraints. *Control and Optimization in Applied Mathematics*, 6 (2021), 23-35.

[4] T. Hoheisel, C. Kanzow, Stationarity conditions for mathematical programs with vanishing constraints using weak constraint qualifications, *Journal of Mathematical Analysis and Application*, 337 (2008), 292-310.

[5] T. Hoheisel, C. Kanzow, On the Abadie and Guignard constraint qualifications for mathematical programs with vanishing constraints, *Optimization*, 58 (2009), 431-448.

[6] C. Kanzow, P. Mehlitz and D. Steck, Relaxation schemes for mathematical programs with switching constraints, *Optimization Methods and Softwar*e, (2019) 1-36.

[7] S. Kazemi, N. Kanzi, Constraint qualifications and stationary conditions for mathematical programming with non-differentiable vanishing constraints. *Journal of Optimization Theory and Applications*, 179 (2018), 800-819.

[8] S. Kazemi, N. Kanzi, A. Ebadian, Estimating the Frechet Normal Cone in Optimization Problems with Nonsmooth Vanishing Constraints. *Iranian Journal of Science and Technology*, Transactions A: Science 43 (2019), 2299-2306.

[9] G. Li , L. Guo, Mordukhovich stationarity for mathematical programs with switching constraints under weak constraint qualifications, *Optimization*, (2022) DOI: 10.1080/02331934.2022.2038151.

[10] YC. Liang, JJ. Ye, Optimality conditions and exact penalty for mathematical programs with switching constraints, *Journal of Optimization Theory and Applications*, 191 (2021), 1-31.

[11] J. Lv, Z. Peng and Z. Wan, Optimality Conditions, Qualifications and Approximation Method for a Class of Non-Lipschitz Mathematical Programs with Switching Constraints, *Mathematics*, (2016) DOI: 10.3390/math9222915.

[12] P. Mehlitz, Stationarity conditions and constraint qualifications for mathematical programs with switching constraints, *Mathematical Programming*, 181 (2020), 149-186.

[13] N. Movahedian, Bounded Lagrange multiplier rules for general nonsmooth problems and application to mathematical programs with equilibrium constraints, *Journal of Global Optimization*, 67 (2017), 829-850.

[14] Y. Pandey , SK. Mishra, Duality for nonsmooth optimization problems with equilibrium constraints, using convexificators, *Journal of Optimization Theory and Applications*, 171 (2016), 694-707.

[15] Y. Pandey, V. Singh, On constraint qualifications for multiobjective optimization problems with switching constraints, V. Laha et al. (Eds.), *Optimization, Variational Analysis and Applications*, 283-306, Springer Proceedings in Mathematics & Statistics, (2021)

[16] RT. Rockafellar, B. Wets, *Variational Analysis*, Springer, Berlin (1998)

[17] V. Shikhman Topological approach to mathematical programs with switching constraints, *Set-Valued and Variational Analysis* 30 (2022), 335-354.

Zahra Jafariani

Ph.D .Candidate of Mathematics
Department of Mathematics
Payame Noor University
P.O. Box 19395-3697
Tehran, Iran
E-mail: z.jafariani@student.pnu.ac.ir

Nader Kanzi
Associate Professor of Mathematics
Department of Mathematics
Payame Noor University
P.O. Box 19395-3697
Tehran, Iran
E-mail: n.kanzi@pnu.ac.ir
E-mail: nad.kanzi@gmail.com

Maryam Naderi Parizi
Assistant Professor of Mathematics
Department of Mathematics
Payame Noor University
P.O. Box 19395-3697
Tehran, Iran
E-mail: M.Naderi.Parizi@pnu.ac.ir
E-mail:mrym_ndr@yahoo.com