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On Nonsmooth Multiobjective Semi-Infinite Programming Problems with Mixed Constraints

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Abstract. We consider the multiobjective semi-infinite programming problems with feasible sets defined by equality and inequality constraints, in which the objective and the constraints functions are locally Lipschitz. First, we introduce an Arrow-Hurwitcz-Uzawa type constraint qualification which is based on the Clarke subdifferential. Then, we derive the strong Karush-Kuhn-Tucker type necessary optimality condition for properly efficient solutions of the considered problems.

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1 Introduction

Suppose that J and T are arbitrary (not necessarily finite) sets with $J \cup T \neq \emptyset$. We consider the following multiobjective semi-infinite programming problem (MSIP, in brief):

$$\begin{array}{lll} (\Theta): & \min \big(g_1(x), ..., g_m(x)\big) & \\ \text{s.t.} & p_j(x) \leq 0, & j \in J, \\ & q_t(x) = 0, & t \in T, \\ & x \in \mathbb{R}^n, & \end{array}$$

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where g_i as $i = 1, ..., m$, p_j as $j \in J$, and q_t as $t \in T$ are locally Lipschitz functions from $\mathbb{R}^n$ to $\mathbb{R}$. If $m = 1$, MSIP coincides to the semi-infinite programming problem, denoted by SIP. Necessary and sufficient optimality conditions for SIP have been studied by many authors; see [7, 8, 10, 18, 21, 25, 26] in the case $T = \emptyset$, and [5, 9, 11] in the case $T \neq \emptyset$.

If the objective and the constraint functions of MSIP are linear, the problem is called "linear multiobjective semi-infinite programming" (LMSIP), and when the objective and the constraint functions of MSIP are convex, the problem is called "convex multiobjective semi-infinite programming" (CMSIP). Weak and strong necessary optimality conditions in Karush-Kuhn-Tucker (KKT) type have been established under various constraint qualifications (CQ) in [4, 17] for LMSIP and in [12, 24] for CMSIP. Also, for nonsmooth MSIP, several CQs and optimality conditions are presented in many articles; see, e.g., [13, 14, 15, 16, 19, 22] in the case $T = \emptyset$ and [20, 23] in the case $T \neq \emptyset$.

As the classic multiobjective optimization, we can consider different kinds of optimality (efficiency) for MSIP, including weakly efficient solution, efficient solution, strictly efficient solution, isolated efficient solution, and properly efficient solution; see, e.g., [3]. In this paper, we focus on properly efficient solutions for MSIPs.

On the other hand, the Abadie, the Basic, the Zangwill, the Mangasarian-Fromovitz, the Slater, and the Guignard type constraint qualifications for MSIP are studied in mentioned references. The aim of this paper is to introduce a new Arrow-Hurwicz-Uzawa type constraint qualification and to provide the strong KKT type condition for properly efficient solutions of nonsmooth MSIP.

The structure of subsequent sections of this paper is as follows: In Sect. 2, we define required definitions and preliminary results which are requested in sequel. Section 3 is devoted to the main results of paper.

2 Notations and Preliminaries

In this section we present some definitions and auxiliary results that will be needed in the sequel from [2, 6, 28].

Given a nonempty set $A \subseteq \mathbb{R}^n$, we denote by $\overline{A}$, $cone(A)$, $\overline{cone}(A)$, and $lin(A)$, the closure of A, the convex cone (containing the origin) generated by A, the closed convex cone of A, and the linear subspace spanned by A, respectively. Also, the polar cone of A, the strict polar cone of A, and orthogonal set to A are respectively defined as:

$$A^{\leqq} := \{x \in \mathbb{R}^n \mid \langle x, a\rangle \le 0, \quad \text{for all} \quad a \in A\},$$

$$A^{<} := \{x \in \mathbb{R}^n \mid \langle x, a\rangle < 0, \quad \text{for all} \quad a \in A\},$$

$$A^{\perp} := A^{\leqq} \cap (-A)^{\leqq} = \{x \in \mathbb{R}^n \mid \langle x, a\rangle = 0, \quad \text{for all} \quad a \in A\},$$

where $\langle x, y\rangle$ shows the standard inner product of $x, y \in \mathbb{R}^n$. Notice that $A^{\leqq}$ and $A^{\perp}$ are always closed convex cones in $\mathbb{R}^n$. It is easy to show that

$$A^{<} \neq \emptyset \quad \Longrightarrow \quad \overline{A^{<}} = A^{\leqq}. \tag{1}$$

$$A^{<} = \big(conv(A)\big)^{<}. \tag{2}$$

We can check the following equalities are true for each sets A_1 and A_2 in $\mathbb{R}^n$ (see e.g., [28], section 2):

$$(A_1 \cup A_2)^{\leqq} = A_1^{\leqq} \cap A_2^{\leqq}, \qquad\qquad (A_1 \cap A_2)^{\leqq} = \overline{A_1^{\leqq} + A_2^{\leqq}}. \tag{3}$$

Theorem 2.1. ***(Bipolar Theorem):*** *[6, 28] Let $A \neq \emptyset$ be a subset of $\mathbb{R}^n$. Then,*

$$(A^{\leqq})^{\leqq} = \overline{cone}(A) \qquad \textit{and} \qquad (A^{\perp})^{\leqq} = lin(A).$$

If $\{C_\alpha \mid \alpha \in \Lambda\}$ is a collection of convex sets in $\mathbb{R}^n$, the following equality is true ([6], section 3):

$$cone\Big(\bigcup_{\alpha\in\Lambda} C_\alpha\Big) = \Big\{\sum_{k=1}^{s} \lambda_{\alpha_k} c_{\alpha_k} \mid c_{\alpha_k} \in C_{\alpha_k},\ s \in \mathbb{N},\ \lambda_{\alpha_k} \ge 0\Big\}. \tag{4}$$

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function. The generalized Clarke directional derivative of f at $\hat{x} \in \mathbb{R}^n$ in the direction $\nu \in \mathbb{R}^n$, and the Clarke subdifferential of f at $\hat{x}$ are respectively defined as

$$f^0(\hat{x}; \nu) := \limsup_{x\to\hat{x},\tau\downarrow 0} \frac{f(x + \tau\nu) - f(x)}{\tau},$$

$$\partial_c f(\hat{x}) := \{\xi \in \mathbb{R}^n \mid \ f^0(\hat{x};\nu) \geq \langle \xi, \nu \rangle, \quad \forall \nu \in \mathbb{R}^n\}.$$

It is worth to observe from [2] that if $f : \mathbb{R}^n \to \mathbb{R}$ is a locally Lipschitz function, then $\partial_c f(\hat{x})$ is a nonempty, compact, and convex subset of $\mathbb{R}^n$. Also, if f is continuously differentiable at $\hat{x}$, we have $\partial_c f(\hat{x}) = \{\nabla f(\hat{x})\}$, and if $\phi : \mathbb{R}^n \to \mathbb{R}$ is a convex function, then $\partial_c \phi(\hat{x}) = \partial \phi(\hat{x})$, in which $\partial \phi(\hat{x})$ denotes the convex subdifferential of ϕ at $\hat{x}$, i.e.,

$$\partial \phi(\hat{x}) := \{\xi \in \mathbb{R}^n \mid \phi(x) - \phi(\hat{x}) \geq \big\langle \xi, x - \hat{x} \big\rangle, \quad \forall x \in \mathbb{R}^n\}.$$

Let us recall the following theorem, named Pshenichnyi-Levin-Valadire Theorem, from [6, Theorem 4.4.2].

Theorem 2.2. *Suppose that Λ is a compact subset of a metric space, $\phi_\beta : \mathbb{R}^n \to \mathbb{R}$ is a convex function for each $\beta \in \Lambda$ and the function $\beta \to \phi_\beta(\hat{x})$ is upper-semicontinuous for a vector $\hat{x} \in \mathbb{R}^n$. Then, the function $\max\limits_{\beta \in \Lambda} \phi_\beta(\cdot)$ is convex and*

$$\partial\big(\max_{\beta \in \Lambda} \phi_\beta\big)(\hat{x}) = conv\Big(\bigcup_{\beta \in \Lambda_0(\hat{x})} \partial \phi_\beta(\hat{x})\Big),$$

where, $\Lambda_0(\hat{x}) := \{\beta_0 \in \Lambda \mid \max\limits_{\beta \in \Lambda} \phi_\beta(\hat{x}) = \phi_{\beta 0}(\hat{x})\}$.

Theorem 2.3. *[6] If the convex function $\phi : \mathbb{R}^n \to \mathbb{R}$ attaints its minimum on a convex set $C \subseteq \mathbb{R}^n$ at $\hat{x} \in C$, then*

$$0_n \in \partial \phi(\hat{x}) + N(C, \hat{x}),$$

where 0_n shows the zero vector in $\mathbb{R}^n$, and $N(C, \hat{x})$ denotes the normal cone of C at $\hat{x}$, defined as

$$N(C, \hat{x}) := \big\{y \in \mathbb{R}^n \mid \langle y, x - \hat{x} \rangle \leq 0 \qquad \forall x \in C\big\}.$$

As a consequence of bipolar theorem, we note from [6, pp. 137] that if $A \subseteq \mathbb{R}^n$ is an arbitrary set, then

$$N(A^{\leq}, 0_n) = (A^{\leq})^{\leq}. \tag{5}$$

Theorem 2.4. *[2] Let $f_1, f_2 : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz functions and $\hat{x} \in \mathbb{R}^n$. Then,*

$$f_1^0(\hat{x};\nu) = \max\{\langle \xi, \nu\rangle \mid \ \xi \in \partial_c f_1(\hat{x})\}, \qquad \forall \nu \in \mathbb{R}^n. \tag{6}$$
$$f_1^0(\hat{x};\tau\nu) = \tau f_1^0(\hat{x};\nu), \qquad \forall \tau \geq 0. \tag{7}$$
$$\nu \to f_1^0(\hat{x};\nu) \text{ is a convex function.} \tag{8}$$
$$\partial_c f_1(\hat{x}) = \partial f_1^0(\hat{x};\cdot)(0_n). \tag{9}$$
$$\partial_c (f_1 + f_2)(\hat{x}) \subseteq \partial_c f_1(\hat{x}) + \partial_c f_2(\hat{x}). \tag{10}$$

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function. f is said to be generalized pseudoconcave at $\hat{x} \in \mathbb{R}^n$ if

$$f^0(\hat{x}; x - \hat{x}) \leq 0 \Rightarrow f(x) \leq f(\hat{x}), \qquad \text{for all } \ x \in \mathbb{R}^n.$$

Note the generalized linear function is a special case of concept of pseudoconcave function, in [27].
Also, the locally Lipshitz function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be generalized linear at $\hat{x}$ if

$$f(x) - f(\hat{x}) = \big\langle \xi, x - \hat{x} \big\rangle, \qquad \text{for some } \ \xi \in \partial_c f(\hat{x}).$$

It is worth to observe that the concept of generalized linear function is a generalization of the concept infine function, considered in [20].

3 Main Results

As the beginning of this section, we introduce some definitions and notations. Assume that the feasible set of problem (Θ), denoted by $\mathcal{S}$, is nonempty, i.e.,

$$\mathcal{S} := \Big\{ x \in \mathbb{R}^n \mid p_j(x) \leq 0, \ q_t(x) = 0, \quad \text{for all } (j,t) \in J \times T \Big\} \neq \emptyset.$$

Definition 3.1. *A feasible point $\hat{x} \in \mathcal{S}$ is called a properly efficient solution for (Θ) when there exist some positive scalars $\alpha_1, ..., \alpha_m > 0$ such that*

$$\sum_{i=1}^{m} \alpha_i g_i(\hat{x}) \leq \sum_{i=1}^{m} \alpha_i g_i(x), \qquad \forall x \in \mathcal{S}.$$

observe that the mention of proper efficiency is and important topic in scalarizations of Multiobjective Programming Problems; see, e.g. [3]. Considering a feasible point $x_0 \in \mathcal{S}$ (this point will be fixed throughout this paper), we define following sets:

$$\begin{aligned}
&J(x_0) := \{j \in J \mid p_j(x_0) = 0\},\\
&J_1(x_0) := \{j \in J(x_0) \mid p_j \text{ is generalized pseudoconcave at } x_0\},\\
&J_2(x_0) := \{j \in J(x_0) \mid p_j \text{ is not generalized pseudoconcave at } x_0\},\\
&\mathcal{P}(x_0) := \bigcup_{j \in J(x_0)} \partial_c p_j(x_0),\\
&\mathcal{P}_1(x_0) := \bigcup_{j \in J_1(x_0)} \partial_c p_j(x_0),\\
&\mathcal{P}_2(x_0) := \bigcup_{j \in J_2(x_0)} \partial_c p_j(x_0),\\
&\mathcal{Q}(x_0) := \bigcup_{t \in T} \partial_c q_t(x_0).
\end{aligned}$$

Also, let

$$\varphi(x) := \sup_{j \in J_2(x_0)} p_j(x), \qquad \forall x \in \mathcal{S}.$$

Note that if $J_2(x_0)$ is finite, then $\varphi(\cdot)$ is a locally Lipschitz function and, from [2, Propisition 2.3.12], we have

$$\partial_c \varphi(x_0) \subseteq conv\big(\mathcal{P}_2(x_0)\big), \tag{11}$$

but in general, (11) does not hold when $J_2(x_0)$ is infinite. The following example shows that if Δ is an infinite index set and ϑ_ℓ is a locally Lipschitz function from R^n to R for all $\ell \in \Delta$, the following inclusion is necessarily true, even in linear case:

$$\partial_c\big(\sup_{\ell \in \Delta} \vartheta_\ell\big)(\hat{x}) \subseteq conv\Big(\bigcup_{\ell \in \Delta_0(\hat{x})} \partial_c \vartheta_\ell(\hat{x})\Big).$$

Example 3.2. *Let* $\Delta := \mathbb{N}$, $\hat{x} := -1$, *and*

$$\begin{aligned}
&\vartheta_\ell(x) := x - \frac{2}{\ell + 1}, \qquad \ell = 1, 3, 5, 7, ...,\\
&\vartheta_\ell(x) := 4x - \frac{\ell}{2}, \qquad \ell = 2, 4, 6, 8, ...
\end{aligned}$$

It is easy to show that

$$\Delta_0(\hat{x}) := \{\ell \in \Delta \mid \vartheta_\ell(\hat{x}) = 0\} = \{1\},$$

$$\partial_c \vartheta_1(\hat{x}) = \{1\} = conv\big(\partial_c \vartheta_1(\hat{x})\big) = conv\Big(\bigcup_{\ell \in \Delta_0(\hat{x})} \partial_c \vartheta_\ell(\hat{x})\Big),$$

$$\sup_{\ell \in \Delta} \vartheta_\ell(x) = \begin{cases} x, & if \;\; x > 0 \\ 4x, & if \;\; x \leq 0 \end{cases},$$

$$\partial_c\big(\sup_{\ell \in \Delta} \vartheta_\ell\big)(\hat{x}) = [1, 4].$$

As a result, we have

$$\partial_c\big(\sup_{\ell \in \Delta} \vartheta_\ell\big)(\hat{x}) \nsubseteq conv\Big(\bigcup_{\ell \in \Delta_0(\hat{x})} \partial_c \vartheta_\ell(\hat{x})\Big).$$

The following definitions are standard in SIP theory, even in differentiable and\or convex cases; see, e.g., [8, 9, 12, 13, 15, 16].

Definition 3.3. *We say that* (Θ) *has the weakly Pshenichnyi-Levin-Valadire (WPLV) property at* x_0*, if* $\varphi(\cdot)$ *is a Lipschitz function around* x_0*, and (11) holds.*

Definition 3.4. *We say that the problem* (Θ) *is continuous at* x_0 *if* $J_2(x_0)$ *is a nonempty compact subset of a metric space, the function* $j \to p_j(x_0)$ *is upper semicontinuous on* $J_2(x_0)$*, and* $j \to \partial_c p_j(x_0)$ *is an upper semicontinuous mapping on* $J_2(x_0)$*; in which they upper semicontinuous of mapping* $j \to p_j(\hat{x})$ *means that*

$$Lim \sup_{x \to \hat{x}} \partial_c p_j(x) = \partial_c p_j(\hat{x}).$$

Remarks 3.5.

1. It is worth mentioning that if the problem (Θ) is continuous at x_0, compactness of $J_2(x_0)$ causes it to reduce from “sup” to “max” in the definition of φ.

2. Obviously, $J_2(x_0) = J(x_0) \setminus J_1(x_0)$ and $\mathcal{P}(x_0) = \mathcal{P}_1(x_0) \cup \mathcal{P}_2(x_0)$. Since it is possible $\partial_c p_{j_1}(x_0) \cap \partial_c p_{j_2}(x_0) \neq \emptyset$ for some $j_1 \in J_1(x_0)$ and $j_2 \in J_2(x_0)$, we can not write $\mathcal{P}_2(x_0) = \mathcal{P}(x_0) \setminus \mathcal{P}_1(x_0)$.

3. We recall that the definition of continuous problems in [8, 12] are based on compactness of J. Since the compactness of index sets J and $J_2(x_0)$ are independent to each other, Definition 3.4 is parallel to the similar definitions given in [8, 12].

4. Since the PLV property, defined in [12, 13, 16], are indexed with all subsets of J and $J_2(x_0) \subseteq J$, WPLV property is weaker than PLV property.

The following theorem presents a relationship between Definitions 3.3 and 3.4.

Theorem 3.6. *If the problem* (Θ) *is continuous at* x_0*, it has the WPLV property at* x_0*.*

Proof. For each $(j,t) \in J_2(x_0) \times T$ and $\nu \in \mathbb{R}^n$ take $\psi_j(\nu) := p_j^0(x_0;\nu)$ and $\phi_t(\nu) := q_t^0(x_0;\nu)$. Observe that $\psi_j(\cdot)$ and $\phi_t(\cdot)$ for $(j,t) \in J_2(x_0)\times T$ are convex functions by (8). Also, by [2, Theorem 2.8.2, Step 1] we know $\varphi(\cdot)$ is locally Lipschitz and

$$\varphi^0(x_0;\nu) \leq \max_{j\in J_2(x_0)} p_j^0(x_0;\nu) = \max_{j\in J_2(x_0)} \psi_j(\nu), \qquad \forall \nu \in \mathbb{R}^n.$$

Note that, $J_2(x_0)$ is a compact set and, by [2, Page 78], $j \to p_j^0(x_0;\nu)$ is an upper-semicontinuous function, so the notation "max" is justified in above inequality, and the function $\nu \to \max_{j\in J_2(x_0)} \psi_j(\nu)$ is convex, by Theorem 2.2. let ζ and ν be arbitrary elements in $\partial_c\varphi(x_0)$ and $\mathbb{R}^n$, respectively. The last inequality and (6) imply that

$$\begin{aligned}\max_{j\in J_2(x_0)} \psi_j(\nu) - \max_{j\in J_2(x_0)} \psi_j(0_n) &= \max_{j\in J_2(x_0)} \psi_j(\nu) \geq \varphi^0(x_0;\nu)\\ &= \max_{\varrho\in\partial_c\varphi(x_0)} \langle \varrho,\nu\rangle \geq \langle \zeta,\nu\rangle = \langle \zeta,\nu - 0_n\rangle.\end{aligned}$$

This inequality and Theorem 2.2 imply that

$$\zeta \in \partial\big(\max_{j\in J_2(x_0)} \psi_j\big)(0_n) = conv\Big(\bigcup_{j\in J^*(0_n)} \partial\psi_j(0_n)\Big),$$

where,

$$J^*(0_n) := \{j \in J_2(x_0) \mid \psi_j(0_n) = \big(\max_{j\in J_2(x_0)} \psi_j\big)(0_n) = 0\} = J_2(x_0).$$

Thus,

$$\zeta \in conv\Big(\bigcup_{j\in J_2(x_0)} \partial\psi_j(0_n)\Big). \tag{12}$$

Since for each $j \in J_2(x_0)$ we have

$$\begin{aligned}\partial\psi_j(0_n) &= \{\xi \in \mathbb{R}^n \mid \langle \xi, d - 0_n\rangle \leq \psi_j(d) - \psi_j(0_n), \quad \forall d \in \mathbb{R}^n\} \\ &= \{\xi \in \mathbb{R}^n \mid \langle \xi, d\rangle \leq p_j^0(x_0; d), \quad \forall d \in \mathbb{R}^n\} = \partial_c p_j(x_0),\end{aligned}$$

the equality (12) concludes that

$$\zeta \in conv\Big(\bigcup_{j\in J_2(x_0)} \partial_c p_j(x_0)\Big),$$

and since ζ is an arbitrary element of $\partial_c\varphi(x_0)$, we deduce that

$$\partial_c\varphi(x_0) \subseteq conv\Big(\bigcup_{j\in J_2(x_0)} \partial_c p_j(x_0)\Big) = conv\big(\mathcal{P}_2(x_0)\big).$$

The proof is complete. □ Now, we introduce a new Arrow-Hurwicz-Uzawa type constraint qualification for MSIPs.

Definition 3.7. *We say that* (Θ) *satisfies the "generalized constraint qualification" (GCQ) at* x_0 *if it has WPLV property at* x_0 *and*

$$\big(\mathcal{P}_1(x_0)\big)^{\leq} \cap \big(\mathcal{P}_2(x_0)\big)^{<} \cap \big(\mathcal{Q}(x_0)\big)^{\perp} \neq \emptyset.$$

Remarks 3.8.

1. If $T = \emptyset$ and for each $j \in J$, the p_j function is differentiable with $|J| < \infty$, the GCQ reduces to the classical Arrow-Hurwicz-Uzawa constraint qualification [1].
2. If $J_1(x_0) = \emptyset$, the GCQ reduces to Mangasarian-Fromotitz constraint qualification [9].
3. If $J_2(x_0) = \emptyset$, the GCQ reduces to Cottle constraint qualification [8].

Now, we can present our main result as follows.

Theorem 3.9. *Assume that* (Θ) *has WPLV property at its properly efficient solution* x_0*. If the GCQ is satisfied at* x_0 *and the* q_t *functions, for* $t \in T$*, are generalized linear at* x_0*, then there exist some* $\alpha_i > 0$ *as* $i = 1, ..., m$ *such that*

$$0_n \in \sum_{i=1}^{m} \alpha_i \partial_c g_i(x_0) + \overline{cone\big(\mathcal{P}(x_0)\big) + lin\big(\mathcal{Q}(x_0)\big)}.$$

Proof. According to GCQ assumption at x_0, we can consider an arbitrary element in $\big(\mathcal{P}_1(x_0)\big)^{\leq} \cap \big(\mathcal{P}_2(x_0)\big)^{<} \cap \big(\mathcal{Q}(x_0)\big)^{\perp}$, named ν. So

$$\nu \in \Big(\bigcap_{j \in J_1(x_0)} \big(\partial_c p_j(x_0)\big)^{\leq} \Big) \cap \big(\mathcal{P}_2(x_0)\big)^{<} \cap \Big(\bigcap_{t \in T} \big(\partial_c q_t(x_0)\big)^{\perp} \Big),$$

and hence

$$\begin{cases} \big\langle \nu, \zeta_1 \big\rangle \leq 0, \quad \text{for all} \quad \zeta_1 \in \partial_c p_j(x_0), \ j \in J_1(x_0), \\ \nu \in \big(\mathcal{P}_2(x_0)\big)^{<}, \\ \big\langle \nu, \xi \big\rangle = 0, \quad \text{for all} \quad \xi \in \partial_c q_t(x_0), \ t \in T. \end{cases} \tag{13}$$

Let $j \in J_1(x_0)$ be given. Using (6), (7) and (13), for all $\lambda > 0$ we have

$$p_j^0\big(x_0; \lambda^{-1}[(x_0 + \lambda\nu) - x_0]\big) = p_j^0(x_0; \nu) = \max_{\varrho \in \partial_c p_j(x_0)} \big\langle \nu, \varrho \big\rangle \leq 0.$$

Consequently,

$$p_j^0\big(x_0; [(x_0 + \lambda\nu) - x_0)]\big) = \lambda p_j^0\big(x_0; \lambda^{-1}[(x_0 + \lambda\nu) - x_0]\big) \leq 0,$$

and the generalized pseudoconcavity of p_j function implies that

$$p_j(x_0 + \lambda\nu) \leq p_j(x_0) \leq 0, \quad \text{for all } \lambda > 0, \ j \in J_1(x_0). \tag{14}$$

On the other hand, the second relation of (13) concludes that $\nu \in \Big(conv\big(\mathcal{P}_2(x_0)\big)\Big)^{<}$, and hence $\nu \in \big(\mathcal{P}_2(x_0)\big)^{<}$ by (2). Thus, WPLV property at x_0 deduces $\nu \in \partial_c \varphi(x_0)$, so $\varphi(x_0; \nu) < 0$ by (6), and there exists a scaler $\delta > 0$ such that $\varphi(x_0 + \lambda\nu) - \varphi(x_0) < 0$ for all $\lambda \in (0, \delta]$. Consequently,

$$p_j(x_0 + \lambda\nu) \leq \varphi(x_0 + \lambda\nu) < \varphi(x_0) \leq 0, \quad \text{for all } \lambda \in (0, \delta],\ j \in J_2(x_0). \tag{15}$$

Note that, generalized linearity of q_t functions at x_0 and the third relation in (13) imply that for each $\lambda > 0$ we have

$$q_t(x_0 + \lambda\nu) - q_t(x_0) = \lambda\langle\xi, \nu\rangle = 0, \quad \text{for some } \xi \in \partial_c q_t(x_0).$$

Thus,

$$q_t(x_0 + \lambda\nu) = 0, \quad \text{for all } \lambda > 0,\ t \in T.$$

The above equality, (14), and (15) deduce that

$$x_0 + \lambda\nu \in \mathcal{S}, \quad \text{for all } \lambda > 0.$$

Since x_0 is a properly efficient solution for (Θ), the last inclusion implies that we can find some $\alpha_i > 0$ for all $i = 1, ..., m$ such that

$$\sum_{i=1}^{m} \alpha_i g_i(x_0) \leq \sum_{i=1}^{m} \alpha_i g_i(x_0 + \lambda\nu), \quad \text{for all } \lambda > 0,$$

and so $\Big(\sum_{i=1}^{m} \alpha_i g_i\Big)^0(x_0; \nu) \geq 0$. Therefore, we proved that

$$\Big(\sum_{i=1}^{m} \alpha_i g_i\Big)^0(x_0; \nu) \geq 0, \quad \text{for all } \nu \in \big(\mathcal{P}_1(x_0)\big)^{\leq} \cap \big(\mathcal{P}_2(x_0)\big)^{<} \cap \big(\mathcal{Q}(x_0)\big)^{\perp}. \tag{16}$$

On the other hand, owing to (3), we conclude that

$$\begin{aligned}
&\overline{\big(\mathcal{P}_1(x_0)\big)^{\leq} \cap \big(\mathcal{P}_2(x_0)\big)^{<} \cap \big(\mathcal{Q}(x_0)\big)^{\perp}} = \\
&\big(\mathcal{P}_1(x_0)\big)^{\leq} \cap \big(\mathcal{P}_2(x_0)\big)^{\leq} \cap \big(\mathcal{Q}(x_0)\big)^{\perp} = \\
&\big(\mathcal{P}_1(x_0) \cup \mathcal{P}_2(x_0)\big)^{\leq} \cap \big(\mathcal{Q}(x_0)\big)^{\perp} = \big(\mathcal{P}(x_0)\big)^{\leq} \cap \big(\mathcal{Q}(x_0)\big)^{\perp},
\end{aligned}$$

and considering the continuity of the function $\nu \to \Big(\sum_{i=1}^{m} \alpha_i g_i\Big)^0(x_0; \nu)$ and (16), we deduce that

$$\Big(\sum_{i=1}^{m} \alpha_i g_i\Big)^0(x_0; \nu) \geq \Big(\sum_{i=1}^{m} \alpha_i g_i\Big)^0(x_0; 0_n), \quad \forall \nu \in \big(\mathcal{P}(x_0)\big)^{\leq} \cap \big(\mathcal{Q}(x_0)\big)^{\perp}.$$

Therefore, $\hat{\nu} := 0_n$ is an optimal solution of the following convex optimization problem:

$$\begin{aligned} &\min \Big(\sum_{i=1}^{m} \alpha_i g_i\Big)^0 (x_0; \nu) \\ \text{s.t.} \quad &\nu \in \big(\mathcal{P}(x_0)\big)^{\leq} \cap \big(\mathcal{Q}(x_0)\big)^{\perp}. \end{aligned}$$

Employing Theorem 2.3, we get

$$0_n \in \partial\Big(\Big(\sum_{i=1}^{m} \alpha_i g_i\Big)^0 (x_0; \cdot)\Big)(0_n) + N\Big(\big(\mathcal{P}(x_0)\big)^{\leq} \cap \big(\mathcal{Q}(x_0)\big)^{\perp}, 0_n\Big). \quad (17)$$

Notice, (9) and (10) imply that

$$\partial\Big(\Big(\sum_{i=1}^{m} \alpha_i g_i\Big)^0 (x_0; \cdot)\Big)(0_n) = \partial_c\Big(\sum_{i=1}^{m} \alpha_i g_i\Big)(x_0) \subseteq \sum_{i=1}^{m} \alpha_i \partial_c g_i(x_0). \quad (18)$$

Also, (3), the bipolar Theorem 2.1, and (5) conclude that

$$\begin{aligned} &N\Big(\big(\mathcal{P}(x_0)\big)^{\leq} \cap \big(\mathcal{Q}(x_0)\big)^{\perp}, 0_n\Big) = \\ &N\Big(\big(\mathcal{P}(x_0)\big)^{\leq} \cap \big(\mathcal{Q}(x_0)\big)^{\leq} \cap \big(-\mathcal{Q}(x_0)\big)^{\leq}, 0_n\Big) = \\ &N\Big(\big(\mathcal{P}(x_0) \cup \mathcal{Q}(x_0) \cup \big(-\mathcal{Q}(x_0)\big)\big)^{\leq}, 0_n\Big) \\ &\Big(\big(\mathcal{P}(x_0) \cup \mathcal{Q}(x_0) \cup \big(-\mathcal{Q}(x_0)\big)\big)^{\leq}\Big)^{\leq} = \\ &\Big(\big(\mathcal{P}(x_0)\big)^{\leq} \cap \big(\mathcal{Q}(x_0)\big)^{\perp}\Big)^{\leq} = \\ &\overline{\Big(\big(\mathcal{P}(x_0)\big)^{\leq}\Big)^{\leq} + \Big(\big(\mathcal{Q}(x_0)\big)^{\perp}\Big)^{\leq}} = \\ &\overline{\overline{cone}\big(\mathcal{P}(x_0)\big) + lin\big(\mathcal{Q}(x_0)\big)} = \overline{cone\big(\mathcal{P}(x_0)\big) + lin\big(\mathcal{Q}(x_0)\big)}. \end{aligned} \quad (19)$$

Now, relations (17), (18), and (19) imply that

$$0_n \in \sum_{i=1}^{m} \alpha_i \partial_c g_i(x_0) + \overline{cone}\big(\mathcal{P}(x_0)\big) + lin\big(\mathcal{Q}(x_0)\big),$$

as required. □

The following theorem presents the strong KKT optimality condition for MSIPs.

Theorem 3.10. *Suppose that* (Θ) *has WPLV property at its properly efficient solution* x_0 *and the GCQ is satisfied at* x_0*. If the* q_t *functions, for* $t \in T$*, are generalized linear at* x_0 *and* $cone\big(\mathcal{P}(x_0)\big)+lin\big(\mathcal{Q}(x_0)\big)$ *is a closed subset of* $\mathbb{R}^n$*, then there exist some scalars* $\alpha_i > 0$ *as* $i = 1, ..., m$*,* $\mu_j \geq 0$ *as* $j \in J(x_0)$*, and* η_t *as* $t \in T$*, with* $\mu_j \neq 0 \neq \eta_t$ *for finitely many indexes and*

$$0_n \in \sum_{i=1}^{m} \alpha_i \partial_c g_i(x_0) + \sum_{j \in J(x_0)} \mu_j \partial_c p_j(x_0) + \sum_{t \in T} \eta_t \partial_c q_t(x_0).$$

Proof. The result is direct consequent of Theorem 3.9, equality (4), and the structure of linear subspaces of $\mathbb{R}^n$. $\square$

The following corollary follows from Theorems 3.6 and 3.10.

Corollary 3.11. *If in the Theorems 3.9 and 3.10 the WPLV property is replaced by continuous property, then the results hold.*

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