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Original Research Paper

On the Ricci Curvature of a Homogeneous Finsler Space with Generalized m-Kropina Metric

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Abstract. In this paper, we consider homogeneous Finsler spaces with generalized m-Kropina metric. First, we find the formula for the Ricci curvature of a homogeneous Finsler space with aforesaid metric. Further, we derive the formula of the Ricci curvature for generalized m-Kropina metric having vanishing S-curvature. Finally, we derive the formula for the projective Ricci curvature of generalized m-Kropina Finsler metric.

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1 Introduction

Finsler geometry is just Riemannian geometry without quadratic restriction [8]. An important and interesting class of Finsler metrics is the class

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of (α, β)-metrics which has many applications in biology, physics and information geometry [1, 2, 3]. These metrics in Finsler geometry were first introduced by Matsumoto [17] in 1972. Some important examples of (α, β)-metric are Randers metric, Kropina metric, Matsumoto metric, square metric, generalized Kropina metric, etc. An (α, β)-metric on a Finsler manifold can be written in the form

$$F = \alpha\phi\left(\frac{\beta}{\alpha}\right),$$

where $\alpha := \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta := b_i(x)y^i$ is a one-form on a connected smooth manifold M. In particular, a Finsler metric in the form

$$F = \frac{\alpha^{m+1}}{\beta^m}\ (m \neq 0, -1)$$

is called generalized m-Kropina metric.

The main purpose of the present paper is to study the Ricci curvature of homogeneous Finsler spaces with generalized m-Kropina metric. Homogeneous Finsler spaces are an important class of Finsler spaces influenced by group theory. A Finsler space (M, F) in which a group of isometries $I(M, F)$ acts transitively on M is said to be homogenous. A homogeneous Finsler manifold M can be considered in the form $M = G/H$, where G is Lie group of isometries acting transitively on M and H is isotropy subgroup. The origin of homogeneous Riemannian spaces can be traced back to the Myers-Steenrod theorem in 1939 [18], which says that a group of isometries of a connected Riemannian manifold admits a differentiable structure that forms a Lie transformation group of the manifold. This theorem was groundbreaking since it broadened the application of Lie theory to all homogeneous Riemannian manifolds. Homogeneous spaces are a natural generalization of symmetric spaces, and they maintain many of their remarkable properties. Later, Deng and Hou [10] generalized the Myers-Steenrod theorem to the Finslerian case. This finding opens the door to applying Lie theory to the study of Finsler geometry. One of the central problems in Finsler spaces is to study curvatures such as flag curvature, Ricci curvature, S-curvature and mean Berwald curvature. In the recent past, many authors have worked on curvatures and other geometrical properties [11, 13, 15, 19].

The Ricci curvature plays an important role in Riemann-Finsler geometry. It provides control over the growth rate of volume of a metric ball in a manifold. Ricci curvature of a Finsler space (M, F) can be written as

$$Ric(x, y) = (n-1)\lambda(x, y)F^2,$$

where $\lambda(x, y)$ is a scalar function on tangent space TM, called Einstein scalar. Einstein metric is a Finsler metric whose Einstein scalar function $\lambda(x, y)$ depends only on x. Bao and Robles [5] have shown that every Randers metric of dimension $(n \geq 3)$ is Ricci constant. Zhou [25] first constructed the formula of the Riemannian curvature and Ricci curvature for (α, β)-metrics. Later, Cheng et al. [7] found some errors in those formulas given by Zhou and provided the correct formulas of Ricci curvature. They also proved that if $\phi(s)$ is a polynomial in s, then the (α, β)-metric is Einstein if and only if it is Ricci flat. Zhang and Shen [24] investigated Ricci curvature on Einstein Kropina metric. Yan and Deng [23] have studied homogeneous Einstein (α, β)-metric and obtained a formula for the Ricci curvature of a homogeneous Einstein (α, β)-metric. Deng and Liu [12] have shown that a homogeneous square Einstein Finsler metric is either Riemannian or flat.

As the S-curvature and the Ricci curvature are deeply related, it is obvious to consider the geometric quantities defined by Ricci curvature and S-curvature. Shen [20] showed that the local behaviour of the Busemann-Hausdorff measure of small metric balls around a point is determined by the S-curvature and the Ricci curvature. In this connection, Shen [21] also introduced the concept of projective Ricci curvature in Finsler geometry. The projective Ricci curvature of geodesic spray G is defined as

$$\mathbf{PRic}_G = Ric + (n-1)\tilde{S}_{|l}y^l + (n-1)\tilde{S}^2,$$

where $\tilde{S} = \dfrac{1}{n+1}S$ is projective S-curvature, $\tilde{G} = G + \dfrac{2S}{n+1}Y$ is projective geodesic spray and $Y = y^i\dfrac{\partial}{\partial y^i}$ is a vertical radial field on TM. It is easy to prove that if two Finsler metrics are projected pointwise on a manifold with a fixed volume form, then their projective Ricci curvatures are equal, i.e., the projective Ricci curvature of Finsler metrics on a manifold is projectively invariant with respect to a fixed volume form. Cheng, et. al. [6] investigated the formula of the projective Ricci curvature and characterized projectively Ricci-flat Randers metrics. Gabrani, et. al. [14] studied Finsler metrics with isotropic, weak and flat projectively Ricci curvature.

This paper is organize as follows:
The second section contains an overview of some basic definitions and results about Ricci curvature of Finsler manifolds. In section 3, we give an explicit formula for Ricci curvature of a homogeneous Finsler space with generalized m-Kropina metric. In the fourth section, we find a necessary and sufficient condition for a homogeneous Finsler space with generalized Kropina metric having vanishing S-curvature. Further, we find the formula for Ricci curvature of aforesaid metric with vanishing S-curvature. In section 5, we obtain the formula for the projective Ricci curvature of generalized m-Kropina metric and as a natural application, we characterize projectively Ricci-flat aforesaid metric defined by a Riemannian metric and constant length Killing 1-form.

2 Preliminaries

In this section, we give some basic definitions and results required for further study.

Definition 2.1. [4] A Finsler metric on an n-dimensional manifold M is a C^∞ function on slit tangent bundle $TM\backslash\{0\}$ which has the following properties:

- F is positively one-homogeneous, i.e., $F(x, \lambda y) = \lambda F(x, y)\ \forall\ \lambda > 0$,
- For each $y \in T_xM$, the Hessian metric $g_{ij} = \left[\frac{1}{2}F^2_{y^iy^j}\right]$ is positive-definite.

If F is a Finsler metric on a smooth manifold M, then the pair (M, F) is called Finsler space.

Lemma 2.2. *[4] Let $F = \alpha\phi(s)$; $s = \dfrac{\beta}{\alpha}$, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric, $\beta = b_i(x)y^i$ is a one-form and ϕ is a smooth function on an open interval $(-b_0, b_0)$. Then F is a Finsler metric if and if it has the following properties:*

$$\phi(s) > 0, \quad \phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0 \quad \forall \quad |s| < b \leq b_0.$$

Definition 2.3. [20] Let (M, F) be an n-dimensional Finsler space and G^i be the geodesic spray coefficients defined as

$$G^i = \frac{g^{ip}}{4}\Big[\big(F^2\big)_{x^k y^p} y^k - \big(F^2\big)_{x^p}\Big], \quad i = 1, 2, ..., n.$$

Also, Riemannian curvature is a linear map $R_y : T_xM \longrightarrow T_xM$ for non-zero $y \in T_xM$ defined by

$$R_y(w) = R^i_j(y) w^j \frac{\partial}{\partial x^i}, \quad w = w^i \frac{\partial}{\partial x^i},$$

where

$$R^i_j(y) = 2\frac{\partial G^i}{\partial x^j} - \frac{\partial^2 G^i}{\partial x^l \partial y^j} y^l + 2G^l \frac{\partial^2 G^2}{\partial y^l \partial y^j} - \frac{\partial G^i}{\partial y^l}\frac{\partial G^l}{\partial y^j}.$$

Definition 2.4. [5] Let M be an n-dimensional Finsler manifold with Finsler metric F. Ricci curvature of M is the trace of Riemannian curvature, i.e., the map $Ric : TM \longrightarrow \mathbb{R}$ defined as

$$Ric(y) = tr(R_y) \ \forall \ y \in T_xM,$$

where $x \in M, y(\neq 0) \in T_xM$.

Definition 2.5. [9] Let (M, F) be a Finsler space of dimension n. A Finsler metric F is called an Einstein metric if it satisfies

$$Ric = (n-1)\lambda F^2,$$

where $\lambda = \lambda(x)$ is a scalar function.
In particular, F is called Ricci constant if scalar function λ is a constant, and Ricci-flat if $\lambda = 0$.

For an (α,β)-metric $F=\alpha\phi(s),\ s=\dfrac{\beta}{\alpha}$ on a Finsler manifold M, let

$$
\begin{array}{lll}
r_{ik}=\frac{1}{2}(b_{i;k}+b_{k;i}), & s_{ik}=\frac{1}{2}(b_{i;k}-b_{k;i}), & r^i_k=a^{ip}r_{pk},\\
s^i_k=a^{ip}s_{pk}, & r_k=b^pr_{pk}=b_kr^k_i, & s_k=b^ps_{pk}=b_ks^k_i,\\
r=r_{ik}b^ib^k=b^ir_i, & r_{00}=r_{ik}y^iy^k, & r_{i0}=r_{ik}y^k,\\
s_{i0}=s_{ik}y^k, & r_0=r_iy^i, & s_0=s_iy^i,\\
q_{ij}=r_{ik}s^k_j, & t_{ij}=s_{ik}s^k_j, & q_j=b^iq_{ij}=r_ks^k_j,
\end{array}
$$

where $a^{ik}=(a_{ik})^{-1},\ b^i=a^{ik}b_k$ and " ; "denotes the covariant derivative with respect to Levi-Civita connection on Riemannian metric α.

Definition 2.6. [14] For a Finsler manifold (M,F) of dimension n, if $r_{ij}=0$, then the 1-form beta is said to be a Killing form on the Riemannian manifold (M,α). If the 1-form β is a Killing form and has a constant length with respect to α, then it is said to have a constant length Killing 1-form, i.e., $r_{ij}=0, s_j=0$.

Theorem 2.7. *[7] Let M be a Finsler manifold with an (α,β)-metric. Then Ricci curvature of F is given by $Ric={}^{\alpha}Ric+RT^j_j$, where*

$$
\begin{aligned}
RT^j_j=&\frac{1}{\alpha^2}\Big[(n-1)\xi_1+\xi_2\Big]r^2_{00}+\frac{1}{\alpha}\Big\{\Big[(n-1)\xi_3+\xi_4\Big]r_{00}s_0+\Big[(n-1)\xi_5\\
&+\xi_6\Big]r_{00}r_0+\Big[(n-1)\xi_7+\xi_8\Big]r_{00;0}\Big\}+\Big[(n-1)\xi_9+\xi_{10}\Big]s^2_0-\xi_{11}\big(r^2_{00}\\
&-rr_{00}\big)+\Big[(n-1)\xi_{12}+\xi_{13}\Big]r_0s_0+\xi_{14}\Big[r_{00;j}b^j-r_{0j;0}b^j+r_{00}r^j_j\\
&-r_{0j}r^j_0\Big]+\Big[(n-1)\xi_{15}+\xi_{16}\Big]r_{0j}s^j_0+\Big[(n-1)\xi_{17}+\xi_{18}\Big]s_{0;0}+\xi_{19}s_{0j}\\
&s^j_0+\alpha\Big\{\xi_{20}rs_0+\Big[(n-1)\xi_{21}+\xi_{22}\Big]s_js^j_0\Big\}+\alpha\Big[\xi_{23}\big(2r_js^j_0+3r^j_0s_j\\
&-2r^j_js_0-2s_{0;j}b^j+s_{j;0}b^j\big)+\xi_{24}s^j_{0;j}\Big]+\alpha^2\big(\xi_{25}s_js^j+\xi_{26}s^i_js^j_i\big),
\end{aligned}
\tag{1}
$$

where

$$\xi_1 = 2\Phi\Theta_s(B-s^2) - \Theta_s - 2s\Theta\Phi + \Theta^2,$$
$$\xi_2 = -(\Phi_s^2 - 2\Phi\Phi_{ss})(B-s^2)^2 - (\Phi_{ss} + 6s\Phi\Phi_s)(B-s^2) + 2s\Phi_s,$$
$$\xi_3 = -4(Q_s\Theta\Phi_s + 2Q\Phi\Theta_s)(B-s^2) - 4Q\Theta(\Theta - s\Phi) + 2(2Q\Theta_s + Q_s\Theta),$$
$$\begin{aligned}\xi_4 = {} & 4\big(\Phi_s^2 - Q_s\Phi\Phi_s - 2Q\Phi^2\Phi_{ss} - Q_{ss}\Phi^2\big)(B-s^2)^2 + 2(2Q\Phi_{ss} + 2Q_{ss}\Phi \\ & + Q_s\Phi s - 2Q\Phi^2 + 2sQ_s\Phi^2 - \Phi_{sB} + 10sQ\Phi\Phi_s)(B-s^2) - 10sQ\Phi_s \\ & + 2\Phi(Q - sQ_s) - 4\Phi s - Q_{ss},\end{aligned}$$
$$\xi_5 = -2\Theta_B + 4\Phi\Theta,$$
$$\xi_6 = -2(\Phi_{sB} - 2\Phi\Phi_s)(B-s^2) - 2\Phi_s,$$
$$\xi_7 = -\Theta,$$
$$\xi_8 = -\Phi_s(B-s^2),$$
$$\xi_9 = 8Q\Phi\big(Q\Theta_s + Q_s\Theta\big)(B-s^2) + 4Q\big(\Theta_B - Q_s + Q\Theta^2 - Q\Theta_s\big),$$
$$\begin{aligned}\xi_{10} = {} & \Big[2Q\Phi\big(Q_s\Phi_s + Q\Phi_{ss} + Q_{ss}\Phi\big) - Q^2\Phi_s^2 - Q_s^2\Phi^2\Big](B-s^2)^2 \\ & + 4\Big[Q_s\Phi_B - Q(Q\Phi_{ss} + Q_s\Phi_s) + Q\Phi_{sB} - \Phi(2QQ_{ss} - Q_s^2) \\ & - 4sQ\Phi(Q_s\Phi + Q_s\Phi_s)\Big](B-s^2) - 4Q^2\Phi(2 + s^2\Phi) \\ & + 2Q(Q_{ss} + 2s\Phi_B) + 4(2 + 3sQ)(Q\Phi_s + Q_s\Phi)\Phi s - Q_s^2,\end{aligned}$$
$$\xi_{11} = 4\Phi^2 + 4\Phi_B,$$
$$\xi_{12} = -4Q(2\Phi\Theta - \Theta_B),$$
$$\begin{aligned}\xi_{13} = {} & 4\big[Q\Phi_{sB} + Q_s\Phi_B - 2\Phi(Q\Phi_s - Q_s\Phi)\big](B-s^2) + 8sQ\Phi^2 + 4Q\Phi_s \\ & - 4(1 - sQ)\Phi_B,\end{aligned}$$
$$\xi_{14} = 2\Phi,$$
$$\xi_{15} = 4Q\Theta,$$
$$\xi_{16} = 4(Q\phi_s - Q_s\Phi)(B-s^2) - 2\phi(1 + 2sQ) + 2Q_s,$$
$$\xi_{17} = 2Q\Theta,$$
$$\xi_{18} = 2(Q_s\Phi + Q\phi_s)(B-s^2) - Q_s + 2sQ\Phi,$$
$$\xi_{19} = -2Q^2 + 2(1 + sQ)Q_s,$$
$$\xi_{20} = -8Q(\Phi^2 + \Phi_B),$$
$$\xi_{21} = -4Q^2\Theta,$$
$$\xi_{22} = -4Q^2\Phi_s(B-s^2) + 2Q\Phi,$$

$\xi_{23} = 2Q\Phi,$
$\xi_{24} = 2Q,$
$\xi_{25} = -4Q^2\Phi,$
$\xi_{26} = -Q^2,$

$$Q = \frac{\phi'}{\phi - s\phi'}, \Theta = \frac{\phi\phi' - s\phi\phi'' - s\phi'\phi'}{2\phi[\phi - s\phi' + (B - s^2)\phi'']}, \Phi = \frac{\phi''}{2[\phi - s\phi' + (B - s^2)\phi'']}.$$

Lemma 2.8. *[6] Let $F = \alpha\phi\left(\frac{\beta}{\alpha}\right)$ be an (α,β)-metric on a Finsler manifold M. Then the geodesic coefficient G^i of F is defined by*

$$G^i = \bar{G}^i + \alpha Q s^i_0 + \Theta(-2\alpha Q s_0 + r_{00})\frac{y^i}{\alpha} + \Phi(-2\alpha Q s_0 + r_{00})b^i, \qquad (2)$$

where $\bar{G}^i$ is geodesic coefficient of α.

3 Ricci Curvature of a Homogeneous Finsler Space with Generalized m-Kropina Metric

Let G/H be a homogeneous Finsler manifold with decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ and u be a G-invariant vector field of length c corresponding to 1-form β. Consider $\left\{u_1, u_2, u_3, ..., u_n = \frac{u}{c}\right\}$ be an orthonormal basis of $\mathfrak{m}$ w.r.t. $\langle\ ,\ \rangle$. The Christoffel symbol Γ^p_{ik} and the structure constants $\mathcal{C}^j_{ik}$ of $\mathfrak{g}$ are defined by

$$\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^k} = \Gamma^p_{ik}\frac{\partial}{\partial x^p}, \quad \mathcal{C}^j_{ik} = \langle u_j, [u_i, u_k]\rangle,$$

respectively. Let $\langle X, [u_i, u_k]\rangle$ is denoted by $\mathcal{C}^0_{ik}$. We require the following lemma for further use:

Lemma 3.1. *[11] At the origin eH, for the structure constant $\mathcal{C}^j_{ik} = \langle [u_i, u_k]_{\mathfrak{m}}, u_j\rangle$ of Lie algebra $\mathfrak{g}$ and*

$$f(i,k) = \begin{cases} 1, & i < k, \\ 0, & i \geq k, \end{cases}$$

where $u_i, u_j, u_k, u_p, u_q \in \mathfrak{m}$, *we have the following values:*

$$\Gamma^p_{ik} = f(i,k)\mathcal{C}^p_{ik} + \langle \nabla_{\hat{u}_i}\hat{u}_k, \hat{u}_p\rangle, \quad \langle \nabla_{\hat{u}_i}\hat{u}_k, \hat{u}_j\rangle = -\frac{1}{2}\big(\mathcal{C}^i_{kp} + \mathcal{C}^k_{ip} + \mathcal{C}^p_{ik}\big),$$
$$\langle \nabla_{\hat{u}_i}\hat{u}_k, \hat{u}_j\rangle \hat{u}_q = \frac{1}{2}\Big(\mathcal{C}^i_{qp}\mathcal{C}^p_{kj} + \mathcal{C}^k_{qp}\mathcal{C}^p_{ij} + \mathcal{C}^j_{qp}\mathcal{C}^p_{ik} + \mathcal{C}^p_{kj}\mathcal{C}^p_{qi} + \mathcal{C}^p_{ij}\mathcal{C}^p_{qk} + \mathcal{C}^p_{ik}\mathcal{C}^p_{qj}\Big),$$
$$b_k = c\delta_{nk}, \qquad s_{ik} = \frac{c}{2}\mathcal{C}^n_{ik}, \qquad s_i = \frac{c^2}{2}\mathcal{C}^n_{ni}, \qquad r_{ik} = -\frac{c}{2}(\mathcal{C}^k_{ni} + \mathcal{C}^i_{nk}),$$
$$s_{ik;j} = \frac{c}{2}\mathcal{C}^p_{ki}\mathcal{C}^n_{jp} + \frac{c}{4}\mathcal{C}^n_{ip}\big(\mathcal{C}^p_{qk} + \mathcal{C}^q_{kp} + \mathcal{C}^k_{qp}\big) + \frac{c}{4}\mathcal{C}^n_{pk}\big(\mathcal{C}^p_{qi} + \mathcal{C}^q_{ip} + \mathcal{C}^i_{qp}\big),$$
$$b_{i;k;q} = c\big(-\Gamma^i_{np}\langle \nabla_{\hat{u}_q}\hat{u}_k, \hat{u}_p\rangle - \Gamma^p_{nk}\langle \nabla_{\hat{u}_q}\hat{u}_i, \hat{u}_p\rangle + \mathcal{C}^p_{qn}\langle \nabla_{\hat{u}_p}\hat{u}_k, \hat{u}_i\rangle$$
$$+ \hat{u}_q\langle \nabla_{\hat{u}_n}\hat{u}_k, \hat{u}_i\rangle\big),$$
$$s_{i;k} = c\big(s_{ni;k} + \frac{a}{2}\mathcal{C}^n_{pi}\Gamma^p_{nk}\big), \quad r_{ik;q} = s_{ik;q} + b_{k;i;q}.$$

Using the above lemma, we calculate the quantities used in theorem 2.7 at the origin as follows:

$$r_{00} = -\frac{c}{2}(\mathcal{C}^0_{n0} + \mathcal{C}^0_{n0}) = -c\mathcal{C}^0_{n0}, \quad s_k = cs_{nk}, \quad s_0 = cs_{n0} = \frac{c^2}{2}\mathcal{C}^n_{n0},$$
$$r_k = b^l r_{lk} = a^{lj} r_{lk} = cr_{nk}, \quad r_0 = cr_{n0} - \frac{c^2}{2}\big(\mathcal{C}^n_{n0} + \mathcal{C}^0_{nn}\big) = -\frac{c^2}{2}\mathcal{C}^n_{n0},$$
$$r = r_{ik}b^i b^k = cr_n = -\frac{c^3}{2}\big(\mathcal{C}^n_{nn} + \mathcal{C}^n_{nn}\big) = 0, \;\; r_{00;0} = c\mathcal{C}^0_{k0}\big(\mathcal{C}^0_{nk} + \mathcal{C}^k_{n0}\big),$$
$$r^j_j = a^{pj} r_{jp} = -\frac{c}{2}a^{jp}\big(\mathcal{C}^p_{nj} + \mathcal{C}^j_{np}\big) = -\frac{c}{2}\big(\mathcal{C}^j_{nj} + \mathcal{C}^j_{nj}\big) = -c\mathcal{C}^j_{nj},$$
$$r_{0j}r^j_0 = r_{0j}r_{j0} = \frac{c^2}{4}\big(\mathcal{C}^0_{nj} + \mathcal{C}^j_{n0}\big)\big(\mathcal{C}^j_{n0} + \mathcal{C}^0_{nj}\big) = \frac{c^2}{4}\big(\mathcal{C}^0_{nj} + \mathcal{C}^j_{n0}\big)^2,$$
$$r_{00;j}b^j = r_{00;n}b^n = cr_{00;n} = \frac{c^2}{2}\big(\mathcal{C}^0_{qn} + \mathcal{C}^q_{0n}\big)\big(\mathcal{C}^0_{nq} + \mathcal{C}^q_{0n} + \mathcal{C}^n_{0q}\big),$$
$$r_{0j;0}b^j = r_{0n;0}b^n = cr_{0n;0} = \frac{c^2}{2}\Big[\mathcal{C}^0_{q0}\mathcal{C}^n_{nq} + \frac{1}{2}\big(\mathcal{C}^q_{0n} + \mathcal{C}^0_{qn} + \mathcal{C}^n_{q0}\big)\big(\mathcal{C}^q_{n0} + \mathcal{C}^0_{nq}\big)\Big],$$
$$r_{0j}s^j_0 = r_{0j}s_{j0} = -\frac{c^2}{4}\mathcal{C}^n_{j0}\big(\mathcal{C}^j_{n0} + \mathcal{C}^0_{nj}\big),$$
$$r_j s^j_0 = cr_{nj}s_{k0} = -\frac{c^3}{4}\mathcal{C}^n_{j0}\big(\mathcal{C}^0_{nj} + \mathcal{C}^j_{n0}\big),$$
$$s_{0;0} = cs_{n0;0} + \frac{c^2}{2}\Gamma^q_{n0}\mathcal{C}^n_{q0} = \frac{c^2}{2}\mathcal{C}^0_{0p}\mathcal{C}^n_{np}, \quad s_{oj}s^j_0 = s_{oj}s_{j0} = -\frac{c^2}{4}\big(\mathcal{C}^n_{0j}\big)^2,$$

$$
\begin{aligned}
&s_j s^j_0 = s_j s_{0j} = \frac{c^3}{4}\mathcal{C}^n_{j0}\mathcal{C}^n_{nj}, \quad s_j r^j_0 = s_j r_{j0} = \frac{c^3}{4}\mathcal{C}^n_{jn}\big(\mathcal{C}^0_{nj} + \mathcal{C}^j_{n0}\big),\\
&s_j s^j = \frac{c^4}{4}\big(\mathcal{C}^n_{nj}\big)^2, \quad s^j_{0;j} = s_{j0;j} = \frac{c}{2}\Big[\mathcal{C}^j_{jq}\mathcal{C}^n_{q0} + \frac{1}{2}\mathcal{C}^n_{jq}\big(\mathcal{C}^j_{0q} + \mathcal{C}^q_{0j} + \mathcal{C}^0_{jq}\big)\Big],\\
&s_{0;j}b^j = cs_{0;n} = \frac{c^3}{4}\mathcal{C}^n_{nq}\big(\mathcal{C}^0_{nq} + \mathcal{C}^q_{0n} + \mathcal{C}^n_{0q}\big),\\
&s_{j;0}b^j = cs_{n;0} = -\frac{c^3}{4}\mathcal{C}^n_{qn}\big(\mathcal{C}^0_{nq} + \mathcal{C}^q_{n0} + \mathcal{C}^n_{0q}\big),\\
&s^j_{0;j} = a^{ji}s_{i0;j} = s_{j0;j} = \frac{c}{4}\Big[\mathcal{C}^n_{jq}\big(\mathcal{C}^j_{0q} + \mathcal{C}^q_{0j} + \mathcal{C}^0_{jq}\big) + 2\mathcal{C}^j_{jq}\mathcal{C}^n_{q0}\Big],\\
&s_j s^j = s_j s_j = \frac{c^4}{4}\big(\mathcal{C}^n_{nj}\big)^2, \quad s^i_j s^j_i = s_{ji}s_{ij} = -\frac{c^4}{4}\big(\mathcal{C}^n_{ij}\big)^2.
\end{aligned}
$$

Using the above equations, (1) can be rewritten as:

$$
\begin{aligned}
RT^j_j =\ &\frac{c^2}{\alpha^2(z)}\big(\mathcal{C}^0_{0n}\big)^2\Big[(n-1)\xi_1 + \xi_2\Big] + \frac{c^3}{2\alpha(z)}\mathcal{C}^n_{n0}\mathcal{C}^0_{0n}\Big[(n-1)\big(\xi_3 - \xi_5\big)\\
&+ \xi_4 - \xi_6\Big] + \frac{c}{\alpha(z)}\mathcal{C}^0_{j0}\big(\mathcal{C}^0_{nj} + \mathcal{C}^j_{n0}\big)\Big[(n-1)\xi_7 + \xi_8\Big] + \frac{c^4}{4}\big(\mathcal{C}^n_{n0}\big)^2\\
&\Big[(n-1)\big(\xi_9 - \xi_{12}\big) + \xi_{10} - \xi_{11} - \xi_{13}\Big] + \frac{c^2}{4}\Big[4\mathcal{C}^j_{jn}\mathcal{C}^0_{0n} + 2\mathcal{C}^0_{0j}\mathcal{C}^j_{nj} + \big(\mathcal{C}^0_{jn}\\
&+ \mathcal{C}^j_{0n}\big)\big(\mathcal{C}^n_{0j} + 2\mathcal{C}^0_{nj} + 2\mathcal{C}^k_{0n}\big)\Big]\xi_{14} + \frac{c^2}{4}\mathcal{C}^n_{0j}\big(\mathcal{C}^0_{nj} + \mathcal{C}^j_{n0}\big)\Big[(n-1)\xi_{15} + \xi_{16}\Big]\\
&+ \frac{c^2\mathcal{C}^0_{0j}\mathcal{C}^n_{nj}}{2}\Big[(n-1)\xi_{17} + \xi_{18}\Big] - \frac{c^2}{4}\big(\mathcal{C}^n_{j0}\big)^2\xi_{19} + \frac{c^3}{4}\alpha(z)\big(\mathcal{C}^n_{nj}\mathcal{C}^n_{j0}\big)^2\\
&\Big[(n-1)\xi_{21} + \xi_{22}\Big] + \frac{c^3}{4}\alpha(z)\Big[4\mathcal{C}^n_{n0}\mathcal{C}^j_{nj} + \mathcal{C}^n_{jn}\big(\mathcal{C}^n_{j0} + 4\mathcal{C}^0_{nj}\big)\Big]\xi_{23} + \frac{c}{4}\alpha(z)\\
&\Big[\mathcal{C}^0_{jq}\mathcal{C}^n_{jq} + 2\mathcal{C}^j_{jq}\mathcal{C}^n_{q0}\Big]\xi_{24} + \frac{c^2}{4}\alpha^2(z)\Big[c^2\big(\mathcal{C}^n_{nj}\big)^2\xi_{25} - \big(\mathcal{C}^n_{ij}\big)^2\xi_{26}\Big].
\end{aligned}
\tag{3}
$$

Next, we have

$$
Q = -\frac{m}{(m+1)s}, \qquad Q_s = \frac{m}{(m+1)s^2}, \qquad Q_{ss} = -\frac{2m}{(m+1)s^3},
$$

$$
\Theta = -\frac{ms}{s^2 + (B - s^2)m}, \qquad \Phi = \frac{m}{2\big[s^2 + (B - s^2)m\big]},
$$

$$\Theta_s = \frac{m\big[s^2-(B+s^2)m\big]}{\big[s^2+(B-s^2)m\big]^2}, \qquad \Theta_B = \frac{m^2 s}{\big[s^2+(B-s^2)m\big]^2},$$

$$\Phi_s = \frac{m(m-1)s}{\big[s^2+(B-s^2)m\big]^2}, \qquad \Phi_B = \frac{-m^2}{2\big[s^2+(B-s^2)m\big]^2},$$

$$\Phi_{ss} = \frac{m(m-1)\big[3(m-1)s^2+Bm\big]}{\big[s^2+(B-s^2)m\big]^3}, \quad \Phi_{sB} = -\frac{-2m^2(m-1)s}{\big[s^2+(B-s^2)m\big]^3}.$$

Now, we calculate ξ_1 to ξ_{26} for generalized m-Kropina metric as follows:

$$\xi_1 = \frac{m^2\big[s^2-(B+s^2)m\big]}{\big[s^2+(B-s^2)m\big]^3}(B-s^2) - \frac{m\big[s^2-(B+3s^2)m\big]}{\big[s^2+(B-s^2)m\big]^2},$$

$$\xi_2 = -\frac{m^2(m-1)\big[2s^2-(B+2s^2m)\big]}{\big[s^2+(B-s^2)m\big]^4}(B-s^2)^2 - \frac{m(m-1)\big[6s^2m-3s^2+Bm\big]}{\big[s^2+(B-s^2)m\big]^3}(B-s^2) + \frac{2m(m-1)s^2}{\big[s^2+(B-s^2)m\big]},$$

$$\xi_3 = \frac{4m^3\big[s^2-(B+s^2)m\big]}{(m+1)s\big[s^2+(B-s^2)m\big]^3}(B-s^2) - \frac{4m^2s\big[2s^2-(B+2s^2)m\big]}{(m+1)s^2\big[s^2+(B-s^2)m\big]^2},$$

$$\xi_4 = \left[\frac{-2m^3(m-1)\big[5s^2-(B+5s^2)m\big]}{s\big[s^2+(B-s^2)m\big]^4} + \frac{2m^3}{s^3\big[s^2+(B-s^2)m\big]^2}\right]\frac{(B-s^2)^2}{(m+1)} - \left[\frac{10m^3(m-1)s}{\big[s^2+(B-s^2)m\big]^4} + \frac{4m^2(m-1)\big[4s^2+(B-2s^2)m\big]}{s\big[s^2+(B-s^2)m\big]^3} + \frac{2m^2\big[3s^2+2(B-2s^2)m\big]}{s^3\big[s^2+(B-s^2)m\big]^2}\right]\frac{(B-s^2)}{(m+1)} + \frac{2m}{(m+1)s^3} + \frac{2m\big[2(2m-1)(m-1)s^2-Bm^2\big]}{(m+1)s\big[s^2+(B-s^2)m\big]^2},$$

$$\xi_5 = -\frac{4m^2 s}{\big[s^2+(B-s^2)m\big]^2},$$

$$
\begin{aligned}
\xi_6 &= \frac{6m^2(m-1)s}{\big[s^2+(B-s^2)m\big]^3}(B-s^2) - \frac{2m(m-1)s}{\big[s^2+(B-s^2)m\big]^2},\\
\xi_7 &= \frac{ms}{\big[s^2+(B-s^2)m\big]},\\
\xi_8 &= -\frac{m(m-1)s}{\big[s^2+(B-s^2)m\big]^2}(B-s^2),\\
\xi_9 &= -\frac{8m^4(m-1)(B-s^2)}{(m+1)\big[s^2+(B-s^2)m\big]^3} - \frac{4m^3\big[2s^2-(B+s^2)m\big]}{(m+1)^2s^2\big[s^2+(B-s^2)m\big]^2}\\
&\quad + \frac{4m^2}{(m+1)^2s^3},\\
\xi_{10} &= \left[\frac{12m^4(m-1)^2s^2}{s^2\big[s^2+(B-s^2)m\big]^4} - \frac{3m^4}{s^4\big[s^2+(B-s^2)m\big]^2}\right]\frac{(B-s^2)^2}{(m+1)^2}\\
&\quad + \left[\frac{4m^3(m-1)\big[5s^2-(B+3s^2)m\big]}{s^2\big[s^2+(B-s^2)m\big]^3} + \frac{2m^3\big[(4m-3)s^2-3\big]}{s^4\big[s^2+(B-s^2)m\big]^2}\right]\\
&\quad \frac{(B-s^2)}{(m+1)^2} + \frac{m^2\big[5m^2-10m+8\big]}{(m+1)^2\big[s^2+(B-s^2)m\big]^2} - \frac{2m^2(3m-2)}{(m+1)s^2\big[s^2+(B-s^2)m\big]}\\
&\quad + \frac{3m^2}{(m+1)^2s^4},\\
\xi_{11} &= -\frac{m^2}{\big[s^2+(B-s^2)m\big]^2},\\
\xi_{12} &= -\frac{8m^3}{(m+1)\big[s^2+(B-s^2)m\big]^2},\\
\xi_{13} &= \frac{12m^3(m-1)}{(m+1)\big[s^2+(B-s^2)m\big]^3}(B-s^2) - \frac{2m^2(m-3)}{(m+1)\big[s^2+(B-s^2)m\big]^2},\\
\xi_{14} &= \frac{m}{\big[s^2+(B-s^2)m\big]},\\
\xi_{15} &= \frac{4m^2}{(m+1)\big[s^2+(B-s^2)m\big]},\\
\xi_{16} &= \frac{2m^2\big[s^2-(B+s^2)m\big](B-s^2)}{(m+1)s^2\big[s^2+(B-s^2)m\big]^2} + \frac{m(m-1)}{(m+1)\big[s^2+(B-s^2)m\big]}\\
&\quad + \frac{2m}{(m+1)s^2},\\
\xi_{17} &= \frac{2m^2}{(m+1)\big[s^2+(B-s^2)m\big]},
\end{aligned}
$$

$$\xi_{18} = \frac{m^2\big[3s^2+(B-3s^2)m\big](B-s^2)}{(m+1)s^2\big[s^2+(B-s^2)m\big]^2} - \frac{m^2}{(m+1)\big[s^2+(B-s^2)m\big]} - \frac{m}{(m+1)s^2},$$

$$\xi_{19} = -\frac{2m(m-1)}{(m+1)^2 s^2},$$

$$\xi_{20} = -\frac{2m^3}{(m+1)s\big[s^2+(B-s^2)m\big]^2},$$

$$\xi_{21} = \frac{4m^3}{(m+1)^2 s\big[s^2+(B-s^2)m\big]},$$

$$\xi_{22} = -\frac{4m^3(m-1)(B-s^2)}{(m+1)^2 s\big[s^2+(B-s^2)m\big]^2} - \frac{m^2}{(m+1)s\big[s^2+(B-s^2)m\big]},$$

$$\xi_{23} = -\frac{m^2}{(m+1)s\big[s^2+(B-s^2)m\big]},$$

$$\xi_{24} = -\frac{2m}{(m+1)s},$$

$$\xi_{25} = -\frac{2m^3}{(m+1)^2 s^2\big[s^2+(B-s^2)m\big]},$$

$$\xi_{26} = -\frac{m^2}{(m+1)^2 s^2}.$$

Using the above calculations, we get the following result:

Theorem 3.2. *Let F be a generalized m-Kropina metric on a homogeneous Finsler space (M,F). Then its Ricci curvature is given by*

$$\boldsymbol{Ric(z)} = {}^{\alpha}Ric(z) - \frac{mc^2\big(\mathcal{C}^0_{0n}\big)^2}{A^4\alpha^2(z)}\Big[m(m-1)A_2(B-s^2)^2 - A\big\{m(n-1)A_1 + (m-1)(A_6-3s^2)\big\}(B-s^2) - 2(m-1)s^2A^2 + (n-1)A^2(A_3-2s^2)\Big] + \frac{mc^3\mathcal{C}^n_{n0}\mathcal{C}^0_{0n}}{(m+1)s^3A^4\alpha(z)}\Big[m^2\big\{A^2-(m-1)s^2A_5\big\}(B-s^2)^2 - m\big\{5m(m-1)s^4 + 3(m^2-1)s^4A - 2ms^2AA_1 + A^2(2A_2-s^2) + 2(m-1)s^2A(A_2+2s^2)\big\}(B-s^2) + 2m(m+1)(n-1)s^4A^2 + 2(2m-1)(m-1)s^4A^2 - Bs^4A^2$$

$$+ A^4 + (m-1)s^4A^2\Big] + \frac{mcs\mathcal{C}^0_{j0}}{\alpha(z)A}\big(\mathcal{C}^0_{nj} + \mathcal{C}^j_{n0}\big)\Big[(n-1)A - (m-1)(B-s^2)\Big]$$

$$+ \frac{m^2c^4\big(\mathcal{C}^n_{n0}\big)^2}{4(m+1)^2s^4A^4}\Big[3m^2\big\{4(m-1)^2s^4 - A^2\big\}(B-s^2)^2 + 2A\big\{2m(m-1)s^2$$

$$(A_3 + 2s^2) - 4m^2(m^2-1)(n-1)s^4 + A(4m-3)s^2 - 3A - 6m(m^2-1)s^4\big\}$$

$$(B-s^2) + 3A^4 + 2(m-3)(m-1)s^2A^2 - 4(n-1)s^2A^2(A_1+s^2) + 8m$$

$$(m+1)(n-1)s^4A^2 + (m+1)^2s^4A^2 + (5m^2-10m+8)s^4A^4 + 4(n-1)s$$

$$A^4 - 2(3m-2)s^2A^3\Big] + \frac{mc^2}{4A}\Big[4\mathcal{C}^j_{jn}\mathcal{C}^0_{0n} + 2\mathcal{C}^0_{0j}\mathcal{C}^j_{nj} + \big(\mathcal{C}^0_{jn} + \mathcal{C}^j_{0n}\big)\big(\mathcal{C}^n_{0j} + 2\mathcal{C}^0_{nj}$$

$$+ 2\mathcal{C}^k_{0n}\big)\Big] + \frac{m(m-1)c^2\big(\mathcal{C}^n_{j0}\big)^2}{2(m+1)s^2} + \frac{mc^2\mathcal{C}^n_{0j}\big(\mathcal{C}^0_{nj} + \mathcal{C}^j_{n0}\big)}{4(m+1)s^2A^2}\Big[2A^2 + (4mn - 3m$$

$$-1)s^2A + 2mA_1(B-s^2)\Big] + \frac{mc^2\mathcal{C}^0_{0j}\mathcal{C}^n_{nj}}{2(m+1)s^2A^2}\Big[(2n-3m)s^2A + m(B-s^2)$$

$$A_3 - A^2\Big] + \frac{m^2c^3\alpha(z)\big(\mathcal{C}^n_{nj}\mathcal{C}^n_{j0}\big)^2}{4(m+1)^2sA^2}\Big[(4n-5)A - 4m(m-1)(B-s^2)\Big]$$

$$- \frac{m^2c^3\alpha(z)}{4(m+1)sA}\Big[4\mathcal{C}^n_{n0}\mathcal{C}^j_{nj} + \mathcal{C}^n_{jn}\big(\mathcal{C}^n_{j0} + 4\mathcal{C}^0_{nj}\big)\Big] - \frac{mc\alpha(z)}{2(m+1)s}\Big[\mathcal{C}^0_{jq}\mathcal{C}^n_{jq} + 2\mathcal{C}^j_{jq}\mathcal{C}^n_{q0}\Big]$$

$$- \frac{m^2c^2\alpha^2(z)}{4(m+1)^2s^2A}\Big[2mc^2\big(\mathcal{C}^n_{nj}\big)^2 - \big(\mathcal{C}^n_{ij}\big)^2A\Big],$$

where $z(\neq 0) \in \mathfrak{m}$,

$$A = s^2 + (B - s^2)m,$$

$$A_\eta = \eta s^2 - (B + \eta s^2)m.$$

Proof. As we know that the Ricci curvature of a homogeneous Finsler space (M, F) is defined as

$$Ric = {}^{\alpha}Ric + RT^j_j,$$

where RT^j_j is given by the equation (1).
Then, using the equation (3) and the values of $\xi_1, \xi_2, ..., \xi_{26}$ for generalized m-Kropina metric in the above equation, we get the required result. □

4 Ricci Curvature of a Homogeneous Generalized m-Kropina Metric with Vanishing S-Curvature

In this section, we first give a condition for a homogeneous Finsler space with G-invariant generalized Kropina metric to have vanishing S-curvature. Next, we calculate the formula for Ricci curvature of aforesaid metric having vanishing S-curvature.

Theorem 4.1. *Let $(G/H, F)$ be a homogeneous Finsler space with G-invariant generalized m-Kropina metric F. Then G/H has vanishing S-curvature if and only if $\langle [X,u]_{\mathfrak{m}}, X\rangle = 0 \ \forall\ X \in \mathfrak{m}$.*

Proof. Let F has vanishing S-curvature. As Shen and Cheng [22] showed that a Finsler metric F has vanishing S-curvature if and only if $r_{ik} = 0$ and $s_i = 0 \ \forall\ 1 \leq i, k \leq n$. Therefore,

$$\frac{c}{2}\left(\mathcal{C}^k_{in} + \mathcal{C}^i_{kn}\right) = 0, \tag{4}$$

and

$$\frac{c^2}{2}\mathcal{C}^n_{ni} = 0.$$

Taking $i = k$ in the equation (4), we get

$$\langle [u_i, u]_{\mathfrak{m}}, u_i\rangle = 0.$$

Therefore, for orthonormal basis of $\mathfrak{m}$, we have

$$\langle [X, u]_{\mathfrak{m}}, X\rangle = 0 \ \forall\ X \in \mathfrak{m}.$$

Conversely, Suppose that $\langle [X, u]_{\mathfrak{m}}, X\rangle = 0 \ \forall\ X \in \mathfrak{m}$, i.e.,

$$\langle [u_i, u]_{\mathfrak{m}}, u_i\rangle = 0 \ \forall\ 1 \leq i \leq n,$$

$$\langle [u_i + u_k, u]_{\mathfrak{m}}, u_i + u_k\rangle = 0 \ \forall\ 1 \leq i, k \leq n,$$

$$\langle [u_i + u, u]_{\mathfrak{m}}, u_i + u, \rangle = 0 \ \forall\ 1 \leq i \leq n,$$

$$\langle [u, u_i], u\rangle = 0.$$

Using the above equations, we get $r_{ik} = 0$ and $s_i = 0 \ \forall\ 1 \leq i, k \leq n$, which implies that F has vanishing S-curvature. This completes the proof. $\square$

Theorem 4.2. *Let $(G/H, F)$ be a homogeneous Finsler space with generalized m-Kropina metric $F = \dfrac{\alpha^{m+1}}{\beta^m}$ $(m \neq 0, -1)$ having vanish S-curvature, then its Ricci curvature is given by*

$$Ric(z) = {}^{\alpha}Ric(z) + \frac{m(m-1)c^2\alpha^2}{2(m+1)^2\beta^2}\left(\mathcal{C}^n_{j0}\right)^2 - \frac{mc\alpha^2}{2(m+1)\beta}\left(\mathcal{C}^0_{jq}\mathcal{C}^n_{jq} + 2\mathcal{C}^j_{jq}\mathcal{C}^n_{q0}\right) + \frac{c^2m^2\alpha^4}{4(m+1)^2\beta^2}\left(\mathcal{C}^n_{ij}\right)^2. \tag{5}$$

Proof. Let F has vanishing S-curvature, then by Theorem 4.1, we have

$$\langle [X, u]_{\mathfrak{m}}, X\rangle = 0 \ \forall \ X \in \mathfrak{m}.$$

Now

$$\mathcal{C}^0_{0n} = \langle [X, u_n]_{\mathfrak{m}}, X\rangle = \left\langle \left[X, \frac{u}{c}\right]_{\mathfrak{m}}, X\right\rangle = 0,$$

$$\mathcal{C}^j_{nj} = \langle [u_n, u_j]_{\mathfrak{m}}, u_j\rangle = \frac{1}{c}\langle [u, u_j]_{\mathfrak{m}}, u_j\rangle = 0,$$

$$\begin{aligned}\mathcal{C}^n_{n0} &= \langle [u_n, X]_{\mathfrak{m}}, u_n\rangle = \frac{1}{c^2}\Big(\langle [u, X]_{\mathfrak{m}}, u\rangle + \langle [u, X]_{\mathfrak{m}}, X\rangle\Big)\\ &= \frac{1}{c^2}\langle [u, u+X]_{\mathfrak{m}}, u+X\rangle = 0,\end{aligned}$$

$$\begin{aligned}\mathcal{C}^n_{nj} &= \langle [u_n, u_j]_{\mathfrak{m}}, u_n\rangle = \left\langle \left[\frac{u}{c}, u_j\right]_{\mathfrak{m}}, \frac{u}{c}\right\rangle = \frac{1}{c^2}\Big(\langle [u, u_j]_{\mathfrak{m}}, u\rangle + \langle [u, u_j]_{\mathfrak{m}}, u_j\rangle\Big)\\ &= \frac{1}{c^2}\langle [u, u+u_j]_{\mathfrak{m}}, u+u_j\rangle = 0,\end{aligned}$$

and

$$\begin{aligned}\mathcal{C}^j_{n0} + \mathcal{C}^0_{nj} &= \langle [u_n, X]_{\mathfrak{m}}, u_j\rangle + \langle [u_n, u_j]_{\mathfrak{m}}, X\rangle\\ &= \frac{1}{c}\Big(\langle [u, X]_{\mathfrak{m}}, u_j\rangle + \langle [u, u_j]_{\mathfrak{m}}, X\rangle + \langle [u, X]_{\mathfrak{m}}, X\rangle + \langle [u, u_j]_{\mathfrak{m}}, u_j\rangle\Big)\\ &= \frac{1}{c}\Big(\langle [u, X]_{\mathfrak{m}}, X+u_j\rangle + \langle [u, u_j]_{\mathfrak{m}}, X+u_j\rangle\Big)\\ &= \frac{1}{c}\Big(\langle [u, X+u_j]_{\mathfrak{m}}, X+u_j\rangle\Big) = 0.\end{aligned}$$

Substituting these values in Theorem 3.2, we get the equation (5). This completes the proof. □

5 Projective Ricci Curvature of Generalized m-Kropina Metric

In this section, we derive the formula for projective Ricci curvature of generalized m-Kropina Finsler metric.

Let M be an n-dimensional Finsler manifold with generalized m-Kropina metric F. From the equation (2), the geodesic coefficient of a generalized m-Kropina metric $F = \dfrac{\alpha^{m+1}}{\beta^m}$ is defined by

$$\begin{aligned} G^i =& \bar{G}^i - \frac{\alpha m}{(m+1)s} s^i_0 - \frac{m}{(m+1)A}\left[\frac{(m+1)sr_{00}}{\alpha} + 2ms_0\right] y^i \\ &+ \frac{m}{2(m+1)sA}\big[(m+1)sr_{00} + 2m\alpha s_0\big] b^i, \end{aligned}$$

where $A = s^2 + (B - s^2)m$. Therefore

$$\begin{aligned} G^i_l =& \bar{G}^i_l - \frac{\alpha_{y^l} m}{(m+1)s} s^i_0 - \frac{\alpha m}{(m+1)s} s^i_l + \frac{\alpha m}{(m+1)s^2} s^i_0 s_{y^l} - \frac{m}{(m+1)A} \\ &\left[\frac{2(m+1)sr_{0l}}{\alpha} + \frac{(m+1)s_{y^l} r_{00}}{\alpha} - \frac{(m+1)sr_{00}\alpha_{y^l}}{\alpha^2} + 2ms_l\right] y^i \\ &- \frac{m}{(m+1)A}\left[\frac{(m+1)sr_{00}}{\alpha} + 2ms_0\right]\delta^i_l + \frac{2sm(1-m)s_{y^l}}{(m+1)A^2}\left[\frac{(m+1)sr_{00}}{\alpha}\right. \\ &\left. + 2ms_0\right] y^i + \frac{m}{2(m+1)sA}\big[2(m+1)sr_{0l} + (m+1)r_{00}s_{y^l} + 2m\alpha s_l \\ &+ 2ms_0\alpha_{y^l}\big] b^i - \frac{m\big(A + 2s^2 - 2ms^2\big)s_{y^l}}{2(m+1)s^2A^2}\big[(m+1)sr_{00} + 2m\alpha s_0\big] b^i. \end{aligned}$$

Then using the above equation, we get

$$\begin{aligned} S_{|l} y^l &= y^l \frac{\partial S}{\partial x^l} - G^k_l y^l \frac{\partial S}{\partial y^k} \\ &= S_{;l} y^l + \left[\frac{2m\alpha}{(m+1)s} s^l_0 - \frac{m}{(m+1)sA}\big[(m+1)sr_{00}\right. \\ &\left. + 2m\alpha s_0\big] b^i\right] S_{y^l} + \frac{2m}{(m+1)A}\left[\frac{(m+1)sr_{00}}{\alpha} + 2ms_0\right] S, \end{aligned} \tag{6}$$

where $|_l$ denotes the horizontal covariant derivative of S along geodesic.

The formula of S-curvature for generalized m-Kropina metric is given by [16]

$$S = -\frac{m\big[(n-nm)s^2+(nm+1)b^2\big]}{(m+1)\alpha\big[(1-m)s^2+b^2m\big]^2}\Big[m\alpha\langle[v,y],v\rangle-(m+1)s\langle[v,y],y\rangle\Big].$$

Now, the previous equation can be rewritten as

$$S = -\frac{m(nA+b^2)}{(m+1)\alpha A^2}\Big[(m+1)sr_{00}-2m\alpha r_0\Big]. \tag{7}$$

Further, the following relations hold:

$$\begin{aligned}
&b^2_{;l}=2(r_l+s_l), && b^2_{;0}=2(r_0+s_0), && s_{;l}=\frac{r_{0l}+s_{0l}}{\alpha}, && s_{;0}=\frac{r_{00}}{\alpha},\\
&s_{y^l}=\frac{\alpha b_l-sy_l}{\alpha^2}, && s^l_0 s_{y^l}=\frac{s_0}{\alpha}, && b^l s_{y^l}=\frac{b^2-s^2}{\alpha}, && \alpha_{;0}=0,\\
&s^l_0 r_{0l}=q_{00}, && s^l_0 r_l=q_0, && s^l_0 y_l=0, && b^l\alpha_{y^l}=s.
\end{aligned}$$

Therefore

$$\begin{aligned}
S_{;l}y^l =& -\frac{m(nA+b^2)}{(m+1)\alpha A^2}\Big[(m+1)sr_{00;0}+\frac{(m+1)r^2_{00}}{\alpha}-2m\alpha r_{0;0}\Big]\\
&+\frac{2m}{(m+1)\alpha A^3}\Big[(m+1)sr_{00}-2m\alpha r_0\Big]\Big[(nA+2b^2)\\
&\Big(\frac{sr_{00}}{\alpha}+\big(r_0+s_0-\frac{sr_{00}}{\alpha}\big)m\Big)-A(r_0+s_0)\Big],
\end{aligned} \tag{8}$$

$$\begin{aligned}
S_{y^l}s^l_0 =& -\frac{m(nA+b^2)}{(m+1)\alpha A^2}\Big[\frac{(m+1)}{\alpha}r_{00}s_0+2(m+1)sq_{00}-2m\alpha q_0\Big]\\
&+\frac{2m(1-m)ss_0}{(m+1)\alpha^2A^3}(nA+b^2)\Big[(m+1)sr_{00}-2m\alpha r_0\Big],
\end{aligned} \tag{9}$$

and

$$\begin{aligned}
S_{y^l}b^l =& -\frac{m(nA+b^2)}{(m+1)\alpha A^2}\Big[(m+1)r_{00}\Big(\frac{b^2-s^2}{\alpha}\Big)-2m\alpha r+2sr_0\Big]\\
&+\frac{m}{(m+1)\alpha^2A^3}\Big[(m+1)sr_{00}-2m\alpha r_0\Big]\\
&\Big[2s(1-m)(nA+2b^2)(b^2-s^2)+sb^2A+nsA^2\Big].
\end{aligned} \tag{10}$$

Using the equations (7), (8), (9) and (10) in equation (6), we have

$$
\begin{aligned}
S_{|l}y^l = &-\frac{m(nA+b^2)}{(m+1)\alpha A^2}\Big[(m+1)sr_{00;0}+\frac{(m+1)r_{00}^2}{\alpha}-2m\alpha r_{0;0}\Big] \\
&+\frac{2m}{(m+1)\alpha A^3}\Big[(m+1)sr_{00}-2m\alpha r_0\Big]\Big[(nA+2b^2)\Big(\frac{sr_{00}}{\alpha}+\big(r_0+s_0 \\
&-\frac{sr_{00}}{\alpha}\big)m\Big)-A(r_0+s_0)\Big]-\frac{2m^2(nA+b^2)}{(m+1)^2 sA^2}\Big[\frac{(m+1)}{\alpha}r_{00}s_0+2(m+1) \\
&sq_{00}-2m\alpha q_0\Big]+\frac{4m^2(1-m)s_0}{(m+1)^2\alpha A^3}(nA+b^2)\Big[(m+1)sr_{00}-2m\alpha r_0\Big] \\
&+\frac{m^2(nA+b^2)}{(m+1)^2\alpha sA^3}\Big[(m+1)r_{00}\Big(\frac{b^2-s^2}{\alpha}\Big)-2m\alpha r+2sr_0\Big]\Big[(m+1)sr_{00} \\
&+2m\alpha s_0\Big]-\frac{m^2}{(m+1)^2\alpha^2 sA^4}\Big[(m+1)sr_{00}-2m\alpha r_0\Big]\Big[2s(1-m)(nA \\
&+2b^2)(b^2-s^2)+sb^2A+nsA^2\Big]\Big[(m+1)sr_{00}+2m\alpha s_0\Big] \\
&-\frac{2m^2(nA+b^2)}{(m+1)^2\alpha^2A^3}\Big[(m+1)sr_{00}-2m\alpha r_0\Big]\Big[(m+1)sr_{00}+2m\alpha s_0\Big].
\end{aligned}
\tag{11}
$$

From the above calculations, we obtain the following result:

Theorem 5.1. *Suppose (M,F) is a homogeneous Finsler space with generalized m-Kropina metric F of dimension n and Ric denotes the Ricci curvature of generalized m-Kropina metric. Then the projective curvature* ***PRic*** *of F is given by*

$$
\begin{aligned}
\boldsymbol{PRic} = \ &{}^{\alpha}Ric-\frac{n-1}{n+1}\Bigg\{\frac{m(nA+b^2)h_5}{(m+1)\alpha^2A^2}-\frac{2mt_1}{(m+1)\alpha A^3}\Big[(nA+2b^2)\Big(h_0 \\
&(1-m)+\frac{h_2-h_1}{4\alpha}\Big)-\frac{A(h_2-h_1)}{4m\alpha}\Big]+\frac{2m^2(nA+b^2)h_4}{(m+1)^2 s\alpha A^2}-\frac{2m(1-m)}{(m+1)^2\alpha A^3} \\
&(nA+b^2)h_1[h_2-(m+1)h_0]-\frac{m^2(nA+b^2)h_3h_2}{(m+1)^2\alpha^2 sA^3}+\frac{m^2h_1h_2}{(m+1)^2\alpha^2 sA^4}\Big[3s \\
&b^2A+3nsA^2+2s(1-m)(nA+2b^2)(b^2-s^2)\Big]-\frac{m^2(nA+b^2)^2\ h_1^2}{(m+1)^2(n+1)\alpha^2A^4}\Bigg\}.
\end{aligned}
\tag{12}
$$

where

$$\begin{cases} h_0 = sr_0, \\ h_1 = (m+1)sr_{00} - 2m\alpha r_0, \\ h_2 = (m+1)sr_{00} + 2m\alpha s_0, \\ h_3 = (m+1)r_{00}(b^2 - s^2) - 2m\alpha^2 r + 2s\alpha r_0, \\ h_4 = (m+1)r_{00}s_0 + 2\alpha(m+1)sq_{00} - 2m\alpha^2 q_0, \\ h_5 = (m+1)s\alpha r_{00;0} + (m+1)r_{00}^2 - 2m\alpha^2 r_{0;0}. \end{cases} \tag{13}$$

Proof. Since the projective Ricci curvature **PRic** of F is given by

$$\mathbf{PRic} = \mathbf{Ric} + \frac{n-1}{n+1} S_{|l} y^l + \frac{n-1}{(n+1)^2} S^2, \tag{14}$$

where **Ric** denotes the Ricci curvature of generalized m-Kropina metric, using the equations (7), (11) and (13) in the equation (14), completes the proof. □

5.1 Applications of projective Ricci curvature

As we know, projective Ricci curvature is an important projective invariant with respect to a fixed volume form in Finsler geometry. It plays an important role in establishing the notion of projective Ricci flat sprays. With the help of this, some important global rigidity results with non-negative Ricci curvature can be established. The current study of projective Ricci curvature with generalized m-Kropina metric can be extended to the weighted projective Ricci curvature with aforesaid metric. The study of projective Ricci curvature can be extended to warped product/twisted product Finsler manifolds. Next, as a natural application, we characterize projectively Ricci-flat generalized m-Kropina metric defined by a Riemannian metric and constant length Killing 1-form.

Let M be an n-dimensional Finsler manifold and F be a generalized m-Kropina metric with constant length Killing 1-form β, equivalently, $r_{ij} = 0, s_j = 0$. Therefore, from theorem 4.1 and 4.2, we have vanishing S-curvature and

$$\mathcal{C}^0_{0n} = \mathcal{C}^j_{nj} = \mathcal{C}^n_{nj} = \mathcal{C}^n_{n0} = \mathcal{C}^0_{0n} = \mathcal{C}^j_{n0} + \mathcal{C}^0_{nj} = 0.$$

Making use of above values, we get

$$\begin{aligned} r_{00} &= r_{00;0} = r_{00;j}b^j = r_{0j;j}b^j = s_0 = s_{0;0} = s_{0;j}b^j = s_{j;0}b^j \\ &= s_j s^j = s_j s^j_0 = r_{0j}s^j_0 = r_j s^j_0 = r^j_0 s_j = r^j_j s_0 = 0. \end{aligned}$$

Using above values in the equation (12), we obtain

$$\mathbf{PRic} = {}^{\alpha}Ric - \frac{2m(m-1)\alpha^2}{(m+1)^2\beta^2}s_{0j}s^j_0 - \frac{2m\alpha^2}{(m+1)\beta}s^j_{0;j} - \frac{m^2\alpha^4}{(m+1)^2\beta^2}s^i_j s^j_i.$$

Now, suppose F is projectively Ricci flat Finsler metric, i.e., $\mathbf{PRic} = 0$, we get

$${}^{\alpha}Ric - \frac{2m(m-1)\alpha^2}{(m+1)^2\beta^2}s_{0j}s^j_0 - \frac{2m\alpha^2}{(m+1)\beta}s^j_{0;j} - \frac{m^2\alpha^4}{(m+1)^2\beta^2}s^i_j s^j_i = 0,$$

i.e.,

$$(m+1)^2\beta^2\ {}^{\alpha}Ric = 2m(m-1)\alpha^2 s_{0j}s^j_0 + 2m(m+1)\alpha^2\beta s^j_{0;j} + m^2\alpha^4 s^i_j s^j_i.$$

The above equation is equivalent to

$$\Pi_4\alpha^4 + \Pi_2\alpha^2 + \Pi_0 = 0, \tag{15}$$

where

$$\begin{aligned} \Pi_0 &= (m+1)^2\beta^2\ {}^{\alpha}Ric(z), \\ \Pi_2 &= 2m(m-1)s_{0j}s^j_0 + 2m(m+1)\beta s^j_{0;j} \\ \Pi_4 &= m^2 s^i_j s^j_i \end{aligned}$$

Since α^2 and β^2 are relatively prime polynomials in y, then from equation (15) and Π_0, there exists a scalar function $\mu = \mu(x)$ on M such that

$${}^{\alpha}Ric = \mu\alpha^2 \tag{16}$$

Substituting (16) into (15), we get

$$(m+1)\beta\big[(m+1)\beta\mu - 2ms^j_{0;j}\big] - 2m(m-1)s_{0j}s^j_0 - m^2\alpha^2 s^i_j s^j_i = 0. \tag{17}$$

Therefore, for a Finsler manifold M of dimension n with constant length Killing 1-form β, F is projectively Ricci flat Finsler metric if and only if there exist a function $\mu = \mu(x)$ such that α is an Einstein metric and β satisfies (17).

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