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A Direct Analytic Method for Solution of Linear Fredholm Integral and Integro-Differential Equations of the Second Kind

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Abstract. In this paper, a direct analytic method is given for the solution of the linear Fredholm integral and integro-differential equations of the second kind, which is based on the span of the known function, under the action of the operator defined by the kernel. The necessary conditions for using this method are so weak that extends its applicability. The solved examples show the strength of this method.

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1. Introduction

There are several methods for approximating the solutions of integral and integro-differential equations [2, 6-12, 14, 17], but there are few direct methods for finding the solutions of such equations. The method

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for solution of the integral equations with separable kernels is one of them [11]. Some other such methods are given by Babolian, Masouri and Hatamzadeh in 2008 [3] and by Navbpour, Maalek Ghaini, Hosseini and Mohyud-Din in 2011 [12].
The method presented in this paper is close to the method of undetermined coefficients for solving nonhomogeneous linear ODEs. We will consider linear Fredholm integral equations of the second kind (FIE) of the form:

$$u(x) = f(x) + \lambda \int_a^b k(x,t)u(t)dt, \qquad a \leqslant x \leqslant b \tag{1}$$

in which $k : [a,b] \times [a,b] \longrightarrow \mathbb{R}$ is square integrable, $f : [a,b] \longrightarrow \mathbb{R}$ is a known function and λ is a known parameter. Moreover the equations are supposed to be unisolvable.

2. The Method

We will begin this section by the following theorem:

Theorem 2.1. *Given any FIE in the form (1) and any integrable function $u_1(x)$, there is another FIE with the same kernel, $k(x,t)$ that $v(x) = u_1(x) - u(x)$ is its solution and*

$$f_1(x) = u_1(x) - \lambda \int_a^b k(x,t)u_1(t)dt - f(x)$$

is its known part.

Proof. Consider a FIE in the form (1) with the unique solution $u(x)$, and let $u_1(x)$ be a known integrable function and put $v(x) = u_1(x) - u(x)$. Then $u(x) = u_1(x) - v(x)$ and we have:

$$\begin{aligned} u_1(x) - v(x) &= f(x) + \lambda \textstyle\int_a^b k(x,t)(u_1(t) - v(t))dt \\ &= f(x) + \lambda \textstyle\int_a^b k(x,t)u_1(t) - \lambda \int_a^b k(x,t)v(t)dt. \end{aligned}$$

So we obtain

$$-v(x) = f(x) - u_1(x) + \lambda \int_a^b k(x,t)u_1(t)dt - \lambda \int_a^b k(x,t)v(t)dt,$$

and so

$$v(x) = [(u_1(x) - \lambda \int_a^b k(x,t)u_1(t)dt) - f(x)] + \lambda \int_a^b k(x,t)v(t)dt. \quad (2)$$

Now $u(x)$ satisfies (1) if and only if $v(x)$ satisfies (2) and $u(x) = u_1(x) - v(x)$. So for any given equation of the form (1) and for any given integrable function $u_1(x)$, we have an integral equation with the same kernel of the form (2). □

Remark 2.2. *We can also put $v(x) = u_1(x) + u(x)$ as $v(x) = u_1(x) - u(x)$ in theorem below and get the same result (not exactly).*

As a consequence of Theorem 2.1 we obtain a method for constructing the exact solution of (1).

Corollary 2.3. *Suppose $\{u_j(x)\}_{j=1}^n$ is a given finite sequence of integrable functions and $u(x)$ is the solution of (1). Then*

$$u(x) = \sum_{j=1}^{n} (-1)^{j-1} u_j(x), \quad (3)$$

if and only if

$$f(x) = \sum_{j=1}^{n} (-1)^{j-1} (u_j(x) - \lambda \int_a^b k(x,t)u_j(t)dt). \quad (4)$$

Proof. For a given $\{u_j(x)\}_{j=1}^n$ we can use Theorem 2.1, n times and construct a sequence of functions $\{v_j(x)\}_{j=1}^n$ such that $v_1(x) = u_1(x) - u(x)$ and $v_j(x) = u_j(x) - v_{j-1}(x)$, $2 \leqslant j \leqslant n$. Then we have

$$u(x) = u_1(x) - v_1(x) = u_1(x) - [u_2(x) - v_2(x)] = \ldots =$$

$$\textstyle\sum_{j=1}^{n} (-1)^{j-1} u_j(x) + (-1)^n v_n(x),$$

and

$$v_n(x) = \textstyle\sum_{j=0}^{n-1} \{(-1)^j [u_{n-j}(x) - \lambda \int_a^b k(x,t)u_{n-j}(t)dt]\} + (-1)^n f(x) +$$

$$\lambda \textstyle\int_a^b k(x,t)v_n(t)dt. \quad (5)$$

Now it is trivial that (5) has solution $v_n(x) = 0$ if and only if (4) is satisfied, and then (3) is also true. □

We can also consider linear Fredholm integro-differential equations of the second kind (FIDE) of the form:

$$\begin{gathered} L[u](x) = f(x) + \lambda \int_a^b k(x,t)u(t)dt, \quad a \leqslant x \leqslant b, \\ u^{(j)}(a) = b_j, \quad 0 \leqslant j \leqslant n-1 \end{gathered} \tag{6}$$

such that $L = \sum_{i=0}^{n} a_i D^i$, where D is the differentiation operator. By a similar discussion we have the same result for Fredholm integro-differential equations of the form (6).

Theorem 2.4. *Given any Fredholm integro-differential equation in the form (6) and any integrable function $u_1(x)$. There is another Fredholm integro-differential equation with the same kernel $k(x,t)$ and initial conditions $v^{(j)}(a) = u_1^{(j)}(a) - u^{(j)}(a)$, that $v(x) = u_1(x) - u(x)$ is its solution and*

$$f_1(x) = u_1(x) - \lambda \int_a^b k(x,t)u_1(t)dt - f(x)$$

is its known part.

Proof. Let $u_1(x)$ be a known integrable function and $v(x) = u_1(x) - u(x)$. For integral part of (6) we have:

$$\lambda \int_a^b k(x,t)u_1(t)dt = \lambda \int_a^b k(x,t)(u(t)+v(t))dt = \lambda \int_a^b k(x,t)u(t)dt +$$

$$\lambda \int_a^b k(x,t)v(t)dt = L[u](x) - f(x) + \lambda \int_a^b k(x,t)v(t)dt = L[u_1 - v](x) -$$

$$f(x) + \lambda \int_a^b k(x,t)v(t)dt = L[u_1](x) - L[v](x) - f(x) + \lambda \int_a^b k(x,t)v(t)dt,$$

So we obtain

$$\begin{gathered} L[v](x) = (L[u_1](x) - \lambda \int_a^b k(x,t)u_1(t)dt) - f(x) + \lambda \int_a^b k(x,t)v(t)dt, \\ v^{(j)}(a) = u_1^{(j)}(a) - u^{(j)}(a), \qquad 0 \leqslant j \leqslant n-1. \end{gathered} \tag{7}$$

Now $v(x)$ is the solution of integro-differential equation (7) if and only if $u(x)$ is the solution of integro-differential equation (6). □

Corollary 2.5. *Suppose* $\{u_j(x)\}_{j=1}^m$ *is a given sequence of integrable functions and* $u(x)$ *is the unique solution of (6). Then*

$$u(x) = \sum_{j=1}^{m} (-1)^{j-1} u_j(x),$$

if and only if

$$f(x) = \sum_{j=1}^{m} (-1)^{j-1} [u_j(x) - \lambda \int_a^b k(x,t) u_j(t) dt],$$

and

$$u^{(i)}(a) = \sum_{j=1}^{m} (-1)^{j-1} u_j^{(i)}(a), \qquad 0 \leqslant i \leqslant (n-1).$$

Proof. By Theorem 2.4, proof is trivial. □

3. Structure of the Method

Let $[Ku](x) = \int_a^b k(x,t)u(t)dt$. To solve integral equations by this method, first we choose a suitable sequence of functions $\{u_r(x)\}_{r\in J}$, where J is an index set, as basis functions and then find the new sequence of functions

$$w_r(x) = [(I - \lambda K)u_r](x), r \in J.$$

Then we choose a finite subset $w_1(x), w_2(x), \cdots, w_m(x)$ of $\{w_r(x)\}_{r\in J}$ and corresponding functions $u_1(x), u_2(x), \cdots, u_m(x)$ for some positive integer m. Now by Corollary 2.3 we have

$$f(x) = \sum_{j=1}^{m} (-1)^{j-1} c_j w_j(x), \tag{8}$$

if and only if

$$u(x) = \sum_{j=1}^{m} (-1)^{j} c_j u_j(x), \tag{9}$$

where $c_j, j = 1, 2, \cdots, m$ are constants.
Also for solving the integro-differential equation (6), if we put

$$z_i(x) = [(L - \lambda K)u_i](x), i = 1, 2, \cdots, m \quad L = \sum_{j=0}^{n} a_j D^j,$$

for some positive integer m, then by Corollary 2.5 we have

$$f(x) = \sum_{j=1}^{m} (-1)^{j-1} c_j z_j(x), \tag{10}$$

if and only if

$$\begin{aligned} u(x) &= \textstyle\sum_{j=1}^{m} (-1)^{j-1} c_j u_j(x), \\ b_i = u^{(i)}(a) &= \textstyle\sum_{j=1}^{m} (-1)^{j-1} c_j u_j^{(k)}(a), \qquad 0 \leqslant i \leqslant n-1. \end{aligned} \tag{11}$$

Note that in some problems, $u(x)$ is the solution of equation (6) and also $u(x)$ is a solution of $L[y](x) = 0$. So we have

$$f(x) = [(L - \lambda K)u](x) = L[u](x) - \lambda K[u](x) = -\lambda K[u](x).$$

Let $\{u_1, \cdots, u_m\}$ be a set of basis functions and

$$f(x) = -\lambda \sum_{j=1}^{m} (-1)^j c_j K[u_j](x).$$

If $v(x) = \sum_{j=1}^{m} (-1)^j c_j u_j(x)$ satisfies the initial conditions (11) then $v(x)$ is the solution of (6). If not, we need to adding some new functions to the set of basis functions which these functions are the solutions of equation, $L[y](x) = 0$ (see examples (4.6) and (4.7)).
We will state more details in examples in the next section.

4. Examples

In this section we demonstrate the strength of our method by using it to solve some examples. To explain the method we will start with some simple examples.

Example 4.1. Consider the equation appearing in [16],

$$u(x) = cosh(x) + (sinh(1))x + e^{-1} - 1 - \int_0^1 (x-t)u(t)dt, \qquad (12)$$

with the exact solution $u(x) = cosh(x)$.
The sequence of functions $u_r(x) = e^{rx}$, $r \in \mathrm{R} - \{0\}$ seems to be suitable, and we have

$$[(I - \lambda K)u_r](x) = u_r(x) - \int_0^1 (x-t)u_r(t)dt = e^{rx} - \int_0^1 (x-t)e^{rt}dt$$

$$= e^{rx} + \frac{x}{r} - \frac{e^r}{r}(1 - \frac{1}{r}) - \frac{1}{r^2}, \qquad r \in \mathrm{R} - \{0\}.$$

Now, we must choose those r's which are necessary for generating the function

$$f(x) = cosh(x) + xsinh(1) + e^{-1} - 1 = \frac{e^x}{2} + \frac{e^{-x}}{2} + \frac{e}{2}x - \frac{e^{-1}}{2}x + e^{-1} - 1.$$

So r's must be such that, $w_r(x)$'s expand e^x and e^{-x} , and this gives $r = \pm 1$. For $r = 1$ and $r = -1$ we have

$$w_1(x) = [(I - \lambda K)u_1](x) = (I - K)e^x = e^x + x(e-1) - 1,$$

and

$$w_{-1}(x) = [(I - \lambda K)u_{-1}](x) = (I - K)e^{-x} = e^{-x} - x(e^{-1} - 1) + 2e^{-1} - 1,$$

respectively. Rename $w_{-1}(x)$ by $w_2(x)$ and $u_{-1}(x)$ by $u_2(x)$. So according to (8) we choose c_1 and c_2 such that

$$f(x) = c_1 w_1(x) - c_2 w_2(x) = c_1[(I - \lambda K)u_1](x) - c_2[(I - \lambda K)u_2](x),$$

i.e.

$$\frac{e^x}{2} + \frac{e^{-x}}{2} + \frac{e}{2}x - \frac{e^{-1}}{2}x + e^{-1} - 1 = c_1 w_1(x) - c_2 w_2(x) =$$

$$c_1[e^x + x(e-1) - 1] - c_2[e^{-x} - x(e^{-1} - 1) + 2e^{-1} - 1],$$

which gives $c_1 = -c_2 = \frac{1}{2}$ and so by (9) we have

$$u(x) = c_1u_1(x) - c_2u_2(x) = \frac{1}{2}e^x + \frac{1}{2}e^{-x} = cosh(x).$$

The exact solution of (12).

Example 4.2. Consider the integro-differential equation appearing in [16],

$$u'(x) = xe^x + e^x - x + \int_0^1 xu(t)dt, \qquad u(0) = 1.$$

So $\lambda = 1$, and for finding the solution $u(x)$ we apply the operator $D-K$ on suitable functions to span the known function $f(x) = xe^x + e^x - x$. It seems that the functions $v_{r,m}(x) = x^m e^{rx}, m = 0, 1, \cdots, r \in$ R are suitable for generating the other part of the known function, $(xe^x + e^x)$, and we have

$$z_{r,m}(x) = (D-K)(x^m e^{rx}) = (mx^{m-1}+rx^m)e^{rx} - x(\frac{1}{r}t^m e^{rt} - \frac{m}{r}t^{m-1}e^{rt} + \cdots +$$

$$(-1)^m \frac{m!}{r^{m+1}}e^{rt})|_{t=0}^1 = (mx^{m-1}+x^m)e^{rx} - \frac{1}{r^{m+1}}x(r^m e^r - mr^{m-1}e^r + \cdots +$$

$$(-1)^m m!e^r - (-1)^m m!), \qquad r \neq 0.$$

For $r = 0$ we have, $v_{0,m} = x^m$,

$$z_{0,0}(x) = [(D - K)v_{0,0}](x) = -x$$

and

$$z_{0,m}(x) = [(D-K)v_{0,m}](x) = mx^{m-1} - \int_0^1 xt^m dt = mx^{m-1} - \frac{x}{m+1}, \qquad m \neq 0.$$

For $m \geqslant 3$, the degree of $z_{0,m}(x)$ is grater than 2. So to span the polynomial part of the known function, we only need $m = 0, 1, 2$, and moreover,

$$z_{0,1}(x) = (D - K)x = 1 - \frac{x}{2} and z_{0,2}(x) = (D - K)x^2 = \frac{5}{3}x.$$

But, for expanding $(xe^x + e^x)$ by $z_{r,m}(x)$'s

$$(r, m) = (0, 0), (0, 1), (0, 2), (1, 0), (1, 1).$$

Now, according to (10) we have

$$xe^x + e^x - x = c_1 z_{0,0}(x) - c_2 z_{0,1}(x) + c_3 z_{0,2}(x) - c_4 z_{1,0}(x) + c_5 z_{1,1}(x)$$

$$= c_1(-x) - c_2(1 - \tfrac{x}{2}) + c_3(\tfrac{5}{3}x) - c_4(e^x - x(e-1)) + c_5(xe^x + e^x - x). \tag{13}$$

According to (11) we should have

$$u(x) = c_1 v_{0,0}(x) - c_2 v_{0,1}(x) + c_3 v_{0,2}(x) - c_4 v_{1,0}(x) + c_5 v_{1,1}(x)$$

$$= c_1 - c_2 x + c_3 x^2 - c_4 e^x + c_5 xe^x, \tag{14}$$

and

$$1 = u(0) = c_1 - c_4.$$

Now from (13) and (15) we obtain that

$$c_1 = 1, c_2 = 0, c_3 = \frac{3}{5}, c_4 = 0, c_5 = 1,$$

and so

$$u(x) = 1 + \frac{3}{5}x^2 + xe^x,$$

is the exact solution of the integro-differential equation.

Example 4.3. Consider the integral equation appearing in [1],

$$u(x) = -\frac{1}{6}(2x^3 - 9x + 2) + \int_0^1 |x - t| u(t) dt. \tag{16}$$

We see that the known function is a polynomial and the kernel is polynomial in parts of interval $[0, 1]$. So to solve the problem, we use the polynomial functions as follows.

Putting $u_r(x) = x^r, r \in \mathbb{N} \cup \{0\}$, we have

$$w_r(x) = [(I - \lambda K)u_r](x) = (I - \lambda K)x^r = x^r - \int_0^1 |x - t| t^r dt = x^r -$$

$$\int_0^x (x - t)t^r dt - \int_x^1 (t - x)t^r dt = \tfrac{-2}{(r+1)(r+2)} x^{r+2} + x^r + \tfrac{x}{r+1} - \tfrac{1}{r+2}. \tag{17}$$

The known function is a polynomial of degree 3. So by (17), $r = 0, r = 1$ are sufficient, and

$$w_0(x) = -x^2 + x - \frac{1}{2}, w_1(x) = -\frac{1}{3}x^3 + \frac{3}{2}x - \frac{1}{3}.$$

So we put

$$u(x) = c_1 x^0 - c_2 x^1,$$

and

$$f(x) = -\frac{1}{6}(2x^3 - 9x + 2) = c_1 w_0 - c_2 w_1 = c_1(-x^2 + x - \frac{1}{2}) - c_2(-\frac{1}{3}x^3 + \frac{3}{2}x - \frac{1}{3}).$$

Thus we obtain $c_1 = 0$ and $c_2 = -1$ and so

$$u(x) = x,$$

which is the exact solution of (16).
Now we can solve a special integral equation via our new method, which is not easily solvable by old methods.

Example 4.4. Consider the integral equation:

$$\varphi(x) = \lambda \int_0^\infty e^{-|x-y|} \varphi(y) dy. \tag{18}$$

which is a homogeneous singular integral equation [15, 16].
If we choose the sequence $\{u_r(x)\} = \{e^{rx}\}, r \in R$ then we have

$$\lambda \int_0^\infty k(x,y) u_r(y) dt = \lambda \int_0^\infty e^{-|x-y|} e^{ry} dy = \lambda \int_0^x e^{-(x-y)} e^{ry} dy +$$

$$\lambda \int_x^\infty e^{+(x-y)} e^{ry} dy = (\frac{\lambda}{r+1} - \frac{\lambda}{r-1}) e^{rx} - \frac{\lambda}{r+1} e^{-x}, r \neq -1, r < 1,$$

and then

$$[(I - \lambda K) u_r](x) = e^{rx} - \frac{-2\lambda}{r^2 - 1} e^{rx} + \frac{\lambda}{r+1} e^{-x} = (\frac{r^2 + 2\lambda - 1}{r^2 - 1}) e^{rx} + \frac{\lambda}{r+1} e^{-x}.$$

So according to (8) we must choose those r's which are necessary for generating $f(x) = 0$. So r's must be such that the coefficients of e^{rx}'s become zero, which gives $r^2 + 2\lambda - 1 = 0$, with the roots

$$r_1 = \sqrt{1 - 2\lambda}, \qquad r_2 = -\sqrt{1 - 2\lambda}.$$

Moreover for $r_1 = \sqrt{1-2\lambda}$ and $r_2 = -\sqrt{1-2\lambda}$ we have

$$u_1(x) = e^{x\sqrt{1-2\lambda}} \quad , \quad u_2(x) = e^{-x\sqrt{1-2\lambda}}$$

respectively. Now

$$u(x) = c_1u_1(x) - c_2u_2(x) = c_1e^{x\sqrt{1-2\lambda}} - c_2e^{-x\sqrt{1-2\lambda}},$$

and according to (8) we have

$$0 = f(x) = c_1(I-\lambda K)u_1(x) - c_2(I-\lambda K)u_2(x) = c_1(I-\lambda K)e^{x\sqrt{1-2\lambda}}$$

$$-c_2(I-\lambda K)e^{-x\sqrt{1-2\lambda}} = -c_1\frac{\lambda}{1+\sqrt{1-2\lambda}}e^{-x} + c_2\frac{\lambda}{1-\sqrt{1-2\lambda}}e^{-x},$$

which implies that

$$-c_1\frac{\lambda}{1+\sqrt{1-2\lambda}} + c_2\frac{\lambda}{1-\sqrt{1-2\lambda}} = 0.$$

Thus for any real β we have $c_1 = \beta(1+\sqrt{1-2\lambda})$ and $c_2 = \beta(1-\sqrt{1-2\lambda})$. So according to (9),

$$u(x) = c_1u_1(x) - c_2u_2(x) = \beta[(\sqrt{1-2\lambda}+1)e^{x\sqrt{1-2\lambda}} + (\sqrt{1-2\lambda}-1)e^{-x\sqrt{1-2\lambda}}]$$

is the exact solution of (18).

Example 4.5. Consider the integral equation appearing in [1],

$$u(x) = (1-\frac{1}{\pi^2})sin\pi x + (2-\frac{8}{\pi^2})cos(\frac{\pi}{2}x) + \frac{8(1-x)}{\pi^2} + \int_0^1 k(x,t)u(t)dt \tag{19}$$

where

$$k(x,t) = \begin{cases} x(1-t), & x \leqslant t \\ t(1-x), & t < x \end{cases} .$$

If we choose the sequences $u_r(x) = cos(rx)$ and $v_s(x) = sin(sx)$, $r,s \in \mathbb{R} - \{0\}$, then we have

$$w_r(x) = [(I-\lambda K)u_r](x) = (I-\lambda K)cos(rx) = cos(rx) - \int_0^1 k(x,t)cos(rt)dt =$$

$cos(rx) - \int_0^x t(1-x)cos(rt)dt - \int_x^1 x(1-t)cos(rt)dt = (1-\frac{1}{r^2})cosrx-$

$\frac{1}{r^2}(xcosr + 1 - x)$. Also,

$z_s(x) = (I - \lambda K)v_s(x) = sinsx - \int_0^1 k(x,t)sin(st)dt = (1 - \frac{1}{s^2})sinsx + \frac{x}{s^2}sin(s).$

First of all, by comparison between $w_r(x)$ and $z_s(x)$ with

$$f(x) = (1-\frac{1}{\pi^2})sin\pi x - \frac{8}{\pi^2}cos(\frac{\pi}{2}x) + \frac{8(1-x)}{\pi^2},$$

we must have $s = \pi$ and $r = \frac{\pi}{2}$. So

$$u(x) = c_1 cos(rx) - c_2 sin(sx) = c_1 cos(\frac{\pi}{2}x) - c_2 sin(\pi x),$$

if and only if

$$f(x) = (1-\frac{1}{\pi^2})sin(\pi x) - \frac{8}{\pi^2}cos(\frac{\pi}{2}x) + \frac{8(1-x)}{\pi^2} =$$

$c_1[(1-\frac{4}{\pi^2})cos(\frac{\pi}{2}x) - \frac{4}{\pi^2}(xcos(\frac{\pi}{2}) + 1 - x)] - c_2[(1-\frac{1}{\pi^2})sin(\pi x) + \frac{x}{\pi}sin(\pi)],$
which gives $c_1 = 2, c_2 = -1$. Then

$$u(x) = sin(\pi x) + 2cos(\frac{\pi}{2}x),$$

is the exact solution of (19).
Note that a part of $f(x)$ is a polynomial, so in general we must use the sequence $\{x^n\}_{n=0}^{\infty}$, but for this problem two sequences $\{u_r(x)\}$ and $\{v_s(x)\}$ were necessary.

The following example shows that considering solution of $L[u](x) = 0$ is essential.

Example 4.6. Consider the integro-differential equation

$$u'' - u = 1 - e + \int_0^1 u(t)dt, \quad u(0) = 1, u'(0) = 1.$$

The known function and the kernel of equation are polynomial of degree zero, so the set of functions $u_n(x) = x^n, n = 0, 1, \cdots$ seems to be suitable, and we have

$$w_n(x) = [(L - \lambda K)u_n](x) = [D^2 - I - K](x^n) =$$

$$n(n-1)x^{n-2} - x^n - \frac{1}{n+1}, \qquad n \geqslant 2,$$

$$w_0(x) = [(L - \lambda K)](1) = -2, \qquad w_1(x) = [(L - \lambda K)](x) = -x - \frac{1}{2}.$$

By comparison between $w_n(x)$ and $f(x) = 1 - e$, only $w_0(x)$ can span $f(x)$, so only $u_0(x) = 1$ must span $u(x)$. But by considering initial conditions, we find that $u_0(x) = 1$ can not span $u(x)$. Therefore we consider the linear independent functions $\{e^x, e^{-x}\}$, the solutions of $0 = L[u] = u'' - u$ as the basis functions and hence

$$z_1(x) = [(L - \lambda K)](e^x) = [D^2 - I - K](e^x) = 1 - e,$$

$$z_2(x) = [(L - \lambda K)](e^{-x}) = [D^2 - I - K](e^x) = e^{-1} - 1.$$

Now we have

$$1-e = f(x) = c_1 w_0(x) - c_2 z_1(x) + c_3 z_2(x) = c_1(-2) - c_2(1-e) + c_3(e^{-1} - 1),$$

$$u(x) = c_1 - c_2 e^x + c_3 e^{-x},$$

$$1 = u(0) = c_1 - c_2 + c_3,$$

$$1 = u'(0) = -c_2 - c_3.$$

So obviously $c_1 = c_3 = 0$, $c_2 = -1$ and then

$$u(x) = e^x,$$

is the exact solution of equation.

Example 4.7. Consider the integro-differential equation appearing in [8] and [17],

$$u'(x) - u(x) = f(x) + \int_0^1 sin(4\pi x + 2\pi t)u(t)dt, \quad u(0) = 1, \qquad (20)$$

with

$$f(x) = (6\pi - 1)cos(2\pi x) - (2\pi + 3)sin(2\pi x) - \frac{1}{2}[sin(4\pi x) + 3cos(4\pi x)].$$

To solve this equation we choose two sequences of functions

$$u_r(x) = cos(rx), \quad r \in \mathbb{R} - \{0\}$$

$$v_s(x) = sin(sx), \quad s \in \mathbb{R} - \{0\},$$

and obtain

$$w_r(x) = [(D-I-K)u_r](x) = (D-I-K)cos(rx) = -rsin(rx) - cos(rx) -$$

$$\int_0^1 sin(4\pi x + 2\pi t)cos(rt)dt = -rsin(rx) - cos(rx) + \frac{1}{2(2\pi + r)}cos(4\pi x + r) -$$

$$\frac{1}{2(2\pi - r)}cos(4\pi x - r) + [\frac{1}{2(2\pi - r)} - \frac{1}{2(2\pi + r)}]cos(4\pi x), \qquad r \neq \pm 2\pi.$$

Also we have

$$w_{\pm 2\pi}(x) = (D - I - \lambda K)u_{\pm 2\pi}(x) = (D - I - \lambda K)cos(2\pi x)$$

$$= -2\pi sin(2\pi x) - cos(2\pi x) - \tfrac{1}{2}sin(4\pi x).$$

and

$$z_s(x) = (D-I-\lambda K)v_s(x) = (D-I-\lambda K)sin(sx) = scos(sx) - sin(sx) -$$

$$\int_0^1 sin(4\pi x + 2\pi t)sin(st)dt = scos(sx) - sin(sx) - \frac{1}{2(2\pi - s)}sin(4\pi x - s) +$$

$$\frac{1}{2(2\pi + s)}sin(4\pi x + s) + [\frac{1}{2(2\pi - s)} - \frac{1}{2(2\pi + s)}]sin(4\pi x), \qquad s \neq \pm 2\pi.$$

Moreover we have $z_{\pm 2\pi} = \pm z_{2\pi}$ and $z_{2\pi}(x) = (D - I - K)v_{2\pi}(x)$

$$= (D - I - K)sin(2\pi x) = 2\pi cos(2\pi x)$$
$$-sin(2\pi x) - \tfrac{1}{2}cos(4\pi x).$$

The special function that must be considered separately is $y_1(x) = e^x$, which is the solution of the homogeneous ordinary differential equation $y' - y = 0$. So we put

$$\begin{aligned} p_1(x) &= [(D - I - K)y_1](x) \\ &= (D - I - K)e^x \\ &= -\int_0^1 sin(4\pi x + 2\pi t)e^t dt \\ &= \tfrac{1-e}{1+4\pi^2}[sin(4\pi x) - 2\pi cos(4\pi x)]. \end{aligned}$$

Now by comparison between $f(x)$, $w_r(x)$, $z_s(x)$ and $p_1(x)$, the functions that can span $f(x)$ as a linear combination are

$$\begin{gathered} w_{2\pi}(x) = (D - I - K)cos(2\pi x) = -2\pi sin(2\pi x) - cos(2\pi x) - \tfrac{1}{2}sin(4\pi x), \\ w_{4\pi}(x) = (D - I - K)cos(4\pi x) = -4\pi sin(4\pi x) - cos(4\pi x), \\ z_{2\pi}(x) = (D - I - K)sin(2\pi x) = 2\pi cos(2\pi x) - sin(2\pi x) - \tfrac{1}{2}cos(4\pi x), \\ z_{4\pi}(x) = (D - I - K)sin(4\pi x) = 4\pi cos(4\pi x) - sin(4\pi x), \\ p_1(x) = (D - I - K)e^x = \tfrac{1-e}{1+4\pi^2}[sin(4\pi x) - 2\pi cos(4\pi x)]. \end{gathered}$$

So, corresponding functions which can span $u(x)$ are:

$$\begin{aligned} u_{2\pi}(x) &= cos(2\pi x), \\ u_{4\pi}(x) &= cos(4\pi x), \\ v_{2\pi}(x) &= sin(2\pi x), \\ v_{4\pi}(x) &= sin(4\pi x), \\ y_1(x) &= e^x. \end{aligned}$$

Now from (9) we have

$$f(x) = c_1 w_{2\pi}(x) - c_2 w_{4\pi}(x) + c_3 z_{2\pi}(x) - c_4 z_{4\pi}(x) + c_5 p_1(x) \qquad (21)$$

if and only if

$$u(x) = c_1 v_{2\pi}(x) - c_2 v_{4\pi}(x) + c_3 z_{2\pi}(x) - c_4 z_{4\pi}(x) + c_5 y_1(x), u(0) = 1. \qquad (22)$$

By using (21) and $u(0) = 1$ in (22) we have

$$\begin{aligned} &f(x) = (6\pi - 1)cos(2\pi x) - (2\pi + 3)sin(2\pi x) - \tfrac{1}{2}[sin(4\pi x) + 3cos(4\pi x)] \\ &= c_1[-2\pi sin(2\pi x) - cos(2\pi x) - \tfrac{1}{2}sin(4\pi x)] - c_2[-4\pi sin(4\pi x) - cos(4\pi x)] \\ &+c_3[2\pi cos(2\pi x) - sin(2\pi x) - \tfrac{1}{2}cos(4\pi x)] - c_4[4\pi cos(4\pi x) - sin(4\pi x)] \\ &+c_5\tfrac{1-e}{1+4\pi^2}[sin(4\pi x) - 2\pi cos(4\pi x)], \\ &1 = u(0) = c_1 - c_2 + c_5. \end{aligned}$$

By equality the coefficients we obtain the linear system of equations:

$$\begin{cases} -2\pi c_1 - c_3 = -(2\pi + 3) \\ -c_1 + 2\pi c_3 = 6\pi - 1 \\ -\frac{1}{2}c_1 + 4\pi c_2 + c_4 + \frac{1-e}{1+4\pi^2}c_5 = -\frac{1}{2} \\ c_2 - \frac{1}{2}c_3 - 4\pi c_4 - \frac{2\pi(1-e)}{1+4\pi^2}c_5 = -\frac{3}{2} \\ c_1 - c_2 + c_5 = 1 \end{cases}$$

which has the unique solution

$$c_1 = 1, c_3 = 3, c_2 = c_4 = c_5 = 0.$$

So the exact solution of integro-differential equation is given by

$$u(x) = cos(2\pi x) + 3sin(2\pi x).$$

5. Conclusion

In this paper we have proposed a new method for solution of linear FIE's and FIDE's of the second kind. This method is not a general method for solving these equations. Moreover, this method can simply find the solutions of the equations with known functions and kernels, which can be polynomials or exponential functions or $sin(rx)$ or $cos(rx)$, $r \in \mathbb{R}$ or products of such functions. We also hope to be able to use this method for solving other kinds of linear integral and integro-differential equations (such as Volterra-Fredholm integral and integro-differential equations).

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