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Original Research Paper

The Flexible Transmuted Record Type-Scale Mixture of Normal Family and its AR(1) Extension

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Abstract. In the present paper, a flexible skew version of the scale mixture of normal family is introduced based on the transmuted record type, called TRT-SMN, which seems suitable for handling any skewness and kurtosis in real data sets. Several properties of the TRT-SMN family are provided, including the moment generating function and r-th moments. The parameters of the new family are estimated through the ECME algorithm. Further to the elegant properties of the proposed family, the paper considers, in the time series context, a first-order autoregressive process with TRT-SMN distributed innovations. Some Monte Carlo simulation experiments are executed to assess the consistency of the ECME estimates. To further motivate its purpose, the proposed process is applied to analyze the series of COVID-19 incidence in Bavaria. The proposed AR(1) with TRT-SMN innovations yields superior fitting criteria compared to AR(1) process with Gaussian innovations.

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1 Introduction

Parametric distributions play an important role in statistical modeling and analyses. In the classical regression and time series frameworks, random errors are assumed to follow a normal distribution. The normal distribution has been applied widely due to its attractive features, such as simplicity, tractability, and convenient mathematical properties. However, some real data sets are not satisfied with the normality assumption and follow an asymmetric distribution with heavier tails. A feasible solution for this issue is data transformation, which confronts some difficulties, as pointed out by Azzalini and Capitanio [7]. The alternative procedure has been attracted by several researchers in the recent statistical literature, which fits an asymmetric model to real data sets.

In the last decades, introducing the skew or asymmetric distributions has been considered by several researchers. For recent accounts of the literature on skew distributions, see Arellano-Valle and Azzalini [4] and Adcock and Azzalini [2].

The scale mixture of normal (SMN) family is a broad class of continuous, unimodal, and symmetric distributions that can cover kurtosis by modifying the scale parameters of the normal distribution. The SMN family of distributions is used for modeling symmetric data with the thick-tailed that provides robust estimates, easy generation, and efficient computation of the EM algorithm for ML estimates. The SMN family can regulate the thickness of its tails, including the lighter or heavier tails than the normal distribution. However, it is common to find skewed and heavy-tailed structures in many actual phenomena. For this reason, it is necessary to have flexible distributions with suitable properties for fitting such kinds of data.

Moravveji et al. [19] concentrated on the Bayesian methodology for linear regression model according to the class of two-piece SMN distributions, which captures some common properties such as heavy-tails, asymmetric, and types of heteroscedasticity.

The SMN family handles only the thickness of tails of the distributions, thus they are still symmetric. Some developments have considered adopting an asymmetric distribution as the baseline distribution, for example, scale mixtures of the skew-normal family ([9]), skew scale mixtures of normal ([11]), and the generalization of them. Maleki and Arellano-Valle [17] considered the autoregressive (AR) model with the finite mixture of SMN and skew-normal innovations.

Shaw and Buckley [21] introduced a new family based on the quadratic rank transmutation map (QTRM) by the order statistics. The QRTM was applied to produce a new distribution via the upper order statistics, that generate the skewed probability distributions. The generated distributions using QRTM are called transmuted distributions, such as transmuted Weibull distribution (Aryal and Tsokos [6]), transmuted generalized inverse Weibull distribution (Khan and King [13]), transmuted log-logistic distribution (Aryal [5]), transmuted Weibull Lomax (Afify et al. [3]), and transmuted generalized Lomax (Abu El Azm et al. [1]).

Balakrishnan and He [8] introduced the transmuted record type (TRT) method to generate new distributions,and provided the TRT-exponential and TRT-Weibull distributions. By using distributions of the two lower record statistics of the inverse Rayleigh distribution, Tanış [22] considered the inverse Rayleigh baseline distribution with the transmuted lower record type transmution and provided several statistical properties of the mentioned distribution.

Tsai et al. [23] proposed the EM estimation of the parameters in a mixture model with the Weibull, generalized exponential and generalized Rayleigh distributions, under the type-I hybrid censored samples. They checked the performance of the EM estimators through the Monte Carlo simulations approach, and used three real data sets to illustrate the applications of the proposed models.

Hintz et al. [12] provided the randomized quasi-Monte Carlo and adaptive randomized quasi-Monte Carlo algorithms for computing the distribution and log-density functions of multivariate normal variance mixtures, respectively. They derived the EM-type algorithm for the estimation of parameters, called expectation conditional maximization either (ECME). The ECME-type algorithms break the optimization part into two steps and thus handle the parameters separately.

This paper focuses on the TRT construction strategy because it leads to skewed distributions and is compatible with one-side long-tailed data. The transmuted family of distributions is a specific case of extremal distributions [14]. Based on SMN and TRT families, a new distribution is introduced, which is applied for asymmetric (left-skewed or right-skewed) and tailed (flatty and peaky) data sets. The stochastic representation of both SMN and TRT families facilitates the statistical analysis of the SMN-TRT family. The other aspect of the paper is dedicated to the flexible AR(1) process with TRT-SMN innovations. The applicability of the proposed process is checked via the weekly incidence of COVID-19 data in Bavaria.

The rest of the paper is organized as follows. We discuss the SMN and TRT families used in constructing the new distribution in section 2. The introduction of the TRT-SMN family of skewed distributions and the estimation of the parameters based on the ECME algorithm are discussed in section 3. A new AR(1) process with TRT-SMN innovations is introduced in section 4, and several statistical properties of the proposed process are provided, comprehensively. In section 5, based on a certain case of the TRT-SMN family, we focus on the TRT-N-AR(1) process in the simulation approach and confirm the consistency of the estimates via the Monte Carlo simulation procedure. Finally, in section 6, the actual data sets of incidence of COVID-19 are considered for some illustrated proposes.

2 Some Elementary Concepts

In this section, some elementary definitions of the SMN and TRT families are reviewed that will be used for a new distribution construction with more flexible features compatible with the nature of real phenomena.

2.1 The Scale Mixture Normal Family

In the symmetric context, a family of thick-tailed distributions is introduced by Lange and Sinsheimer [15], which has the normal distribution as a particular case. A random variable follows the SMN family with the notation $M \sim \mathrm{SMN}(\mu, \sigma, \boldsymbol{\nu})$ if its probability density function (PDF) assumes the form

$$f_{\text{SMN}}(m, \mu, \sigma, \boldsymbol{\nu}) = \int_0^\infty \phi(m, \mu, u^{-1}\sigma^2)\, \mathrm{d}\mathrm{H}(u|\boldsymbol{\nu}), \tag{1}$$

where $\phi(., \mu, u^{-1}\sigma^2)$ is the normal PDF with mean μ and variance $u^{-1}\sigma^2$, also $\mathrm{H}(u|\boldsymbol{\nu})$ is the cumulative distribution function (CDF) of a positive random variable U and $\boldsymbol{\nu}$ is the vector of parameters.

The CDF of the SMN family is represented as

$$\begin{aligned} F_{\text{SMN}}(t, \mu, \sigma, \boldsymbol{\nu}) &= \int_{-\infty}^t \int_0^\infty \phi(m, \mu, u^{-1}\sigma^2)\, \mathrm{d}\mathrm{H}(u|\boldsymbol{\nu})\, \mathrm{d}m \qquad (2) \\ &= \int_0^\infty \Big[\int_{-\infty}^t \phi(m, \mu, u^{-1}\sigma^2)\, \mathrm{d}m \Big] \mathrm{d}\mathrm{H}(u|\boldsymbol{\nu}) \\ &= \int_0^\infty \Phi(t, \mu, u^{-1}\sigma^2)\, \mathrm{d}\mathrm{H}(u|\boldsymbol{\nu}) = E_U\big(\Phi(t, \mu, u^{-1}\sigma^2)\big), \end{aligned}$$

where $E_{_U}(.)$ means expectation over the random variable U.

Proposition 2.1. *Let* $M \sim \mathrm{SMN}(\mu, \sigma, \boldsymbol{\nu})$*,*

(i) If $U \sim \mathrm{H}(u|\boldsymbol{\nu})$*, then its stochastic representation is given by*

$$M|U = u \sim \mathrm{N}(\mu, u^{-1}\sigma^2),$$

(ii) The moment generating function (MGF) of M *is given by*

$$M_{_{\mathrm{SMN}}}(t) = e^{\mu t} E_{_U}\Big(e^{\frac{U^{-1}\sigma^2 t^2}{2}}\Big), \qquad t \in \mathbb{R}. \tag{3}$$

Proof. *(i) The joint PDF of* M *and* U *is obtained as*

$$f_{_{M,U}}(m, u) = \phi(m, \mu, u^{-1}\sigma^2) h(u|\boldsymbol{\nu}),$$

where $h(u|\boldsymbol{\nu})$ *is the PDF of the random variable* U*. So, the conditional probability function of* M *given* U *is computed as follows*

$$f_{M|U}(m|u) = \frac{\phi(m, \mu, u^{-1}\sigma^2) h(u|\boldsymbol{\nu})}{h(u|\boldsymbol{\nu})} = \phi(m, \mu, u^{-1}\sigma^2).$$

Hence, given $U = u$*, the random variable* M *has a normal distribution with parameters* $(\mu, u^{-1}\sigma^2)$*.*
(ii) Based on the iterated expectation, we have

$$\begin{aligned} M_{_{\mathrm{SMN}}}(t) &= E\big(E(e^{tM}|U)\big) = E\Big(e^{\mu t + \frac{U^{-1}\sigma^2 t^2}{2}}\Big) = e^{\mu t} \int_0^\infty e^{\frac{u^{-1}\sigma^2 t^2}{2}} \mathrm{d}\mathrm{H}(u|\boldsymbol{\nu}) \\ &= e^{\mu t} E_{_U}\Big(e^{\frac{U^{-1}\sigma^2 t^2}{2}}\Big), \end{aligned}$$

which completes the proof. $\square$

2.2 The Transmuted Record Type Family

The transmuted distributions constructed from the QRTM methodology have attracted much attention in recent years. The stochastic representation of the transmuted distributions as a mixture based on order statistics facilitates the generalization of the proposed distributions. We focus on the transmuted family based on the upper records. The stochastic mixture representations of the TRT distributions will be provided based on order statistics.

First, let us recall the definition of record values and their distributions.
Let $Y_1, Y_2, \ldots,$ be a sequence of random variables with absolutely continuous PDF $g_Y(y, \boldsymbol{\delta})$ and CDF $G_Y(y, \boldsymbol{\delta})$. Let $Y_{_{U(1)}}, Y_{_{U(2)}}, \ldots,$ be a sequence of upper record values, such that $U(1) = 1$, and for $n > 1$,

$$U(n) = \min\Big\{i : i > U(n-1), Y_i > Y_{_{U(n-1)}}\Big\},$$

where $\{U(n)\}$ and $\{Y_{U(n)}\}$ called upper record times and upper record sequences, respectively.

Shakil and Ahsanullah [20] obtained the first and second upper records distributions as follows

$$F_{Y_{_{U(1)}}}(y, \boldsymbol{\delta}) = P(Y_{_{U(1)}} \leq z) = G_{_Y}(z, \boldsymbol{\delta}), \tag{4}$$

$$F_{Y_{_{U(2)}}}(y, \boldsymbol{\delta}) = P(Y_{_{U(2)}} \leq z) = 1 - (1 - G_{_Y}(z, \boldsymbol{\delta}))\big(1 - \ln(1 - G_{_Y}(z, \boldsymbol{\delta}))\big), \tag{5}$$

where $G_{_Y}(.)$ is an arbitrary baseline distribution of Y with the vector of parameters $\boldsymbol{\delta}$. The PDF of $Y_{U(2)}$ is given by

$$f_{Y_{_{U(2)}}}(y, \boldsymbol{\delta}) = -g_{_Y}(z, \boldsymbol{\delta}) \ln\Big(1 - G_{_Y}(z, \boldsymbol{\delta})\Big).$$

The TRT method can be summarized as follows.
Let $Y_1, \ldots, Y_n$ be a random sample from the baseline distribution $G_Y(y, \boldsymbol{\delta})$. Let $Y_{U(1)}$ and $Y_{U(2)}$ be the first and second upper records based on the sample $Y_1, \ldots, Y_n$. Define the TRT random variable Z as follows

$$Z \stackrel{d}{=} \begin{cases} Y_{U(1)}, & \text{w.p. } \gamma \\ Y_{U(2)}, & \text{w.p. } 1-\gamma \end{cases}, \qquad 0 \leq \gamma \leq 1.$$

Hence, the CDF and PDF of the TRT family are shown as

$$f_{_{\text{TRT}}}(z, \boldsymbol{\delta}, \gamma) = \gamma g_{_Y}(z, \boldsymbol{\delta}) - (1-\gamma) g_{_Y}(z, \boldsymbol{\delta}) \ln(1 - G_{_Y}(z, \boldsymbol{\delta})),$$

$$F_{_{\text{TRT}}}(z, \boldsymbol{\delta}, \gamma) = G_{_Y}(z, \boldsymbol{\delta}) + (1-\gamma)(1 - G_{_Y}(z, \boldsymbol{\delta})) \ln(1 - G_{_Y}(z, \boldsymbol{\delta})).$$

In the following, we compute the MGF, mean and variance of the TRT family.

Proposition 2.2. *If $Z \sim TRT(\boldsymbol{\delta}, \gamma)$ with baseline distribution $G_{_Y}(z, \boldsymbol{\delta})$, the MGF of the TRT distribution is provided by*

$$M_Z(t) = \gamma M_{_Y}(t) - (1-\gamma) E_{_Y}\Big(e^{tY} \ln(1 - G_{_Y}(Y, \boldsymbol{\delta}))\Big).$$

Proof. *Consider the definition of the MGF,*

$$
\begin{aligned}
M_Z(t) &= \int_{-\infty}^{\infty} e^{tz} f_Z(z, \boldsymbol{\delta}, \gamma)\, \mathrm{d}z \\
&= \int_{-\infty}^{\infty} e^{tz} \Big(\gamma g_Y(z, \boldsymbol{\delta}) - (1-\gamma) g_Y(z, \boldsymbol{\delta}) \ln(1 - G_Y(z, \boldsymbol{\delta})) \Big)\, \mathrm{d}z \\
&= \gamma M_Y(t) - (1-\gamma) \int_{-\infty}^{\infty} e^{tz} g_Y(z, \boldsymbol{\delta}) \ln(1 - G_Y(z, \boldsymbol{\delta})) \Big)\, \mathrm{d}z \\
&= \gamma M_Y(t) - (1-\gamma) E_Y \Big(e^{tY} \ln(1 - G_Y(Y, \boldsymbol{\delta})) \Big).
\end{aligned}
$$

So, the proof is completed. □

Based on Proposition 2.2, the first and second moments of the TRT family are concluded as follows

$$
\begin{aligned}
E(Z) &= \gamma E_Y(Y) - (1-\gamma) E_Y \Big(Y \ln(1 - G_Y(Y, \boldsymbol{\delta})) \Big), \\
E(Z^2) &= \gamma E_Y(Y^2) - (1-\gamma) E_Y \Big(Y^2 \ln(1 - G_Y(Y, \boldsymbol{\delta})) \Big).
\end{aligned}
$$

Hence the r-th moment is corroborated as below

$$
E(Z^r) = \gamma E_Y(Y^r) - (1-\gamma) E_Y \Big(Y^r \ln(1 - G_Y(Y, \boldsymbol{\delta})) \Big).
$$

Subsequently, the variance of the TRT family is given by

$$
\begin{aligned}
Var(Z) = \gamma E_Y(Y^2) &- (1-\gamma) E_Y \Big(Y^2 \ln(1 - G_Y(Y, \boldsymbol{\delta})) \Big) - \gamma^2 E_Y^2(Y) \\
&- (1-\gamma)^2 E_Y^2 \Big(Y \ln(1 - G_Y(Y, \boldsymbol{\delta})) \Big) + 2\gamma(1-\gamma) E_Y(Y) E_Y \Big(Y \ln(1 - G_Y(Y, \boldsymbol{\delta})) \Big).
\end{aligned}
$$

3 The New TRT-SMN Family with the Estimation of Parameters

In this section, by considering the heavy-tailed SMN and asymmetric TRT families, we introduce a new family of distribution called TRT-SMN, which provides several prominent behaviors.

Regarding (4) and (5), we introduce the TRT-SMN distributions by considering the SMN baseline family. The CDF of TRT-SMN family is represented as

$$\begin{aligned}
F_Z(z,\mu,\sigma,\boldsymbol{\nu},\gamma) &= F_{\text{SMN}}(z,\mu,\sigma,\boldsymbol{\nu}) \\
&\quad + (1-\gamma)\big(1-F_{\text{SMN}}(z,\mu,\sigma,\boldsymbol{\nu})\big)\ln\Big(1-F_{\text{SMN}}(z,\mu,\sigma,\boldsymbol{\nu})\Big) \\
&= \int_0^\infty \Phi(z,\mu,u^{-1}\sigma^2)\,\mathrm{dH}(u|\boldsymbol{\nu}) + (1-\gamma)\Big(1-\int_0^\infty \Phi(z,\mu,u^{-1}\sigma^2)\,\mathrm{dH}(u|\boldsymbol{\nu})\Big) \\
&\quad \ln\Big(1-\int_0^\infty \Phi(z,\mu,u^{-1}\sigma^2)\,\mathrm{dH}(u|\boldsymbol{\nu})\Big) \\
&= \int_0^\infty \Phi(z,\mu,u^{-1}\sigma^2)\,\mathrm{dH}(u|\boldsymbol{\nu}) + (1-\gamma)\int_0^\infty \Phi(-z,\mu,u^{-1}\sigma^2)\,\mathrm{dH}(u|\boldsymbol{\nu}) \\
&\quad \ln\Big(\int_0^\infty \Phi(-z,\mu,u^{-1}\sigma^2)\,\mathrm{dH}(u|\boldsymbol{\nu})\Big),
\end{aligned} \tag{6}$$

where $F_{\text{SMN}}(z,\mu,\sigma,\boldsymbol{\nu})$ is the SMN baseline distribution with the parameters $(\mu,\sigma,\boldsymbol{\nu})$, defined in (2).

The PDF and hazard rate function (HRF) of the TRT-SMN family are represented as follows

$$\begin{aligned}
f_Z(z,\mu,\sigma,\boldsymbol{\nu},\gamma) &= f_{\text{SMN}}(z,\mu,\sigma,\boldsymbol{\nu})\Big[1+\gamma\Big(-\ln(1-F_{\text{SMN}}(z,\mu,\sigma,\boldsymbol{\nu}))-1\Big)\Big] \\
&= \int_0^\infty \phi(z,\mu,u^{-1}\sigma^2)\,\mathrm{dH}(u|\boldsymbol{\nu}) \\
&\quad \Big[1-\gamma\ln\big(\int_0^\infty \Phi(-z,\mu,u^{-1}\sigma^2)\,\mathrm{dH}(u|\boldsymbol{\nu})\big)-\gamma\Big],
\end{aligned} \tag{7}$$

where $f_{\text{SMN}}(z,\mu,\sigma,\boldsymbol{\nu})$ is the PDF of SMN distribution, defined in (1), and

$$\mathrm{HRF}_Z(z,\mu,\sigma,\boldsymbol{\nu},\gamma) = \frac{f_{\text{SMN}}(z,\mu,\sigma,\boldsymbol{\nu})}{1-F_{\text{SMN}}(z,\mu,\sigma,\boldsymbol{\nu})}\Big(\frac{1-\gamma-\gamma\ln\big(1-F_{\text{SMN}}(z,\mu,\sigma,\boldsymbol{\nu})\big)}{1-(1-\gamma)\ln\big(1-F_{\text{SMN}}(z,\mu,\sigma,\boldsymbol{\nu})\big)}\Big).$$

Here, some properties of the TRT-SMN family are investigated, such as moment generating function, non-central moments, first and second moments. It is worth mentioning that moments allow the computation of the skewness and kurtosis coefficients.

Proposition 3.1. *Consider Z from the TRT-SMN family with the parameters $(\mu,\sigma,\boldsymbol{\nu},\gamma)$ and (3), the MGF of the TRT-SMN family is computed as below*

$$M_Z(t) = e^{\mu t}\Big[\gamma E_U\Big(e^{\frac{U^{-1}\sigma^2 t^2}{2}}\Big) + (1-\gamma)\Big(\sum_{k=1}^{\infty}\frac{\int_{-\infty}^{\infty} E_U\Big(e^{t\sigma u^{\frac{-1}{2}}y}\Big)e^{-\frac{y^2}{2}}\Big(1+\mathrm{erf}(\frac{y}{\sqrt{2}})\Big)^k\,\mathrm{d}y}{2^k k\sqrt{2\pi}}\Big].$$

Proof. *First, we set* $\mu = 0, u^{-1}\sigma^2 = 1$*, and obtain the MGF of* $Y_{U(2)}$ *distribution as* $M_{Y_{U(2)}}(t) = E\Big(E(e^{tY_{U(2)}}|U)\Big)$*. Hence we need to compute* $E(e^{tY_{U(2)}}|U)$*. Consider the expansion of the logarithm function, so*

$$\begin{aligned} E(e^{tY_{U(2)}}|U) &= -\int_{-\infty}^{\infty} e^{ty}\phi(y)\ln(1-\Phi(y))\,\mathrm{d}y = \sum_{k=1}^{\infty}\frac{1}{k}\int_{-\infty}^{\infty} e^{ty}\phi(y)\Phi^k(y)\,\mathrm{d}y \\ &= \sum_{k=1}^{\infty}\frac{1}{2^k k\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ty-\frac{y^2}{2}}\Big(1+\mathrm{erf}(\frac{y}{\sqrt{2}})\Big)^k\,\mathrm{d}y, \end{aligned}$$

where $\phi(y), \Phi(y)$ *are PDF and CDF of standard normal, and* $\mathrm{erf}(.)$ *is the error function, defined as* $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,\mathrm{d}t$*. The last integral is solved by mathematical or statistical software. Therefore, the MGF with arbitrary parameters is obtained as*

$$M_{Y_{U(2)}}(t) = e^{\mu t}\Big(\sum_{k=1}^{\infty}\frac{1}{2^k k\sqrt{2\pi}}E_U\Big[\int_{-\infty}^{\infty} e^{t\sigma U^{\frac{-1}{2}}y-\frac{y^2}{2}}\Big(1+\mathrm{erf}(\frac{y}{\sqrt{2}})\Big)^k\,\mathrm{d}y\Big)\Big].$$

Consider the MGF of the TRT family with SMN baseline, so

$$\begin{aligned} M_Z(t) &= \gamma M_Y(t) - (1-\gamma)E_Y\Big(e^{tY}\ln(1-G_Y(Y,\boldsymbol{\delta}))\Big) \\ &= e^{\mu t}\Big[\gamma E_U\Big(e^{\frac{U^{-1}\sigma^2 t^2}{2}}\Big) + (1-\gamma)\Big(\sum_{k=1}^{\infty}\frac{\int_{-\infty}^{\infty}E_U\Big(e^{t\sigma U^{\frac{-1}{2}}y}\Big)e^{-\frac{y^2}{2}}\Big(1+\mathrm{erf}(\frac{y}{\sqrt{2}})\Big)^k\,\mathrm{d}y\Big)}{2^k k\sqrt{2\pi}}\Big], \end{aligned}$$

which completes the proof. $\square$

The r-th non-central moment of the TRT-SMN family is represented by

$$E(Z^r) = \gamma E_U\Big(Y^r|U\Big) + (1-\gamma)\Big(\mu + \sigma\sum_{k=1}^{\infty}\frac{\int_{-\infty}^{\infty}E_U\Big(U^{-1}e^{t\sigma U^{\frac{-1}{2}}y}\Big)e^{-\frac{y^2}{2}}\Big(1+\mathrm{erf}(\frac{y}{\sqrt{2}})\Big)^k\mathrm{d}y}{2^k k\sqrt{2\pi}}\Big).$$

Consequently, for the specified random variable U, we can compute the first to fourth moments of the TRT-SMN family. Based on the first and second moments, the variance of the TRT-SMN family is represented in the closed-form. The skewness and kurtosis are the shape measurements of distribution and are directly derived by the third and fourth moments. The positive values of skewness are witness to the right long-tailed, and negative values indicate the left long-tailed. Kurtosis is the flatty or peaky measurement of a distribution (amount of the probability in tails as heavy or thin tail).

All mentioned statistics for the special case of the random variable U of the TRT-SMN family are provided in section 4.

3.1 The EM-Type Algorithm Estimation Approach of the Parameters

There is a challenge with fitting the TRT-SMN family via the direct optimization method due to the complexity of the likelihood function. One solution is to formulate a hierarchical exhibition of the model and execute the EM-type algorithm in a complete-data structure with the augmented joint distribution expressed as several complete conditional distributions, (Dempster et al. [10]). However, it is not directly applicable to estimating the TRT-SMN family since the M-step involves intractable computations.

The M-step of EM can be modified by a sequence of conditional maximization steps (CM-steps) when individuals confront the intractable analytic, referred as the ECM algorithm (Meng and Rubin [18]). The ECME algorithm is achieved by maximizing the conditional expectation of complete data with some CM-steps that maximize the corresponding constrained log-likelihood function, called the CML-steps, (Liu and Rubin [16]).

In this section, the estimation of the parameters of the TRT-SMN family is investigated via the ECME algorithm.

Note that from Proposition 2.1 and the definition of the TRT family, it follows that

$$Z_i|U_i = u_i, W_i = w_i \sim w_i\phi(z_i, \mu, u^{-1}\sigma^2) - (1 - w_i)\phi(z_i, \mu, u^{-1}\sigma^2)\ln\Big(1 - \Phi(z_i, \mu, u^{-1}\sigma^2)\Big),$$

$$U_i \sim H(U|\boldsymbol{\nu}), \qquad W_i \sim \text{Ber}(\gamma), \qquad i = 1, 2, \ldots, n,$$

where W and U are independent, and the mixture component $W_i = 1$ with probability γ leads to the $Y_{U(1)}$ and $W_i = 0$ includes that $Y_{U(1)}$.

Letting $\boldsymbol{z} = (z_1, \ldots, z_n)'$, $\boldsymbol{u} = (u_1, \ldots, u_n)'$, $\boldsymbol{w} = (w_1, \ldots, w_n)'$ and treating $\boldsymbol{u}$ and $\boldsymbol{w}$ as missing data, it follows that the complete log-likelihood function associated with $\boldsymbol{z}_C = (\boldsymbol{z}', \boldsymbol{u}', \boldsymbol{w}')'$ is given by

$$\begin{aligned}
\ell_C\Big(\boldsymbol{\theta}|\boldsymbol{z}_C\Big) &= \sum_{i=1}^{n} \ln\Big(f(Z_i|U_i, W_i)h(U_i|\boldsymbol{\nu})P(W_i|\gamma)\Big) \\
&= \sum_{i=1}^{n} \ln(\phi(Z_i, \mu, u^{-1}\sigma)) + \sum_{i=1}^{n} \ln\Big(w_i - (1 - w_i)\ln(1 - \Phi(Z_i, \mu, u^{-1}\sigma))\Big) \\
&\quad + \ln(\gamma)\sum_{i=1}^{n} w_i + \ln(1-\gamma)(n - \sum_{i=1}^{n} w_i) + \sum_{i=1}^{n} \ln(h(U_i|\boldsymbol{\nu})
\end{aligned}$$

$$= K - \frac{n}{2}\ln(\sigma^2) - \sum_{i=1}^{n}\frac{(z_i-\mu)^2 u_i}{2\sigma^2} + \sum_{i=1}^{n}\ln\Big(w_i - (1-w_i)\ln\big(1-\Phi(Z_i,\mu,u^{-1}\sigma)\big)\Big)$$
$$+ \ln(\gamma)\sum_{i=1}^{n} w_i + \ln(1-\gamma)(n - \sum_{i=1}^{n} w_i) + \sum_{i=1}^{n}\ln(h(U_i|\boldsymbol{\nu}),$$

where K is a constant independent of unknown parameters and $\boldsymbol{\theta} = (\mu, \sigma, \boldsymbol{\nu}, \gamma)'$ is the vector of unknown parameters. Given the current estimate $\hat{\boldsymbol{\theta}}^{(k)} = (\hat{\mu}^{(k)}, \hat{\sigma}^{(k)}, \hat{\boldsymbol{\nu}}^{(k)}, \hat{\gamma}^{(k)})'$, the E-step calculates the function

$$Q(\boldsymbol{\theta}|\hat{\boldsymbol{\theta}}) = E\Big[\ell_C\Big(\boldsymbol{\theta}|\boldsymbol{z}_C, \hat{\boldsymbol{\theta}}^{(k)}\Big)\Big] = \sum_{i=1}^{n} Q_{1i}(\boldsymbol{\theta}_1, \hat{\boldsymbol{\theta}}^{(k)}) + \sum_{i=1}^{n} Q_{2i}(\boldsymbol{\nu}, \hat{\boldsymbol{\theta}}^{(k)}),$$

with $\boldsymbol{\theta}_1 = (\mu, \sigma^2, \gamma)'$, $Q_{2i}(\boldsymbol{\nu}, \hat{\boldsymbol{\theta}}^{(k)}) = E\Big[\ln\Big(h(U_i|\boldsymbol{\nu})|\boldsymbol{y}, \hat{\boldsymbol{\theta}}^{(k)}\Big)\Big]$ and

$$\begin{aligned}
Q_{1i}(\boldsymbol{\theta}_1, \hat{\boldsymbol{\theta}}^{(k)}) &= -\frac{n}{2}\ln(\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(z_i-\mu)^2 E\Big(u_i|z_i, \hat{\boldsymbol{\theta}}^{(k)}\Big) \\
&\quad + \sum_{i=1}^{n} E\Big(\ln\Big(w_i - (1-w_i)\ln\big(1-\Phi(Z_i,\mu,u^{-1}\sigma)\big)\Big)|z_i, \hat{\boldsymbol{\theta}}^{(k)}\Big) \\
&\quad + \ln(\gamma)\sum_{i=1}^{n} E\big(w_i|z_i, \hat{\boldsymbol{\theta}}^{(k)}\big) + \ln(1-\gamma)(n - \sum_{i=1}^{n} E\big(w_i|z_i, \hat{\boldsymbol{\theta}}^{(k)}\big) \\
&= -\frac{n}{2}\ln(\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(z_i-\mu)^2 E_{1,i} + \sum_{i=1}^{n} E_{2,i} + \ln(\gamma)\sum_{i=1}^{n} E_{3,i} \\
&\quad + \ln(1-\gamma)(n - \sum_{i=1}^{n} E_{3,i}),
\end{aligned}$$

where, we use the following notations

$$\begin{aligned}
E_{1,i} &= E\Big(u_i|z_i, \hat{\boldsymbol{\theta}}^{(k)}\Big), \\
E_{2,i} &= E\Big(\ln\Big(w_i - (1-w_i)\ln(1-\Phi(z_i|\mu, u_i^{-1}\sigma^2))\Big)|z_i, \hat{\boldsymbol{\theta}}^{(k)}\Big), \\
E_{3,i} &= E\big(w_i|z_i, \hat{\boldsymbol{\theta}}^{(k)}\big).
\end{aligned}$$

The CM-step then conditionally maximizes $Q(\boldsymbol{\theta}|\hat{\boldsymbol{\theta}})$ with respect to $\boldsymbol{\theta}$, obtaining a new estimate $\hat{\boldsymbol{\theta}}^{(k+1)}$, as described below:

CM-step. Update $\hat{\mu}^{(k+1)}$, $\hat{\sigma}^{(k+1)}$ and $\hat{\gamma}^{(k+1)}$ as

$$\hat{\mu}^{(k+1)} = \frac{\sum_{i=1}^{n} z_i E_{1,i}}{\sum_{i=1}^{n} E_{1,i}},$$

$$\widehat{\sigma^2}^{(k+1)} = \frac{\sum_{i=1}^{n} (z_i - \hat{\mu}^{(k+1)})^2 E_{1,i}}{n},$$

$$\hat{\gamma}^{(k+1)} = \frac{\sum_{i=1}^{n} E_{3,i}}{n}.$$

So, we need to compute $E_{1,i}$ and $E_{3,i}$.

$$E_{1,i} = E\Big(u_i|z_i, \hat{\boldsymbol{\theta}}^{(k)}\Big) = \int_0^\infty u_i f(u_i|z_i, \hat{\boldsymbol{\theta}}^{(k)})\,\mathrm{d}u_i = \frac{1}{f(z_i)} \int_0^\infty u_i f(z_i|u_i, \hat{\boldsymbol{\theta}}^{(k)})\,\mathrm{d}u_i$$

$$= \frac{1}{f(z_i)}$$

$$\int_0^\infty u_i \phi(z_i|\hat{\mu}^{(k)}, u_i^{-1}\widehat{\sigma^2}^{(k+1)})\Big(\hat{\gamma}^{(k)} - (1-\hat{\gamma}^{(k)})\ln(1-\Phi(z_i|\hat{\mu}^{(k)}, u_i^{-1}\widehat{\sigma^2}^{(k+1)}))\Big)\,\mathrm{dH}(u_i|\boldsymbol{\nu})$$

$$= \frac{\int_0^\infty u_i \phi(z_i|\hat{\mu}^{(k)}, u_i^{-1}\widehat{\sigma^2}^{(k+1)})\Big(\hat{\gamma}^{(k)} - (1-\hat{\gamma}^{(k)})\ln(1-\Phi(z_i|\hat{\mu}^{(k)}, u_i^{-1}\widehat{\sigma^2}^{(k+1)}))\Big)\,\mathrm{dH}(u_i|\boldsymbol{\nu})}{\int_0^\infty \phi(z_i|\hat{\mu}^{(k)}, u_i^{-1}\widehat{\sigma^2}^{(k+1)})\Big(\hat{\gamma}^{(k)} - (1-\hat{\gamma}^{(k)})\ln\Big(1-\Phi(z_i|\hat{\mu}^{(k)}, u_i^{-1}\widehat{\sigma^2}^{(k+1)})\Big)\Big)\,\mathrm{dH}(u_i|\boldsymbol{\nu})},$$

$$E_{3,i} = E\big(w_i|z_i, \hat{\boldsymbol{\theta}}^{(k)}\big) = \sum_{w_i=0}^{1} w_i f(w_i|z_i, \hat{\boldsymbol{\theta}}^{(k)}) = \frac{1}{f(z_i)} \sum_{w_i=0}^{1} w_i f(z_i|w_i, \hat{\boldsymbol{\theta}}^{(k)})$$

$$= \frac{1}{f(z_i)} \sum_{w_i=0}^{1} \int_0^\infty w_i f(z_i|u_i, w_i) P(W_i = w_i)\,\mathrm{dH}(u_i|\boldsymbol{\nu})$$

$$= \frac{\hat{\gamma}^{(k)}}{f(z_i)} \int_0^\infty w_i f(z_i|u_i, w_i = 1)\mathrm{dH}(u_i|\boldsymbol{\nu})$$

$$= \frac{\hat{\gamma}^{(k)}}{f(z_i)} \int_0^\infty \phi(z_i|\hat{\mu}^{(k)}, u_i^{-1}\widehat{\sigma^2}^{(k+1)})\,\mathrm{dH}(u_i|\boldsymbol{\nu})$$

$$= \frac{\hat{\gamma}^{(k)} \int_0^\infty \phi(z_i|\hat{\mu}^{(k)}, u_i^{-1}\widehat{\sigma^2}^{(k+1)})\,\mathrm{dH}(u_i|\boldsymbol{\nu})}{\int_0^\infty \phi(z_i|\hat{\mu}^{(k)} u_i^{-1}\widehat{\sigma^2}^{(k+1)})\Big(\hat{\gamma}^{(k)} - (1-\hat{\gamma}^{(k)})\ln\Big(1-\Phi(z_i|\hat{\mu}^{(k)}, u_i^{-1}\widehat{\sigma^2}^{(k+1)})\Big)\Big)\,\mathrm{dH}(u_i|\boldsymbol{\nu})}.$$

CML-step: The update of $\hat{\nu}^{(k+1)}$ depends on the conditional expectation $Q_{2i}(\boldsymbol{\nu}, \hat{\boldsymbol{\theta}}^{(k)})$, which is quite complicated. However, we can update $\hat{\nu}^{(k+1)}$ by the actual log-likelihood function maximization. Fix $\hat{\mu}^{(k+1)}, \hat{\sigma}^{(k+1)}$ and $\hat{\gamma}^{(k+1)}$, then update $\hat{\boldsymbol{\nu}}^{(k+1)}$ by optimizing the constrained log-likelihood function, i.e.,

$$\hat{\boldsymbol{\nu}}^{(k+1)} = \arg\max_{\boldsymbol{\nu}} \sum_{i=1}^{n} \log \Big[\phi(z_i|\hat{\mu}^{(k+1)}, u_i^{-1}\widehat{\sigma^2}^{(k+1)}) \Big(\hat{\gamma}^{(k+1)} - (1-\hat{\gamma}^{(k+1)}) \ln \big(1 - \Phi(z_i|\hat{\mu}^{(k+1)}, u_i^{-1}\widehat{\sigma^2}^{(k+1)})\big)\Big)\Big].$$

The iterations of the above algorithms are repeated until a suitable convergence rule is satisfied, e.g., $|\hat{\boldsymbol{\theta}}^{(k+1)} - \hat{\boldsymbol{\theta}}^{(k)}|$ is sufficiently small.

4 The New AR(1) Process with TRT-SMN Innovations

This section is dedicated to the new first-order autoregressive process based on the TRT-SMN innovations. The basic AR(1) process is defined as

$$X_t = \alpha X_{t-1} + Z_t, \qquad X_t \in \mathbb{R}, \quad t \geq 1, \tag{8}$$

where Z_t is a sequence of arbitrary distribution with $E(Z_t) = 0$ and $Var(Z_t) < \infty$. The process (8) is stationary if $|\alpha| < 1$.

We introduce a first-order autoregressive process based on the TRT-SMN innovations called TRT-SMN-AR(1), that is obtained based on the observations at the previous time and innovations. The $\{Z_t\}$ is a sequence of non-zero mean with the TRT-SMN family. In the rest of the paper, we consider the vector of the parameters as $\boldsymbol{\eta} = (\alpha, \boldsymbol{\theta})$.

Here, we derive the conditional PDF of the TRT-SMN-AR(1) process. Considering the conditional probability $P(X_t < j | i \leq X_{t-1} < i + d),\ d > 0$ and letting $d \to 0$, so

$$P(X_t < x_t | X_{t-1} = x_{t-1}) = P(Z_t \leq x_t - \alpha x_{t-1}) = F_z(x_t - \alpha x_{t-1}),$$

where $F_z(.)$ is the CDF of the TRT-SMN family, defined by (6). So, the conditional PDF is

$$f(x_t|x_{t-1}) = f_z(x_t - \alpha x_{t-1}), \tag{9}$$

since $f_z(.)$ is the PDF of the TRT-SMN family, defined in (7).

Proposition 4.1. *The conditional expectation and variance of the TRT-SMN-AR(1) process are obtained as follows*

$$E(X_t|X_{t-1}) = \alpha X_{t-1} + \mu_z,$$

and $Var(X_t|X_{t-1}) = \sigma_z^2$.

Proposition 4.2. *Suppose the TRT-SMN-AR(1) process $\{X_t\}$ is stationary, so*

$$E(X_t) = \frac{\mu_Z}{1-\alpha}, \qquad Var(X_t) = \frac{\sigma_Z^2}{1-\alpha^2}.$$

Proposition 4.3. *Let $\{X_t\}$ be a stationary process, satisfying (8),*

(i) The autocovariance and autocorrelation functions of the TRT-SMN-AR(1) process are given, respectively, as

$$\varsigma(k) = Cov(X_t, X_{t-k}) = \alpha^{|k|}\varsigma(0), \qquad \varrho(k) = Corr(X_t, X_{t-k}) = \alpha^{|k|},$$

where $\varsigma(0) = Var(X)$.

(ii) The spectral density function of the TRT-SMN-AR(1) process is represented as

$$f(\omega) = \frac{\varsigma(0)}{2\pi}\left[\frac{1-\alpha^2}{1+\alpha^2-2\alpha\cos(\omega)}\right], \qquad \pi \le \omega \le \pi.$$

Proof. *(i) The first-lag autocovariance of the TRT-SMN-AR(1) process is computed as*

$$\begin{aligned}\varsigma(1) = Cov(X_t, X_{t-1}) &= E\Big(Cov(X_t, X_{t-1}|X_{t-1})\Big) + Cov\Big(E(X_t|X_{t-1}), E(X_{t-1}|X_{t-1})\Big)\\ &= Cov\Big(\alpha X_{t-1} + \mu_Z, X_{t-1}\Big) = \alpha\,\varsigma(0).\end{aligned}$$

By induction, the k-lag autocovariance is included, and the autocorrelation function is obtained, subsequently.
(ii) Based on the autocorrelation function and definition of the spectral density function, we have

$$\begin{aligned}f(\omega) &= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\varsigma(k)e^{-i\omega k} = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\alpha^{|k|}\varsigma(0)e^{-i\omega k}\\ &= \frac{\varsigma(0)}{2\pi}\Big[1 + \sum_{k=1}^{\infty}\alpha^k e^{-i\omega k} + \sum_{k=1}^{\infty}\alpha^k e^{i\omega k}\Big]\\ &= \frac{\varsigma(0)}{2\pi}\left[\frac{1-\alpha^2}{1+\alpha^2-2\alpha\cos(\omega)}\right],\end{aligned}$$

where $i = \sqrt{-1}$ is the imaginary unit. $\square$

The unknown parameters of the model are estimated based on the conditional maximum likelihood method through a realization $X_1, X_2, \ldots, X_n$. Since the TRT-N-AR(1) model is a first-order Markov model, the log conditional likelihood function is derived using conditional density (9). Thus, the conditional log-likelihood function of observed values $x_1, x_2, \ldots, x_n$ with the vector of the parameters $\boldsymbol{\eta}$ is given by

$$\ell(\boldsymbol{\eta}) = \log L(\alpha, \mu, \sigma, \boldsymbol{\nu}, \gamma) = \sum_{t=2}^{n} \log P(X_t = x_t | X_{t-1} = x_{t-1}) = \sum_{t=2}^{n} \log f_Z(x_t - \alpha x_{t-1}).$$

The ECME estimates procedure for parameters $(\hat{\mu}^{(k+1)}, \hat{\sigma}^{(k+1)}, \hat{\boldsymbol{\nu}}^{(k+1)}, \hat{\gamma}^{(k+1)})$ are the same as section 3. The extra parameter α is added to the CML step, where we substitute z_t with $x_t - \alpha x_{t-1}$. So, we have

$$\hat{\alpha}^{(k+1)} = \frac{\sum_{t=1}^{n} x_t x_{t-1} E_{2,t}}{\sum_{t=1}^{n} x_{t-1}^2 E_{2,t}}.$$

5 Simulation of the TRT-SMN-AR(1) Process

In this section, as a submodel of the SMN family, we consider the normal distribution and introduce the TRT-N distribution from the TRT-SMN family, which has a skewed and kurtosis density shape. Also, the AR-TRT-N(1) process is defined.

5.1 A Special Class of TRT-SMN Family

The CDF, PDF and HRF of TRT-N distribution are represented, respectively, as

$$F_Z(z, \mu, \sigma, \gamma) = \Phi(z, \mu, u^{-1}\sigma^2) + (1-\gamma)\Big(1 - \Phi(z, \mu, u^{-1}\sigma^2) \ln\Big(1 - \Phi(z, \mu, u^{-1}\sigma^2)\Big),$$

$$f_Z(z, \mu, \sigma, \gamma) = \phi(z, \mu, u^{-1}\sigma^2)\Big[1 - \gamma \ln\big(1 - \phi(z, \mu, u^{-1}\sigma^2)\big) - \gamma\Big],$$

$$\mathrm{HRF}_Z(z, \mu, \sigma, \gamma) = \frac{\phi(z, \mu, u^{-1}\sigma^2)}{1 - \Phi(z, \mu, u^{-1}\sigma^2)}\Big(\frac{1 - \gamma - \gamma \ln\big(1 - \Phi(z, \mu, u^{-1}\sigma^2)\big)}{1 - (1-\gamma)\ln\big(1 - \Phi(z, \mu, u^{-1}\sigma^2)\big)}\Big).$$

The PDF plots of the TRT-N distribution are depicted in Figure 1, for different combinations of the parameters. Based on Figure 1, the PDF of TRT-N distribution shows unimodal shapes, with different kinds of skewness and kurtosis, which makes it a fascinating choice for any data set.

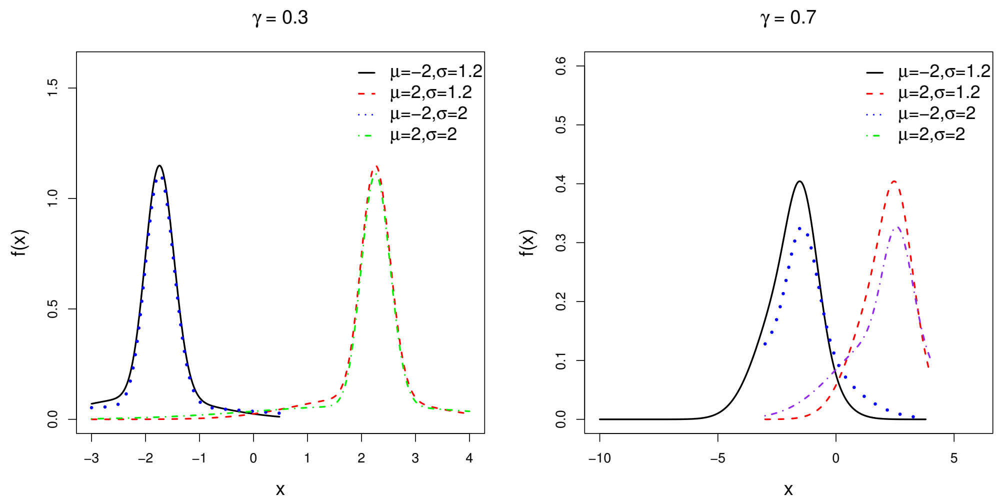

Figure 1: The PDF of TRT-N distribution for different combinations of parameters.

Based on the first to fourth moments, the mean, variance, skewness and kurtosis coefficients of the TRT-N distribution can be obtained in a closed-form. The statistical indices of the TRT-N distribution are provided in Table 1, for different combinations of the parameters. As can be seen, the TRT-N distribution covers the asymmetric data with both left and right-tailed shapes. The kurtosis coefficient less than 3, greater than 3, and almost near 3 indicate flatty (platykurtic), peaked (leptokurtic), and the same as normal distribution (mesokurtic) distribution, respectively. The results of Table 1 confirm the applicability of the TRT-N distribution for both flatty and peaky data sets. By increasing γ, the skewness and kurtosis coefficients are increased.

Table 1: Some statistical indices of the TRT-N distribution.

	$\gamma = 0.3$				$\gamma = 0.7$			
	(μ, σ)				(μ, σ)			
Measures	$(-2, 1.2)$	$(2, 1.2)$	$(-2, 2)$	$(2, 2)$	$(-2, 1.2)$	$(2, 1.2)$	$(-2, 2)$	$(2, 2)$
Mean	-1.242	2.754	-0.739	3.258	-1.675	2.323	-1.459	2.539
Variance	1.457	1.461	4.045	4.052	1.587	1.588	4.409	4.411
Sk. Coef.	2.341	-2.642	0.951	-1.256	5.075	-5.164	2.222	-2.312
Kur. Coef.	5.586	18.562	1.869	7.461	14.863	26.482	3.632	8.437

5.2 Data Generating Process for TRT-N Distribution

In this section, by considering the TRT-N innovation, we introduce a new AR(1) model, abbreviated as TRT-N-AR(1). The data generating process of the TRT-N distribution, for $t = 1, 2, \ldots, n$, is reported as follows, divided into two parts:

A. Generating innovations from TRT-N distribution:
1. Set $t = 1$ and simulate the random variable $Y_1, \ldots, Y_n$ from the normal distribution with parameters (μ, σ^2).
2. Compute $Y_{U(1)} = Y_1$ and $Y_{U(2)}$ form $(Y_1, \ldots, Y_n)$, where $Y_{U(2)}$ is obtained by "upper.record.values(sqnc=Y,k=1)[2]" command in the "Record" package in "R" software.
3. Generate random variable V from the uniform distribution over $(0, 1)$.
4. If $V = v < \gamma$, set $Z_t = Y_{U(1)}$ else, $Z_t = Y_{U(2)}$.

B. Generating data from AR(1) process:
1. Consider $X_0 = 0$.
2. Compute $X_t = \alpha X_{t-1} + Z_t$.

Set $t = t + 1$ and repeat parts A and B.

5.3 Simulation Results of the TRT-N-AR(1) Process

In this section, the performances of estimation of the TRT-N-AR(1) process are investigated. We perform the Monte Carlo simulation studies using different sample sizes with iteration 100 for two different combinations of the parameters as $(\alpha, \gamma, \mu, \sigma) = (0.3, 0.8, 2, 1.2)$ and $(\alpha, \gamma, \mu, \sigma) = (0.8, 0.3, -2, 2)$.

Table 2 represents the mean of the estimates and standard errors in the bracket for several values of the parameters with different sample sizes. As can be seen in Table 2, the estimates are convergent to their actual values, which results in the consistency of estimators. Also, increasing the sample size implies a smaller standard error.

Table 2: Simulation results for estimates of the TRT-N-AR(1) process with the standard errors in brackets.

	$(\alpha,\gamma,\mu,\sigma)=(0.3,0.8,2,1.2)$				$(\alpha,\gamma,\mu,\sigma)=(0.8,0.3,-2,2)$			
n	$\hat{\alpha}$	$\hat{\gamma}$	$\hat{\mu}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\gamma}$	$\hat{\mu}$	$\hat{\sigma}$
100	0.284838	0.784317	1.973515	1.231668	0.824221	0.288921	-1.973978	1.966895
St. err.	(0.059535)	(0.097075)	(0.225612)	(0.155364)	(0.088319)	(0.044388)	(0.263756)	(0.248451)
200	0.287439	0.789637	1.981588	1.212288	0.812611	0.304746	-2.026529	1.977777
St. err.	(0.041033)	(0.081162)	(0.153313)	(0.130632)	(0.076314)	(0.034186)	(0.230195)	(0.213671)
500	0.290398	0.790186	2.002129	1.192597	0.791361	0.302547	-1.983484	1.989686
St. err.	(0.024736)	(0.062409)	(0.142022)	(0.086729)	(0.057256)	(0.015513)	(0.173318)	(0.183376)
1000	0.298564	0.795637	1.996382	1.197897	0.803036	0.297052	-1.990008	2.007055
St. err.	(0.010944)	(0.042416)	(0.091192)	(0.075039)	(0.034775)	(0.088713)	(0.111537)	(0.139016)

6 Real Data Application

We consider the weekly incidence of COVID-19 from August 2021 to August 2022, with $n = 53$, in Bavaria, which is reported by the Robert Koch Institute: SurvStat@RKI 2.0, https://survstat.rki.de.

In this paper, the simulation and real data analysis are provided by "R" statistical software. Several descriptive statistics are provided in Table 3, including the mean, median and variance of the COVID-19 data set, along with the skewness and kurtosis coefficients. The COVID-19 data are asymmetric (skewed to the right) and leptokurtic. So, the non-Gaussian distribution is logical for the COVID-19 data series.

Table 3: Some statistical properties of COVID-19 real data set.

Data	Mean	Median	Variance	Skewness	Kurtosis
First data set	8.538	6.774	50.484	0.994	1.921

The difference transformation of the first order is considered to obtain stationary data set, which is confirmed by the Augmented Dicky Fuller test with p-value 0.031. The sample path and partial autocorrelation function (PACF) of the transformed COVID-19 data are represented in Figure 2. As can be seen in Figure 2, the proposed data have the first-order autoregressive structure.

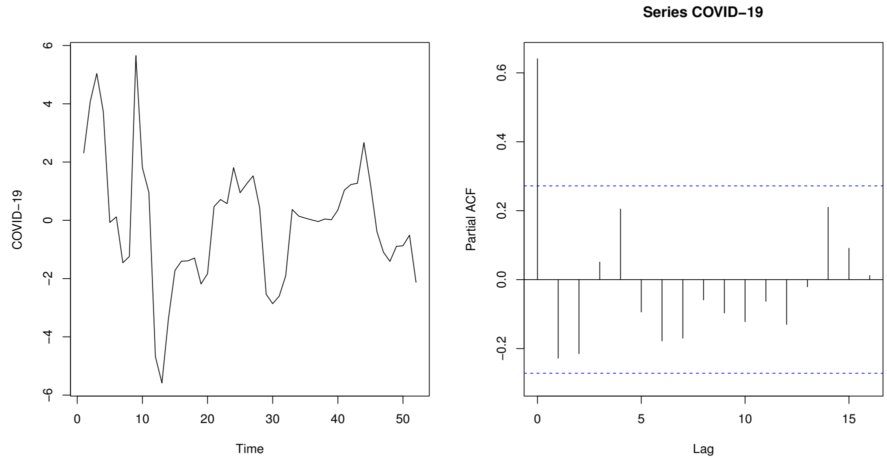

Figure 2: The sample path and PACF of COVID-19 data set.

The test of whether sample data have skewness and kurtosis matching a normal distribution is checked with the "jarque.bera.test" command in package "tseries". The p-value equals 0.0126 rejects the null hypothesis, which confirms the non-normality of the series, at a significant level of 5%.

The ML estimates of the parameters and selection measures (AIC and BIC) of the TRT-N-AR(1) and N-AR(1) are given in Table 4. Based on Table 4, the TRT-N-AR(1) model has the minimum selection measures for the COVID-19 data series. Thus, we can conclude that the TRT-N-AR(1) model has the best fit for the COVID-19 data.

Table 4: Estimated parameters and some selection measures for the COVID-19 data.

Model	MLE	AIC	BIC
TRT-N-AR(1)	$\hat{\alpha} = 0.68216,\ \hat{\gamma} = 0.38767,\ \hat{\mu} = 0.20435,\ \hat{\sigma} = 2.15888$	211.49	235.29
N-AR(1)	$\hat{\alpha} = 0.65355,\ \hat{\mu} = -0.09699,\ \hat{\sigma} = 1.61581$	287.67	305.53

Conclusion

We concentrated on a new family of distributions, which can handle the asymmetric and different kurtosis data sets. Several properties of the proposed family are discussed, along with the estimation of parameters by the ECME algorithm. Based on a certain case of the TRT-SMN family, we introduced a new autoregressive model with TRT-N innovations, called the TRT-N-AR(1) process, which can be applied to different kinds of real data sets. By considering the weekly incidence of COVID-19 in Bavaria, we represented the applicability of the proposed process. Based on the AIC and BIC measures, the TRT-N-AR(1) process is preferred over the N-AR(1).

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