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Original Research Paper

Fixed Point Approximation of Modified k-iteration Method in CAT(κ) Space

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Abstract. The paper aims is to discuss strong and $\Delta-$convergence of modified k-iteration method in CAT(κ) space, where $\kappa > 0$, for nearly asymptotically nonexpansive mappings. Our results generalize the corresponding results of Hussain *et al.* [10] and Ullah *et al.* [21].

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1 Introduction

The first fixed point approximation method is Picard's iteration method introduced by Banach [3] in 1920 in his famous result *Banach contraction*

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principle. After that many researchers worked in this direction and introduced different fixed point approximation methods for different types of mappings. The study convergence of fixed point approximation method in CAT(κ) space is very active research area in present era. Hussain *et al.* [10] introduced the new fixed point approximation method known as k-iteration method which is faster than Picard S-iteration [9] for contraction mapping in Banach space. Since Picard S-iteration is faster than Picard [3], Mann [14], Ishikawa [11], Noor [15], SP [17], S-iteration [2], Abbas [1], Normal S-iteration [19], therefore k-iteration is faster than all those iteration methods. The strong and weak convergence results have been proved by Hussain *et al.* [10] for Sujuki's generalized nonexpansive mapping in uniformly convex Banach space. Then Ullah *et al.* [21] studied the strong and $\Delta-$convergence results for Sujuki's generalized nonexpansive mapping, which is not nonexpansive, in the setting of CAT(0) space. They compared the fastness of the k-iteration method with the S-iteration method and three step Picard S-iteration method.

On the other hand, Sahu [18] introduced the generalization of the class of Lipschitzian mappings which is a class of nearly Lipschitzian mappings. Goebel *et al.* [8] introduced the generalization of nonexpansive mappings which is a class of asymptotically nonexpansive mappings. It is well known that *Every asymptotically nonexpansive mapping is a nearly asymptotically nonexpansive mapping.* The concept of $\Delta-$convergence is introduced by Lim [13] in metric space. Kirk *et al.* [12] extend the concept of $\Delta-$convergence in CAT(0) space.

In this paper, we prove the strong convergence and $\Delta-$convergence results for nearly asymptotically nonexpansive mappings in the setting of CAT(κ) space for $\kappa > 0$ via the modified k-iteration method. The results in this paper generalize the corresponding results of Hussain *et al.* [10] and Ullah *et al.* [21].

2 Basic Definitions and Results

Suppose that (Y, d) is metric space. The geodesic path joining t to u where $t, u \in Y$ is function $c : [0, 1] \subset \mathbb{R} \to Y$ such that $c(0) = t, c(l) = u$

and $d(c(e_1), c(e_2)) = |e_1 - e_2|$ $\forall e_1, e_2 \in [0, l]$. An image α of c is known as a geodesic segment joining t and u. This geodesic is denoted by $[t, u]$, when it is unique. A space (Y, d) is called a unique geodesic if there exists exactly one geodesic joining t and u for each $t, u \in Y$.

Suppose that M_κ represents two-dimensional, simply connected complete spaces whose curvature κ, where κ is considered as constant. We define the diameter of $M_\kappa(\kappa \geq 0)$ by $D_\kappa = \infty$ for $\kappa = 0$ and $D_\kappa = \frac{\pi}{\sqrt{\kappa}}$ for $\kappa > 0$. The ball in Y whose radius less than $D_\kappa/2$ is convex ([4]). The geodesic triangle $\triangle(t, u, v)$ in (Y, d) has three points t, u, v in Y and has three geodesic segments between each pair of vertices. For $\triangle(t, u, v)$ in Y satisfying

$$d(t, u) + d(u, v) + d(v, t) < 2D_\kappa,$$

there are points $t', u', v' \in M_\kappa$ such that $d(t, u) = d_\kappa(t', u'), d(u', v') = d_\kappa(u', v')$ and $d(v, t) = d_\kappa(v', t')$ where d_κ is the metric of M_κ. The triangle having three vertices $t', u', v' \in M_\kappa$ is known as comparison triangle of $\triangle(t, u, v)$. The geodesic triangle $\triangle(t, u, v)$ in Y with $d(t, u)+d(u, v)+d(v, t) < 2D_\kappa$ satisfy the CAT(κ) inequality if, for any $p_1, p_2 \in \triangle(t, u, v)$ and for their comparison points $p'_1, p'_2 \in \overline{\triangle}(t', u', v')$, $d(p_1, p_2) \leq d_\kappa(p'_1, p'_2)$.

Definition 2.1. [16] Suppose that (Y, d) is geodesic space and $\rho \in (0, 2]$. Then Y is called $\rho-$convex for ρ, if

$$d^2(v, \alpha t \oplus (1-\alpha)u) \leq \alpha d^2(v, t) + (1-\alpha)d^2(v, u) - \frac{\rho}{2}\alpha(1-\alpha)d^2(t, u)$$

for all $t, u, v \in Y$.

The geodesic space (Y, d) is a CAT(0) space iff (Y, d) is ρ-convex for $\rho = 2$.

Definition 2.2. The metric space (Y, d) is called CAT(κ) space iff

1. Y is a geodesic space and all of its geodesic triangles satisfy the CAT(κ) inequality (for $\kappa \leq 0$);

2. Y is D_κ-geodesic, geodesic triangle $\triangle(t, u, v)$ in Y with $d(t, u) + d(u, v) + d(v, t) < 2D_\kappa$ satisfies the CAT(κ) inequality (for $\kappa > 0$).

Remark 2.3. In CAT(0) space (Y, d) if $t, u, v \in Y$, then CAT(0) inequality implies

$$d^2\Big(t, \frac{u \oplus v}{2}\Big) \leq \frac{1}{2}d^2(t,u) + \frac{1}{2}d^2(t,v) - \frac{1}{4}d^2(u,v) \tag{CN}$$

which is known as (CN) inequality defined by Bruhat *et al.* [5]. Dhompongsa *et al.* [6] generalized (CN) inequality which is known as (CN*) inequality:

$$d^2(t, \delta v \oplus (1-\delta)u) \leq \delta d^2(t,v) + (1-\delta)d^2(t,u) - \delta(1-\delta)d^2(v,u) \tag{CN*}$$

for all $\delta \in [0,1]$ and $t, u, v \in Y$.

Suppose that Y is geodesic space, then there is equivalency among following statements:

(a) Y satisfies the (CN) inequality;

(b) Y satisfies the (CN*) inequality;

(c) Y is a CAT(0) space.

Suppose that (Y, d) is CAT(κ) space and sequence $\{y_m\}$ is bounded in Y. For $y \in Y$,

$$r(y, \{y_m\}) = \lim_{m\to\infty} \sup d(y, y_m).$$

Asymptotic radius $r(\{y_m\})$ of $\{y_m\}$ is given by

$$r(\{y_m\}) = \inf\{r(y, \{y_m\}) : y \in Y\}.$$

Asymptotic center $A(\{y_m\})$ of $\{y_m\}$ is given by

$$A(\{y_m\}) = \{y \in Y : r(\{y_m\}) = r(y, \{y_m\})\}.$$

Remark 2.4. [7] Suppose that (Y, d) is CAT(κ) space whose $diam(Y) = \frac{\pi}{2\sqrt{\kappa}}$ and sequence $\{y_m\}$ is bounded in Y. Then $A(\{y_m\})$ consists of exactly one point.

Definition 2.5. Suppose that (Y, d) is metric space and D is nonempty subset of Y. The mapping $S : D \to D$ is said to nonexpansive mapping, if $d(St_1, St_2) \leq d(t_1, t_2)$ for all $t_2, t_2 \in D$.

Definition 2.6. [13] The sequence $\{y_m\}$ in Y is called Δ-converge to $y \in Y$ if y is the unique asymptotic center of $\{u_m\}$ for every sub-sequence $\{u_m\}$ of $\{y_m\}$. Denote $\Delta\text{-}\lim_{m\to\infty} y_m = y$, where y is known as the Δ-limit of $\{y_m\}$.

Sahu [18] introduced the nearly Lipschitzian mappings which is generalization of Lipschitzian mapping.

Definition 2.7. Suppose that (Y, d) is metric space and D is nonempty subset of Y. The sequence $\{s_m\} \subset [0, \infty)$ with $\lim_{m\to\infty} s_m = 0$ is fixed. The mapping $S : D \to D$ is called nearly Lipschitzian with respect to $\{s_m\}$ if $\forall\ m \geq 1$, there is a constant $k_m \geq 0$ such that $d(S^m t_1, S^m t_2) \leq k_m(d(t_1, t_2) + s_m\ \forall t_1, t_2 \in D$.

The infimum of the constants k_m is represented by $\eta(S^m)$ and known as nearly Lipschitz constant and

$$\eta(S^m) = \sup\Big\{\frac{d(S^m t_1, S^m t_2)}{d(t_1, t_2) + s_m} : t_1, t_2 \in D, t_1 \neq t_2\Big\}.$$

If S is nearly Lipschitzian mapping with sequence $\{s_m, \eta(S^m)\}$ and if

1. $\eta(S^m) \geq 1\ \forall\ m \geq 1$ and $\lim_{m\to\infty} \eta(S^m) = 1$, then S is called nearly asymptotically nonexpansive;
2. $\eta(S^m) = 1\ \forall\ m \geq 1$, then S is called nearly nonexpansive;
3. $\eta(S^m) \leq k$ for all $m \geq 1$, then S is called nearly uniformly k-Lipschitzian.

Suppose that $F(S) = \{x \in D : Sx = x\}$ represents the set of fixed points of mapping S.

Definition 2.8. Suppose that (Y, d) is metric space and D is nonempty subset of Y. The mapping $S : D \to D$ is called to satisfy condition (I) if there is nondecreasing mapping $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(\tau) > 0\ \forall\, \tau \in (0, \infty)$ such that $d(t, St) \geq f(d(t, F(S)))$ for all $t \in D$.

Definition 2.9. Suppose that (Y, d) is metric space and D is nonempty subset of Y and $S : D \to D$ a mapping. A sequence $\{y_m\}$ in D is called AFPS (approximating fixed point sequence) for S if $\lim_{m\to\infty} d(y_m, Sy_m) = 0$.

Now, we present some important results of $\mathrm{CAT}(\kappa), \kappa > 0$, space which we use in sequel.

Lemma 2.10. *[20] Suppose that D is nonempty, closed and convex subset of complete $\mathrm{CAT}(\kappa)$ space (Y, d) whose diam$(Y) = \frac{\pi/2-\epsilon}{\sqrt{\kappa}}, \kappa > 0$ for $\epsilon \in (0, \pi/2)$ and $S : D \to D$ is uniformly continuous and nearly asymptotically nonexpansive mapping. Then S has a fixed point.*

Lemma 2.11. *[20] Suppose that D is nonempty, closed and convex subset of complete $\mathrm{CAT}(\kappa)$ space (Y, d) whose diam$(Y) = \frac{\pi/2-\epsilon}{\sqrt{\kappa}}, \kappa > 0$ for $\epsilon \in (0, \pi/2)$ and $S : D \to D$ is uniformly continuous and nearly asymptotically nonexpansive mapping. If the sequence $\{z_m\}$ is AFPS for the mapping S with $\Delta - \lim_{m\to\infty} z_m = x$, then $x \in D$ and $x = Sx$.*

Lemma 2.12. *[6] Suppose that (Y, d) is complete $\mathrm{CAT}(\kappa)$ space whose diam$(Y) = \frac{\pi/2-\epsilon}{\sqrt{\kappa}}, \kappa > 0$ for $\epsilon \in (0, \pi/2)$. Assume that $\{z_m\}$ is bounded sequence in Y such that $A(\{z_m\}) = \{z\}$ and $\{w_m\}$ is sub-sequence of $\{z_m\}$ such that $A(\{w_m\}) = \{w\}$ and the sequence $\{d(z_m, w)\}$ converges. Then $z = w$.*

Theorem 2.13. *[4] Suppose that (Y, d) is complete $\mathrm{CAT}(\kappa)$ space whose diam$(Y) = \frac{\pi/2-\epsilon}{\sqrt{\kappa}}, \kappa > 0$ for $\epsilon \in (0, \pi/2)$. Then*

$$d((1-\beta)t \oplus \beta u, v) \leq (1-\beta)d(t, v) + \beta d(u, v)$$

$\forall\, t, u, v \in Y$ and $\beta \in [0, 1]$.

Lemma 2.14. *Suppose that $\{u_m\}_{m=1}^{\infty}$, $\{v_m\}_{m=1}^{\infty}$ and $\{w_m\}_{m=1}^{\infty}$ are sequences of non-negative numbers satisfying the inequality*

$$u_{m+1} \leq (1 + v_m)u_m + w_m, \quad \forall m \geq 1$$

and $\sum_{m=1}^{\infty} v_m < \infty, \sum_{m=1}^{\infty} w_m < \infty$, then $\lim_{m\to\infty} u_m$ exists.

3 Main Results

Theorem 3.1. *Suppose that D is nonempty, closed and convex subset of complete* CAT(κ) *space (Y, d) whose diam$(Y) = \frac{\pi/2-\epsilon}{\sqrt{\kappa}}, \kappa > 0$ for $\epsilon \in (0, \pi/2)$ and $S : D \to D$ is uniformly continuous and nearly asymptotically nonexpansive mapping with sequence $\{s_m, \eta(S^m)\}$. Suppose that $t_0 \in D$ is arbitrary and the sequence $\{t_m\}$ is generated by modified k-iteration defined as*

$$\begin{aligned} v_m &= (1-\alpha_m)t_m \oplus \alpha_m S^m t_m \\ u_m &= S^m\big[(1-\beta_m)S^m t_m \oplus \beta_m S^m v_m\big] \\ t_{m+1} &= S^m u_m \end{aligned} \tag{1}$$

where $\{\alpha_m\}$, $\{\beta_m\}$ are two sequences satisfying the conditions $\sum_{m=1}^{\infty} t_m \leq \infty$ and $\sum_{m=1}^{\infty}(\eta(S^m) - 1) \leq \infty$. Then sequence $\{t_m\}$ generated by (1) $\Delta-$converges to a fixed point of the nearly asymptotically nonexpansive mapping S.

Proof. It is clear that $F(S) \neq \emptyset$, from Lemma 2.10. So let $p \in F(S)$. From Lemma 2.13, (1) and mapping S is nearly asymptotically nonexpansive,

$$\begin{aligned} d(v_m, p) &= d((1-\alpha_m)t_m \oplus \alpha_m S^m t_m, p) \\ &\leq (1-\alpha_m)d(t_m, p) + \alpha_m d(S^m t_m, p) \\ &\leq (1-\alpha_m)d(t_m, p) + \alpha_m \eta(S^m)(d(t_m, p) + s_m) \\ &\leq \eta(S^m)((1-\alpha_m)d(t_m, p) + \alpha_m d(t_m, p) + \alpha_m s_m) \\ &= \eta(S^m)(d(t_m, p) + \alpha_m s_m). \end{aligned} \tag{2}$$

By (1), (2) and Lemma 2.13,

$$\begin{aligned} d(u_m,p) &= d(S^m((1-\beta_m)S^m t_m \oplus \beta_m S^m v_m),p) \\ &\le \eta(S^m)(d((1-\beta_m)S^m t_m \oplus \beta_m S^m v_m),p)+s_m) \\ &\le \eta(S^m)((1-\beta_m)d(S^m t_m,p)+\beta_m d(S^m v_m,p)+s_m) \\ &\le \eta(S^m)((1-\beta_m)\eta(S^m)(d(t_m,p)+s_m)+\beta_m\eta(S^m)d(v_m,p) \\ &\quad +s_m)+s_m) \\ &\le \eta(S^m)((1-\beta_m)\eta(S^m)(d(t_m,p)+s_m)+\beta_m\eta(S^m)(\eta(S^m)d(t_m,p) \\ &\quad +\alpha_m s_m+s_m)+s_m)+s_m) \\ &\le \eta(S^m)((1-\beta_m)\eta(S^m)^2(d(t_n,p)+s_m)+\beta_m\eta(S^m)^2 d(t_m,p) \\ &\quad +(1+\alpha_m)\beta_m s_m\eta(S^m)+s_m) \\ &= \eta(S^m)^3 d(t_m,p)+s_m\eta(S^m)((1-\beta_m)\eta(S^m)^2 \\ &\quad +(1+\alpha_m)\beta_m\eta(S^m)+1) \end{aligned} \tag{3}$$

By (1), (3) and Lemma 2.13,

$$\begin{aligned} d(t_{m+1},p) &= d(S^m u_m,p) \\ &\le \eta(S^m)(d(u_m,p)+s_m) \\ &\le \eta(S^m)(\eta(S^m)^3 d(t_m,p)+s_m\eta(S^m)((1-\beta_m)\eta(S^m)^2 \\ &\quad +(1+\alpha_m)\beta_m\eta(S^m)+1)+s_m) \\ &\le \eta(S^m)^4 d(t_m,p)+s_m\eta(S^m)((1-\beta_m)\eta(S^m)^3 \\ &\quad +(1+\alpha_m)\beta_m\eta(S^m)^2 \\ &\quad +\eta(S^m)+1) \\ &= (1+A_m)d(t_m,p)+B_m \end{aligned}$$

where $A_m=\eta(S^m)^4-1$ and $B_m=s_m\eta(S^m)((1-\beta_m)\eta(S^m)^3+(1+\alpha_m)\beta_m\eta(S^m)^2+\eta(S^m)+1)$. Since $\sum_{m=1}^{\infty}s_m<\infty$ and $\sum_{m=1}^{\infty}(\eta(S^m)-1)<\infty$. Thus $\sum_{m=1}^{\infty}A_m<\infty$, $\sum_{m=1}^{\infty}B_m<\infty$. From Lemma 2.14, $\lim_{m\to\infty}d(t_m,p)$ exists.

Next, we will prove $\lim_{n\to\infty}d(St_m,t_m)=0$. Since $\{t_m\}$ is bounded, there is $\rho>0$ such that $\{t_m\},\{u_m\},\{v_m\}\subset B_\rho(p)\,\forall m\ge 1$ and $\rho<\frac{\delta_\kappa}{2}$.

Using (2.1) and (1),

$$\begin{aligned}
d^2(v_m,p) &= d^2((1-\alpha_m)t_m+\alpha_m S^m t_m,p)\\
&\leq (1-\alpha_m)d^2(t_m,p)+\alpha_m d^2(S^m t_m,p)-\frac{\rho}{2}\alpha_m\\
&\quad (1-\alpha_m)d^2(t_m,S^m t_m)\\
&\leq (1-\alpha_m)d^2(t_m,p)+\alpha_m(\eta(S^m)(d(t_m,p)+s_m))\\
&\quad -\frac{\rho}{2}\alpha_m(1-\alpha_m)d^2(t_m,S^m t_m)\\
&= \eta(S^m)^2(1-\alpha_m)d^2(t_m,p)+\alpha_m\eta(S^m)^2(d^2(t_m,p)+s_m^2)\\
&\quad +2s_m d(t_m,p))-\frac{\rho}{2}\alpha_m(1-\alpha_m)d^2(t_m,S^m t_m)\\
&= \eta(S^m)^2 d^2(t_m,p)+s_m\mathcal{A}-\frac{\rho}{2}\alpha_m(1-\alpha_m)d^2(t_m,S^m t_m)
\end{aligned}$$

where $\mathcal{A}=\eta(S^m)^2(s_m+2d(t_m,p))>0$.

$$d^2(v_m,p) \leq \eta(S^m)^2 d^2(t_m,p)+s_m A-\frac{\rho}{2}\alpha_m(1-\alpha_m)d^2(t_m,S^m t_m)$$

which implies that

$$\frac{\rho}{2}\alpha_m(1-\alpha_m)d^2(t_m,S^m t_m) \leq -d^2(v_m,p)+\eta(S^m)^2 d^2(t_m,p)+s_m\mathcal{A}.$$

Then $\frac{\rho}{2}\alpha_m(1-\alpha_m)d^2(t_m,S^m t_m) < \infty$, because $\sum_{m=1}^{\infty}s_m < \infty$ and $\sum_{m=1}^{\infty}\eta(S^m)^2<\infty$.
Since $\liminf_{m\to\infty}\alpha_m(1-\alpha_m)>0$, therefore

$$\lim_{m\to\infty} d(S^m t_m,t_m)=0. \tag{4}$$

$$\begin{aligned}
d^2(u_m,p) &= d^2(S^m((1-\beta_m)t_m+\beta_m S^m v_m,p))\\
&\leq \eta(S^m)^2(d((1-\beta_m)t_m+\beta_m S^m v_m,p)+s_m)^2\\
&= \eta(S^m)^2(d^2((1-\beta_m)t_m+\beta_m S^m v_m,p)+s_m\mathcal{B})
\end{aligned}$$

where $\mathcal{B} = s_m + 2d((1-\beta_m)t_m + \beta_m S^m v_m, p) > 0$

$$
\begin{aligned}
d^2(u_m,p) &\le \eta(S^m)^2((1-\beta_m)d^2(S^m t_m,p) + \beta_m d^2(S^m v_m,p) \\
&\quad -\frac{\rho}{2}(1-\beta_m)\beta_m d^2(S^m t_m, S^m v_m) + s_m\mathcal{B}) \\
&\le \eta(S^m)^2((1-\beta_m)d^2(S^m t_m,p) + \beta_m(\eta(S^m)^2 d^2(t_m,p) + s_m\mathcal{A} \\
&\quad -\frac{\rho}{2}\alpha_m(1-\alpha_m)d^2(t_m,S^m t_m)) - \frac{\rho}{2}(1-\beta_m) \\
&\quad \beta_m d^2(S^m t_m, S^m v_m) + s_m\mathcal{B}) \\
&\le \eta(S^m)^4 d^2(S^m t_m,p) + s_m(\beta_m\mathcal{A}\eta(S^m)^2 + \mathcal{B}) \\
&\quad -\frac{\rho}{2}\beta_m\alpha_m(1-\alpha_m)d^2(t_m,S^m t_m)\eta(S^m)^2 - \frac{\rho}{2}\eta(S^m)^2\beta_m \\
&\quad (1-\beta_m)d^2(S^m t_m, S^m v_m) \\
&\le \eta(S^m)^4(\eta(S^m)^2(d(t_m,p)+s_m)) + s_m(\beta_m\mathcal{A}\eta(S^m)^2 + \mathcal{B}) \\
&\quad -\frac{\rho}{2}\beta_m\alpha_m(1-\alpha_m)d^2(t_m,S^m t_m)\eta(S^m)^2 - \frac{\rho}{2}\eta(S^m)^2\beta_m \\
&\quad (1-\beta_m)d^2(S^m t_m, S^m v_m) \\
&\le \eta(S^n)^6 d^(t_m,p) + s_m\mathcal{C} + s_m(\beta_m\mathcal{A}\eta(S^m)^2 + \mathcal{B}) \\
&\quad -\frac{\rho}{2}\beta_m\alpha_m(1-\alpha_m)d^2(x_m,S^m t_m)\eta(S^m)^2 - \frac{\rho}{2}\eta(S^m)^2\beta_m \\
&\quad (1-\beta_m)d^2(S^m t_m, S^m v_m)
\end{aligned}
$$

where $\mathcal{C} = (s_m + d(t_m,p))\eta(S^m)^6 + \beta_m\mathcal{A} + \eta(S^m)^2 + \mathcal{B} > 0$. Now,

$$
\begin{aligned}
\frac{\rho}{2}\eta(S^m)^2\beta_m(1-\beta_m)d^2(S^m t_m, S^m v_m) &\le -d^(u_m,p) + \eta(S^m)^6 d^(t_m,p) \\
&\quad +s_m\mathcal{C} + s_m(\beta_m\mathcal{A}\eta(S^m)^2 + \mathcal{B}) \\
&\quad -\frac{\rho}{2}\beta_m\alpha_m(1-\alpha_m)d^2(t_m,S^m t_m)\eta(S^m)^2
\end{aligned}
$$

Since $\sum_{m=1}^{\infty} s_m < \infty, \sum_{m=1}^{\infty}\eta(S^m) < \infty$,

$$
\frac{\rho}{2}\eta(S^m)^2\beta_m(1-\beta_m)d^2(S^m t_m, S^m v_m) < \infty
$$

Since $\beta_m(1-\beta_m) > 0$ and $\lim_{m\to\infty}\eta(S^m)^2 = 0$,

$$
\lim_{m\to\infty} d^2(S^m t_m, S^m v_m) = 0. \tag{5}
$$

From (4) and (5),

$$\begin{aligned} d(S^m v_m, t_m) &\leq d(S^m v_m, S^m t_m) + d(S^m t_m, t_m) \\ &\to 0 \quad \text{as} \quad m \to \infty. \end{aligned} \tag{6}$$

Now, from (1), (4) and (6),

$$\begin{aligned} d(t_m, u_m) &\leq d(t_m, S^m t_m) + d(S^m t_m, u_m) \\ &= d(t_m, S^m t_m) + d(S^m t_m, S^m((1-\beta_m)S^m t_m + \beta_m S^m v_m)) \\ &\leq d(t_m, S^m t_m) + \eta(S^m)(d(t_m, S^m((1-\beta_m)S^m t_m + \beta_m S^m v_m) + s_m) \\ &\leq d(t_m, S^m t_m) + \eta(S^m)((1-\beta_m)d(t_m, S^m t_m) + \beta_m d(S^m v_m, t_m) + s_m) \\ &\to 0 \quad \text{as} \quad m \to \infty. \end{aligned} \tag{7}$$

From (4) and (7),

$$\begin{aligned} d(t_m, t_{m+1}) &= d(t_m, S^m u_m) \\ &\leq d(t_m, S^m t_m) + d(S^m t_m, S^m u_m) \\ &\leq d(t_m, S^m t_m) + \eta(S^m)(d(t_m, u_m) + s_m) \\ &\to 0 \quad \text{as} \quad m \to \infty. \end{aligned} \tag{8}$$

From (4) and (8),

$$\begin{aligned} d(t_m, St_m) &\leq d(t_m, t_{m+1}) + d(t_{m+1}, S^{m+1}t_{m+1}) + d(S^{m+1}x_{m+1}, S^{m+1}t_m) \\ &\quad + d(S^{m+1}t_m, S^m t_m) \\ &\leq d(t_m, t_{m+1}) + d(t_{m+1}, S^{m+1}t_{m+1}) + \eta(S^{m+1})(d(t_{m+1}, t_m) \\ &\quad + s_m) + d(S^{m+1}t_m, S^m t_m) \\ &\leq (\eta(S^{m+1}))d(t_m, t_{m+1}) + d(t_{m+1}, S^{m+1}t_{m+1}) + \eta(S^{m+1})s_m \\ &\quad + d(S^{m+1}t_m, S^m t_m) \\ &\to 0 \quad \text{as} \quad m \to \infty. \end{aligned}$$

Assume that $\mathcal{W}_u(t_m) = \cup A(\{w_m\})$ that is $\mathcal{W}_u(t_m)$ is union which is taken for all sub-sequences $\{w_m\}$ of $\{t_m\}$. Now we will prove that $\mathcal{W}_u(t_m) \subseteq F(S)$ and $\mathcal{W}_u(t_m)$ will consist only one element. Assume that $w \in \mathcal{W}_u(t_m)$, then there is a subsequence $\{a_m\}$ of $\{w_m\}$ such that $\lim_{m\to\infty} a_m = a \in D$. From Lemma 2.11 and (9), $a \in F(S)$.

Since $\lim_{m\to\infty} d(t_m, a)$ exists, and from Lemma 2.12, $a = w$. therefore $u_u(t_m) \subseteq F(S)$. Next, we will prove that $\{t_m\}$ $\Delta-$converges to an element of $F(S)$. For proving this, we will prove $u_u(t_m)$ consists of only one element. Assume that $\{u_m\}$ be sub-sequence in $\{t_m\}$ such that $A(\{u_m\}) = \{u\}$. Also assume that $A(\{t_m\}) = \{x\}$. Since $u \in u_u(t_m) \subseteq F(S)$, thus $\lim_{m\to\infty} d(t_m, a)$ exists and $u_u(t_m) = \{x\}$. Thus $\{t_m\}$ $\Delta-$converges to an element of $F(S)$. Therefore our theorem is proved. $\square$

The following result can be obtained by Theorem 3.1 directly.

Theorem 3.2. *Suppose that D is nonempty, closed and convex subset of complete* CAT(κ) *space (Y, d) whose diam$(Y) = \frac{\pi/2-\epsilon}{\sqrt{\kappa}}, \kappa > 0$ for $\epsilon \in (0, \pi/2)$ and $S : D \to D$ is uniformly continuous and nearly asymptotically nonexpansive mapping with sequence $\{s_m, \eta(S^m)\}$. Suppose that $t_0 \in D$ is arbitrary and the sequence $\{t_m\}$ is generated by modified k-iteration in (1), where $\{\alpha_m\}$, $\{\beta_m\}$ are two sequences satisfying the conditions $\sum_{m=1}^{\infty} s_m \leq \infty$ and $\sum_{m=1}^{\infty}(\eta(S^m) - 1) \leq \infty$. Then sequence $\{t_m\}$ $\Delta-$converges to fixed point of nearly asymptotically nonexpansive mapping S.*

The next results shows the strong convergence result for modified k-iteration method in CAT(κ) space.

Theorem 3.3. *Suppose that D is nonempty, closed and convex subset of complete* CAT(κ) *space (Y, d) whose diam$(Y) = \frac{\pi/2-\epsilon}{\sqrt{\kappa}}, \kappa > 0$ for $\epsilon \in (0, \pi/2)$ and $S : D \to D$ is uniformly continuous and nearly asymptotically nonexpansive mapping with sequence $\{s_m, \eta(S^m)\}$. Suppose that $t_0 \in D$ is arbitrary and the sequence $\{t_m\}$ is generated by modified k-iteration in (1), where $\{\alpha_m\}$, $\{\beta_m\}$ are two sequences satisfying the conditions $\sum_{m=1}^{\infty} s_m < \infty$ and $\sum_{m=1}^{\infty}(\eta(S^m) - 1) < \infty$. Assume that S^r is semi compact for some $r \in \mathbb{N}$. Then sequence $\{t_m\}$ strongly converges to fixed point of nearly asymptotically nonexpansive mapping S.*

Proof. We have $\lim_{m\to\infty} d(t_m, St_m) = 0$ from Theorem 3.1. Due to uniform continuity of S, we have

$$\begin{aligned} d(t_m, S^r t_m) &\leq d(t_m, St_m) + d(S^1 t_m, S^2 t_m) + \cdots + d(S^{r-1} t_m, S^r t_m) \\ &\to 0 \quad \text{as} \quad m \to \infty. \end{aligned} \tag{9}$$

Therefore $\{t_m\}$ is AFPS for S^r. The semi compactness of S^r implies that there is subsequence $\{t_{m_i}\}$ of $\{t_m\}$ with $\lim_{i\to\infty} t_{m_i} = t$, where $t \in D$. Since D is uniform continuous, therefore

$$\begin{aligned} d(t, St) &\leq d(St, St_{m_i}) + d(St_{m_i}, t_{m_i}) + d(t_{m_i}, t) \\ &\to 0 \quad \text{as} \quad m \to \infty. \end{aligned} \tag{10}$$

which shows that $p \in F(S)$. The $\lim_{m\to\infty} d(t_m, t)$ exists from Theorem 3.1. Therefore sequence $\{t_m\}$ converges strongly to fixed point t of S. Therefore our theorem is proved. □

The following result can be obtained by Theorem 3.3 directly.

Theorem 3.4. *Suppose that D is nonempty, closed and convex subset of complete* CAT(κ) *space* (Y, d) *whose diam*$(Y) = \frac{\pi/2-\epsilon}{\sqrt{\kappa}}, \kappa > 0$ *for* $\epsilon \in (0, \pi/2)$ *and* $S : D \to D$ *is uniformly continuous and nearly asymptotically nonexpansive mapping with sequence* $\{s_m, \eta(S^m)\}$*. Suppose that* $t_0 \in D$ *is arbitrary and the sequence* $\{t_m\}$ *is generated by modified k-iteration in (1), where* $\{\alpha_m\}$, $\{\beta_m\}$ *are two sequences satisfying the conditions* $\sum_{m=1}^{\infty} s_m < \infty$ *and* $\sum_{m=1}^{\infty}(\eta(S^m) - 1) < \infty$*. Assume that* S^r *is semi compact for some* $r \in \mathbb{N}$*. Then sequence* $\{t_m\}$ *strongly converges to fixed point of nearly asymptotically nonexpansive mapping* S*.*

Next, we will prove strong convergence theorems for k-iteration method using Condition (I). Before that we will prove the following Theorem which we use in sequel.

Theorem 3.5. *Suppose that D is nonempty, closed and convex subset of complete* CAT(κ) *space* (Y, d) *whose diam*$(Y) = \frac{\pi/2-\epsilon}{\sqrt{\kappa}}, \kappa > 0$ *for* $\epsilon \in (0, \pi/2)$ *and* $S : D \to D$ *is uniformly continuous and nearly asymptotically nonexpansive mapping with sequence* $\{s_m, \eta(S^m)\}$*. Suppose that* $t_0 \in D$ *is arbitrary and the sequence* $\{t_m\}$ *is generated by modified k-iteration in (1), where* $\{\alpha_m\}$, $\{\beta_m\}$ *are two sequences satisfying the conditions* $\sum_{m=1}^{\infty} s_m < \infty$ *and* $\sum_{m=1}^{\infty}(\eta(S^m) - 1) < \infty$*. Assume that* S *satisfies condition* (I)*. Then sequence* $\{t_m\}$ *strongly converges to fixed point of nearly asymptotically nonexpansive mapping* S *iff* $\lim_{m\to\infty} \inf d(t_m, F(S)) = 0$.

Proof. Assume that sequence $\{t_m\}$ converges to fixed point of mapping S, then it is clear that $\lim_{m\to\infty} \inf d(t_m, F(S)) = 0$. Now assume that $\lim_{m\to\infty} \inf d(t_m, F(S)) = 0$, then we have to prove that the sequence $\{t_m\}$ converges to fixed point of mapping S. Using Theorem 3.1, we have

$$d(t_{m+1}, t) \leq d(t_m, t) \quad t \in F(S).$$

Therefore

$$d(t_{m+1}, F(S)) \leq d(t_m, F(S)),$$

which shows that $\lim_{m\to\infty} d(t_{m+1}, F(S))$ exists. By assumption, $\lim_{m\to\infty} \inf d(t_m, F(S)) = 0$, therefore $\lim_{m\to\infty} d(t_{m+1}, F(S)) = 0$. Next, we will prove that $\{t_m\}$ is Cauchy sequence in D. Assume that $\epsilon > 0$ is arbitrary. We have $\lim_{m\to\infty} d(t_{m+1}, F(S)) = 0$, therefore there is m_0 for all $m \geq m_0$ with

$$d(t_m, F(S)) < \frac{\epsilon}{4}.$$

Thus $\inf\{d(t_{m_0}, t) : t \in F(S)\} < \frac{\epsilon}{4}$, therefore there is $t^* \in F(S)$ such that

$$d(t_{m_0}, t^*) < \frac{\epsilon}{2}.$$

For $m, p \geq m_0$, we have

$$\begin{aligned} d(t_{m+p}, t_m) &\leq d(t_{m+p}, t^*) + d(t_m, t^*) \\ &< 2d(t_{m_0}, t^*) \\ &< \epsilon. \end{aligned}$$

This shows that $\{t_m\}$ is Cauchy sequence in D of complete CAT(κ) space Y, therefore $\{t_m\}$ will converge in D. Assume that $\lim_{m\to\infty} t_m = y$. Then by Theorem 3.1, $\lim_{m\to\infty} d(t_m, St_m) = 0$ and uniform continuity of S, we have

$$\begin{aligned} d(y, Sy) &\leq d(y, t_m) + d(t_m, St_m) + d(St_m, TSy) \\ &\to 0 \quad \text{as} \quad m \to \infty. \end{aligned}$$

Thus $y \in F(S)$. Therefore our theorem is proved. □

The following result can be obtained by Theorem 3.6 directly.

Theorem 3.6. *Suppose that D is nonempty, closed and convex subset of complete* CAT(κ) *space (Y, d) whose diam$(Y) = \frac{\pi/2-\epsilon}{\sqrt{\kappa}}, \kappa > 0$ for $\epsilon \in (0, \pi/2)$ and $S : D \to D$ is uniformly continuous and nearly asymptotically nonexpansive mapping with sequence $\{s_m, \eta(S^m)\}$. Suppose that $t_0 \in D$ is arbitrary and the sequence $\{t_m\}$ is generated by modified k-iteration in (1), where $\{\alpha_m\}$, $\{\beta_m\}$ are two sequences satisfying the conditions $\sum_{m=1}^{\infty} s_m < \infty$ and $\sum_{m=1}^{\infty}(\eta(S^m) - 1) < \infty$. Assume that S satisfies condition (I). Then sequence $\{t_m\}$ strongly converges to fixed point of nearly asymptotically nonexpansive mapping S iff* $\lim_{m\to\infty} \inf d(t_m, F(S)) = 0$.

Theorem 3.7. *Suppose that D is nonempty, closed and convex subset of complete* CAT(κ) *space (Y, d) whose diam$(Y) = \frac{\pi/2-\epsilon}{\sqrt{\kappa}}, \kappa > 0$ for $\epsilon \in (0, \pi/2)$ and $S : D \to D$ is uniformly continuous and nearly asymptotically nonexpansive mapping with sequence $\{s_m, \eta(S^m)\}$. Suppose that $t_0 \in D$ is arbitrary and the sequence $\{t_m\}$ is generated by modified k-iteration in (1), where $\{\alpha_m\}$, $\{\beta_m\}$ are two sequences satisfying the conditions $\sum_{m=1}^{\infty} s_m < \infty$ and $\sum_{m=1}^{\infty}(\eta(S^m) - 1) < \infty$. Assume that S satisfies condition (I). Then sequence $\{t_m\}$ strongly converges to a fixed point of the nearly asymptotically nonexpansive mapping S.*

Proof. Using Theorem 3.1, we have $\lim_{m\to\infty} d(t_m, St_m) = 0$ and $\lim_{m\to\infty} d(t_m, F(S))$ exists. The Condition (I) yields

$$\lim_{m\to\infty} g(d(t_m, F(S))) \leq \lim_{m\to\infty} d(t_m, St_m) = 0$$

that is, $\lim_{m\to\infty} g(d(t_m, F(S))) = 0$. The function g is nondecreasing with $g(\tau) > 0, \tau \in (0, \infty)$ and $g(0) = 0$. Thus we have

$$\lim_{m\to\infty} d(t_m, F(S)) = 0.$$

Therefore the theorem is proved by Theorem 3.3. □

The following result can be obtained by Theorem 3.7 directly.

Theorem 3.8. *Suppose that D is nonempty, closed and convex subset of complete* CAT(κ) *space (Y, d) whose diam$(Y) = \frac{\pi/2-\epsilon}{\sqrt{\kappa}}, \kappa > 0$ for $\epsilon \in (0, \pi/2)$ and $S : D \to D$ is uniformly continuous and nearly* *asymptotically nonexpansive mapping with sequence* $\{s_m, \eta(S^m)\}$. *Suppose that* $t_0 \in D$ *is arbitrary and the sequence* $\{t_m\}$ *is generated by modified k-iteration in (1), where* $\{\alpha_m\}$, $\{\beta_m\}$ *are two sequences satisfying the conditions* $\sum_{m=1}^{\infty} s_m < \infty$ *and* $\sum_{m=1}^{\infty}(\eta(S^m) - 1) < \infty$. *Assume that* S *satisfies condition* (I). *Then sequence* $\{t_m\}$ *strongly converges to fixed point of nearly asymptotically nonexpansive mapping* S.

4 Conclusion

Our Theorems improve and extend the corresponding results in the following ways:

1. Hussain *et al.* [10] proved the strong and weak convergence results for Suzuki's generalized nonexpansive mapping in uniformly convex Banach space, whereas we prove the strong and $\Delta-$convergence results for nearly asymptotically nonexpansive mapping in CAT(κ) space.

2. Ullah *et al.* [21] proved the strong and $\Delta-$convergence results for Suzuki's generalized nonexpansive mapping in CAT(0) space, whereas we prove the strong and $\Delta-$convergence results for nearly asymptotically nonexpansive mapping in CAT(κ) space.

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