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Original Research Paper

On Invariant Submanifolds of Lorentz Sasakian Space Forms

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Abstract. In this article, invariant submanifolds of Lorentz-Sasakian space forms on the W_7- curvature tensor are investigated. For the W_7-curvature tensor, the pseudoparallel, $2-$pseudoparallel, Ricci generalized pesudoparallel and $2-$Ricci generalized pseudoparallel properties of the invariant submanifolds of the Lorentz-Sasakian space form are discussed.

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1 Introduction

$\phi-$sectional curvature plays the an important role for Sasakian manifold. If the $\phi-$sectional curvature of a Sasakian manifold is constant,

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then the manifold is a Sasakian-space-form [2]. P. Alegre and D. Blair described generalized Sasakian space forms [11]. P. Alegre and D. Blair obtained important properties of generalized Sasakian space forms in their studies and gave some examples. P. Alegre and A. Carriazo later discussed generalized indefinite Sasakian space forms [10]. Generalized indefinite Sasakian space forms are also called Lorentz-Sasakian space forms, and Lorentz manifolds are of great importance for Einstein's theory of Relativity. Sasakian space forms, generalized Sasakian space forms and Lorentz-Sasakian space forms have been discussed by many scientists and important properties of these manifolds have been obtained ([1],[3],[4],[8],[9]).

Many mathematicians have considered the submanifolds of manifolds such as $K-$paracontak, Lorentzian para-Kenmotsu, almost Kenmotsu and studied their various characterizations ([5],[6],[7]).

In this article, total geodesic submanifolds for Lorentz-Sasakian space forms are investigated. For these submanifolds, pseudoparallel, 2-pseudoparallel, Ricci generalized pseudoparallel and 2-Ricci generalized pseudoparallel invariant submanifolds have been studied and many new results have been obtained. In addition, necessary and sufficient conditions have been obtained for these submanifolds to be total geodesic on the concircular and projective curvature tensors.

2 Preliminary

Let $\tilde{M}$ be a $(2m+1)-$dimensional Lorentz manifold. If the $\tilde{M}$ Lorentz manifold with (ϕ, ξ, η, g) structure tensors satisfies the following conditions, this manifold is called a Lorentz-Sasakian manifold

$$\phi^2 X_1 = -X_1 + \eta(X_1)\xi, \eta(\xi) = 1, \eta(\phi X_1) = 0,$$

$$g(\phi X_1, \phi X_2) = g(X_1, X_2) + \eta(X_1)\eta(X_2), \eta(X_1) = -g(X_1, \xi),$$

$$\left(\tilde{\nabla}_{X_1}\phi\right)X_2 = -g(X_1, X_2)\xi - \eta(X_2)X_1, \tilde{\nabla}_{X_1}\xi = -\phi X_1,$$

where, $\tilde{\nabla}$ is the Levi-Civita connection according to the Riemann metric g.

The plane section Π in $T_x\tilde{M}$. If the Π plane is spanned by X_1 and ϕX_1, this plane is called the ϕ-section. The curvature of the ϕ-section is called the ϕ-sectional curvature. If the Lorentz-Sasakian manifold has a constant ϕ-sectional curvature, this manifold is called the Lorentz-Sasakian space form and is denoted by $\tilde{M}(c)$. The curvature tensor of the Lorentz-Sasakian space form $\tilde{M}(c)$ is defined as

$$\tilde{R}(X_1,X_2)X_3 = \left(\tfrac{c-3}{4}\right)\{g(X_2,X_3)X_1 - g(X_1,X_3)X_2\}$$

$$+\left(\tfrac{c+1}{4}\right)\{g(X_1,\phi X_3)\phi X_2 - g(X_2,\phi X_3)\phi X_1$$

$$+2g(X_1,\phi X_2)\phi X_3 + \eta(X_2)\eta(X_3)X_1 - \eta(X_1)\eta(X_3)X_2$$

$$+g(X_1,X_3)\eta(X_2)\xi - g(X_2,X_3)\eta(X_1)\xi\},$$

for all $X_1, X_2, X_3 \in \chi\left(\tilde{M}\right)$.

Lemma 2.1. *Let $\tilde{M}(c)$ be the $(2m+1)-$dimensional Lorentz-Sasakian space form. The following relations are provided for the Lorentz-Sasakian space forms.*

$$\tilde{\nabla}_{X_1}\xi = -\phi X_1, \tag{1}$$

$$\left(\tilde{\nabla}_{X_1}\phi\right)X_2 = -g(X_1,X_2)\xi - \eta(X_2)X_1, \tag{2}$$

$$\left(\tilde{\nabla}_{X_1}\eta\right)X_2 = g(\phi X_1,X_2),$$

$$\tilde{R}(\xi,X_2)X_3 = -g(X_2,X_3)\xi - \eta(X_3)X_2, \tag{3}$$

$$\tilde{R}(\xi,X_2)\xi = \eta(X_2)\xi - X_2, \tag{4}$$

$$\tilde{R}(X_1,X_2)\xi = \eta(X_2)X_1 - \eta(X_1)X_2, \tag{5}$$

$$S(X_1,\xi) = -\left[\frac{(c+1)-4m}{2}\right]\eta(X_1), \tag{6}$$

where $\tilde{R}, S$ are the Riemann curvature tensor, Ricci curvature tensor of $\tilde{M}(c)$, respectively.

Let M be the immersed submanifold of the $(2m+1)-$dimensional Lorentz-Sasakian space form $\tilde{M}(c)$. Let the tangent and normal subspaces of M in $\tilde{M}(c)$ be $\Gamma(TM)$ and $\Gamma\left(T^{\perp}M\right)$, respectively. Gauss and Weingarten formulas for $\Gamma(TM)$ and $\Gamma\left(T^{\perp}M\right)$ are

$$\tilde{\triangledown}_{X_1}X_2 = \triangledown_{X_1}X_2 + h(X_1, X_2), \tag{7}$$

$$\tilde{\triangledown}_{X_1}X_5 = -A_{X_5}X_1 + \triangledown^{\perp}_{X_1}X_5,$$

respectively, for all $X_1, X_2 \in \Gamma(TM)$ and $X_5 \in \Gamma\left(T^{\perp}M\right)$, where $\triangledown$ and $\triangledown^{\perp}$ are the connections on M and $\Gamma\left(T^{\perp}M\right)$, respectively, h and A are the second fundamental form and the shape operator of M. There is a relation

$$g(A_{X_5}X_1, X_2) = g(h(X_1, X_2), X_5)$$

between the second basic form and shape operator defined as above. The covariant derivative of the second fundamental form h is defined as

$$\left(\tilde{\triangledown}_{X_1}h\right)(X_2, X_3) = \triangledown^{\perp}_{X_1}h(X_2, X_3) - h(\triangledown_{X_1}X_2, X_3) - h(X_2, \triangledown_{X_1}X_3). \tag{8}$$

Specifically, if $\tilde{\triangledown}h = 0$, M is said to be the parallel second fundamental form or $1-$parallel.

Let R be the Riemann curvature tensor of M. In this case, the Gauss equation can be expressed as

$$\tilde{R}(X_1, X_2)X_3 = R(X_1, X_2)X_3 + A_{h(X_1,X_3)}X_2 - A_{h(X_2,X_3)}X_1$$
$$+\left(\tilde{\triangledown}_{X_1}h\right)(X_2, X_3) - \left(\tilde{\triangledown}_{X_1}h\right)(X_1, X_3).$$

Let M be a Riemannian manifold, T is $(0,k)-$type tensor field and A is $(0,2)-$type tensor field. In this case, the tensor field $Q(A,T)$ is defined as

$$Q(A,T)(X_1, ..., X_k; X, Y) = -T((X \wedge_A Y)X_1, ..., X_k)$$
$$-... - T(X_1, ..., X_{k-1}, (X \wedge_A Y)X_k), \tag{9}$$

where

$$(X \wedge_A Y)Z = A(Y,Z)X - A(X,Z)Y,$$

$k \geq 1, X_1, X_2, ..., X_k, X, Y \in \Gamma(TM)$.

Let M be the immersed submanifold of a $(2m+1)-$dimensional Lorentz-Sasakian space form $\tilde{M}(c)$. If $\phi(T_{X_1}M) \subset T_{X_1}M$ in every X_1 point, the M manifold is called invariant submanifold. From this section of the article, we will assume that the manifold M is the invariant submanifold of the Lorentz-Sasakian space form $\tilde{M}(c)$. So it is clear from (3) and (9) that

$$h(X_1, \xi) = 0, h(\phi X_1, X_2) = h(X_1, \phi X_2) = \phi h(X_1, X_2) \tag{10}$$

for all $X_1, X_2 \in \Gamma(TM)$.

The tensor defined as

$$T(X_1, X_2) X_3 = a_0 R(X_1, X_2) X_3 + a_1 S(X_1, X_3) X_2$$

$$+a_2 S(X_1, X_3) X_2 + a_3 S(X_1, X_2) X_3 + a_4 g(X_2, X_3) QX_1$$

$$+a_5 g(X_1, X_3) QX_2 + a_6 g(X_1, X_2) QX_3$$

$$+a_7 [g(X_2, X_3) X_1 - g(X_1, X_3) X_2]$$

in an $n-$dimensional (M, g) semi-Riemann manifold is a $(1,3)-$type tensor field, where $a_0, a_1, ..., a_7$ are smooth functions in M and also R, S, Q and r are $(1,1)-$type curvature tensor, Ricci tensor, Ricci operator and scalar curvature, respectively. If we specifically choose $a_0 = 1, a_1 = -a_4 = \frac{-1}{2m}, a_2 = a_3 = a_5 = a_6 = a_7 = 0$, then we get the W_7-curvature tensor defined as

$$W_7 = R(X_1, X_2) X_3 - \frac{1}{2m} [S(X_2, X_3) X_1 - g(X_2, X_3) QX_1]. \tag{11}$$

For the $(2m+1)-$dimensional $\tilde{M}(c)$ Lorentz Sasakian space form, if we choose $X_1 = \xi, X_2 = \xi$ and $X_3 = \xi$ respectively in (11) from (3), (4), (5), the following relations are obtained.

$$\begin{aligned} W_7(\xi, X_2) X_3 = \tfrac{(c+1)-(m+2)c-(5m+2)}{4m} g(X_1, X_3) \xi \\ -\eta(X_3) X_2 - \tfrac{(c+1)(m+1)}{4m} \eta(X_2) \eta(X_3) \xi, \end{aligned} \tag{12}$$

$$W_7(X_1,\xi)X_3 = \tfrac{(c+1)-(m+2)c+(3m-2)}{4m}\eta(X_3)X_1 \\ +g(X_1,X_3)\xi + \tfrac{(c+1)(m+1)}{4m}\eta(X_3)\eta(X_1)\xi, \tag{13}$$

$$W_7(X_1,X_2)\xi = \tfrac{(c+1)-(m+2)c+(3m-2)}{4m}\eta(X_2)X_1 \\ -\eta(X_1)X_2 + \tfrac{(c+1)(m+1)}{4m}\eta(X_1)\eta(X_2)\xi. \tag{14}$$

3 Invariant Pseudoparalel Submanifolds of Lorentz-Sasakian Space Forms

Now let's first characterize the pseudoparallel submanifold of the $(2m+1)-$dimensional Lorentz-Sasakian space form $\tilde{M}(c)$ on the W_7-curvature tensor.

Definition 3.1. Let M be the invariant submanifold of the $(2m+1)-$dimensional Lorentz-Sasakian space form $\tilde{M}(c)$. If $W_7 \cdot h$ and $Q(g,h)$ are linearly dependent, M is called W_7-pseudoparallel submanifold.

Equivalent to this definition, it can be said that there is a function L_1 on the set $M_1 = \{X_1 \in M | \, h(X_1) \neq g(X_1)\}$ such that

$$W_7 \cdot h = L_1 Q(g,h).$$

If $L_1 = 0$ specifically, M is called a W_7-semiparallel submanifold.

Let us now examine the case of W_7-pseudoparallel for the submanifold M of the $(2m+1)-$dimensional Lorentz-Sasakian space form $\tilde{M}(c)$.

Theorem 3.2. *Let M be the invariant submanifold of the $(2m+1)-$dimensional Lorentz-Sasakian space form $\tilde{M}(c)$. If M is W_7-pseudoparallel submanifold, then M is either a total geodesic or*

$$L_1 = \frac{(m+2)c-(c+1)-(3m-2)}{4m}.$$

Proof. Let's assume that M is a W_7-pseudoparallel submanifold. So, we can write from (9),

$$(W_7(X_1,X_2)\cdot h)(X_4,X_5) = L_1 Q(g,h)(X_4,X_5;X_1,X_2),$$

that is

$$\begin{aligned} &R^{\perp}(X_1,X_2)\,h(X_4,X_5)-h(W_7(X_1,X_2)\,X_4,X_5)\\ &-h(X_4,W_7(X_1,X_2)\,X_5)=-L_1\{h((X_1\wedge_g X_2)\,X_4,X_5)\\ &+h(X_4,(X_1\wedge_g X_2)\,X_5)\}, \end{aligned} \tag{15}$$

for all $X_1,X_2,X_4,X_5\in\Gamma(TM)$. If we choose $X_5=\xi$ in (15) and make use of $(5),(14)$, we get

$$\begin{aligned} &\left(\frac{(c+1)-(m+2)\,c+(3m-2)}{4m}+L_1\right)\{\eta(X_2)\,h(X_4,X_1)\\ &-\eta(X_1)\,h(X_4,X_2)\}=0. \end{aligned} \tag{16}$$

If we choose $X_2=\xi$ in (16), we obtain

$$\left(\frac{(c+1)-(m+2)\,c+(3m-2)}{4m}+L_1\right)h(X_4,X_1)=0.$$

This completes the proof. □

Proposition 3.3. *Let M be the invariant submanifold of the $(2m+1)-$dimensional Lorentz-Sasakian space form $\tilde{M}(c)$. If M is W_7-semiparallel submanifold, then M is either a total geodesic or a real space form with constant section curvature $c=\dfrac{3m-1}{m+1}$.*

Let us now examine the case of W_7-2 pseudoparallel for the submanifold M of the $(2m+1)-$dimensional Lorentz-Sasakian space form $\tilde{M}(c)$.

Definition 3.4. Let M be the invariant submanifold of the $(2m+1)-$dimensional Lorentz-Sasakian space form $\tilde{M}(c)$. If $W_7\cdot\tilde{\nabla}h$ and $Q\left(g,\tilde{\nabla}h\right)$ are linearly dependent, M is called W_7 $2-$pseudoparallel submanifold.

Equivalent to this definition, it can be said that there is a function L_2 on the set $M_2=\left\{X_1\in M|\,\tilde{\nabla}h(X_1)\neq g(X_1)\right\}$ such that

$$W_7\cdot\tilde{\nabla}h=L_2Q\left(g,\tilde{\nabla}h\right).$$

If $L_2 = 0$ specifically, M is called a W_7 $2-$seniparallel submanifold.

Theorem 3.5. *Let M be the invariant submanifold of the $(2m+1)-$dimensional Lorentz-Sasakian space form $\tilde{M}(c)$. If M is $W_7 - 2$ pseudoparallel submanifold, then M is either a total geodesic submanifold or $L_2 = -1$.*

Proof. Let's assume that M is a W_7 $2-$ pseudoparallel submanifold. So, we can write from (9),

$$\left(W_7(X_1, X_2) \cdot \tilde{\nabla} h\right)(X_4, X_5, X_3) = L_2 Q\left(g, \tilde{\nabla} h\right)(X_4, X_5, X_3; X_1, X_2), \tag{17}$$

for all $X_1, X_2, X_4, X_5, X_3 \in \Gamma(TM)$. If we choose $X_1 = X_3 = \xi$ in (17), we can write

$$\begin{aligned} & R^{\perp}(\xi, X_2)\left(\tilde{\nabla}_{X_4} h\right)(X_5, \xi) - \left(\tilde{\nabla}_{W_7(\xi, X_2)X_4} h\right)(X_5, \xi) \\ & - \left(\tilde{\nabla}_{X_4} h\right)(W_7(\xi, X_2)X_5, \xi) - \left(\tilde{\nabla}_{X_4} h\right)(X_5, W_7(\xi, X_2)\xi) \\ & = -L_2\left\{\left(\tilde{\nabla}_{(\xi \wedge_g X_2)X_4} h\right)(X_5, \xi) + \left(\tilde{\nabla}_{X_4} h\right)((\xi \wedge_g X_2)X_5, \xi)\right. \\ & \left. + \left(\tilde{\nabla}_{X_4} h\right)(X_5, (\xi \wedge_g X_2)\xi)\right\}. \end{aligned} \tag{18}$$

Let's calculate all the expressions in (18). So, we can write from $(1), (5), (8), (12), (13), (14)$,

$$\begin{aligned} & R^{\perp}(\xi, X_2)\left(\tilde{\nabla}_{X_4} h\right)(X_5, \xi) = W_7^{\perp}(\xi, X_2)\left\{\nabla^{\perp}_{X_4} h(X_5, \xi)\right. \\ & \left. -h(\nabla_{X_4} X_5, \xi) - h(X_5, \nabla_{X_4}\xi)\right\} \\ & = R^{\perp}(\xi, X_2)\phi h(X_5, X_4), \end{aligned} \tag{19}$$

$$\begin{aligned} & \left(\tilde{\nabla}_{W_7(\xi, X_2)X_4} h\right)(X_5, \xi) = \nabla^{\perp}_{W_7(\xi, X_2)X_4} h(X_5, \xi) - h\left(\nabla_{W_7(\xi, X_2)X_4} X_5, \xi\right) \\ & -h\left(X_5, \nabla_{W_7(\xi, X_2)X_4}\xi\right) \\ & = \eta(X_4)\phi h(X_5, X_2), \end{aligned} \tag{20}$$

$$\begin{aligned}&\left(\tilde{\nabla}_{X_4}h\right)\left(W_7\left(\xi,X_2\right)X_5,\xi\right)=\nabla^{\perp}_{X_4}h\left(W_7\left(\xi,X_2\right)X_5,\xi\right)\\&-h\left(\nabla_{X_4}W_7\left(\xi,X_2\right)X_5,\xi\right)-h\left(W_7\left(\xi,X_2\right)X_5,\nabla_{X_4}\xi\right)\\&=-\eta\left(X_5\right)\phi h\left(X_2,X_4\right),\end{aligned} \tag{21}$$

$$\begin{aligned}&\left(\tilde{\nabla}_{X_4}h\right)\left(X_5,W_7\left(\xi,X_2\right)\xi\right)=\left(\tilde{\nabla}_{X_4}h\right)\left(X_5,\eta\left(X_2\right)\xi-X_2\right)\\&=\left(\tilde{\nabla}_{X_4}h\right)\left(X_5,\eta\left(X_2\right)\xi\right)-\left(\tilde{\nabla}_{X_4}h\right)\left(X_5,X_2\right)\\&=-h\left(X_5,X_4\eta\left(X_2\right)\xi+\eta\left(X_2\right)\nabla_{X_4}\xi\right)-\left(\tilde{\nabla}_{X_4}h\right)\left(X_5,X_2\right)\\&=\eta\left(X_2\right)\phi h\left(X_5,X_4\right)-\left(\tilde{\nabla}_{X_4}h\right)\left(X_5,X_2\right),\end{aligned} \tag{22}$$

$$\begin{aligned}&\left(\tilde{\nabla}_{(\xi\wedge_g X_2)X_4}h\right)\left(X_5,\xi\right)=\nabla^{\perp}_{(\xi\wedge_g X_2)X_4}h\left(X_5,\xi\right)-h\left(\nabla_{(\xi\wedge_g X_2)X_4}X_5,\xi\right)\\&-h\left(X_5,\nabla_{(\xi\wedge_g X_2)X_4}\xi\right)\\&=\eta\left(X_4\right)\phi h\left(X_5,X_2\right),\end{aligned} \tag{23}$$

$$\begin{aligned}&\left(\tilde{\nabla}_{X_4}h\right)\left(\left(\xi\wedge_g X_2\right)X_5,\xi\right)=\nabla^{\perp}_{X_4}h\left(\left(\xi\wedge_g X_2\right)X_5,\xi\right)\\&-h\left(\nabla_{X_4}\left(\xi\wedge_g X_2\right)X_5,\xi\right)-h\left(\left(\xi\wedge_g X_2\right)X_5,\nabla_{X_4}\xi\right)\\&=\eta\left(X_5\right)\phi h\left(X_2,X_4\right),\end{aligned} \tag{24}$$

$$\left(\tilde{\nabla}_{X_4}h\right)(X_5,(\xi\wedge_g X_2)\xi)=\left(\tilde{\nabla}_{X_4}h\right)(X_5,-\eta(X_2)\xi+X_2)$$
$$=-\left(\tilde{\nabla}_{X_4}h\right)(X_5,\eta(X_2)\xi)+\left(\tilde{\nabla}_{X_4}h\right)(X_5,X_2) \tag{25}$$
$$=-\eta(X_2)\phi h(X_5,X_4)+\left(\tilde{\nabla}_{X_4}h\right)(X_5,X_2).$$

If we substitute $(19),(20),(21),(22),(23),(24),(25)$ for (18), we obtain

$$R^{\perp}(\xi,X_2)\phi h(X_5,X_4)-\eta(X_4)\phi h(X_5,X_2)+\eta(X_5)\phi h(X_2,X_4)$$
$$-\eta(X_2)\phi h(X_5,X_4)+\left(\tilde{\nabla}_{X_4}h\right)(X_5,X_2)$$
$$=-L_2\{\eta(X_4)\phi h(X_5,X_2)+\eta(X_5)\phi h(X_2,X_4)$$
$$-\eta(X_2)\phi h(X_5,X_4)-\left(\tilde{\nabla}_{X_4}h\right)(X_5,X_2)\Big\}. \tag{26}$$

If we choose $X_5=\xi$ in (26) from (10), we get

$$\phi h(X_2,X_4)+\left(\tilde{\nabla}_{X_4}h\right)(\xi,X_2)=-L_2\{\phi h(X_2,X_4)$$
$$+\left(\tilde{\nabla}_{X_4}h\right)(\xi,X_2)\Big\}. \tag{27}$$

On the other hand, it is clear that

$$\left(\tilde{\nabla}_{X_4}h\right)(\xi,X_2)=\phi h(X_2,X_4). \tag{28}$$

If (28) is written instead of (27), we obtain

$$(1+L_2)h(X_2,X_4)=0.$$

This completes the proof. □

Proposition 3.6. *Let M be the invariant submanifold of the $(2m+1)-$dimensional Lorentz-Sasakian space form $\tilde{M}(c)$. Then M is W_7-2 semiparallel submanifold if and only if M is totally geodesic.*

Let us now examine the case of W_7-Ricci generalized pseudoparallel for the submanifold M of the $(2m+1)-$dimensional Lorentz-Sasakian space form $\tilde{M}(c)$.

Definition 3.7. Let M be the invariant submanifold of the $(2m+1)-$dimensional Lorentz-Sasakian space form $\tilde{M}(c)$. If $W_7 \cdot h$ and $Q(S,h)$ are linearly dependent, M is called W_7-Ricci generalized pseudoparallel submanifold.

Equivalent to this definition, it can be said that there is a function L_3 on the set $M_3 = \{X_1 \in M | \, h(X_1) \neq S(X_1)\}$ such that

$$W_7 \cdot h = L_3 Q(S,h).$$

If $L_3 = 0$ specifically, M is called a W_7-Ricci generalized semiparallel submanifold.

Theorem 3.8. *Let M be the invariant submanifold of the $(2m+1)-$dimensional Lorentz-Sasakian space form $\tilde{M}(c)$. If M is W_7-Ricci generalized pseudoparallel submanifold, then M is either a total geodesic or $L_3 = \dfrac{2}{4m-(c+1)}$ provided $4m \neq (c+1)$.*

Proof. Let's assume that M is a W_7-Ricci generalized pseudoparallel submanifold. So, we can write from (9),

$$(W_7(X_1,X_2)\cdot h)(X_4,X_5) = L_3 Q(S,h)(X_4,X_5;X_1,X_2),$$

that is

$$\begin{aligned} & R^{\perp}(X_1,X_2)h(X_4,X_5) - h(W_7(X_1,X_2)X_4,X_5) \\ & -h(X_4,W_7(X_1,X_2)X_5) = -\lambda_3\{h((X_1\wedge_g X_2)X_4,X_5) \\ & +h(X_4,(X_1\wedge_g X_2)X_5)\}, \end{aligned} \tag{29}$$

for all $X_1,X_2,X_4,X_5 \in \Gamma(TM)$. If we choose $X_1 = X_5 = \xi$ in (29) and make use of $(10),(12)$, we get

$$\left[1+\frac{(c+1)-4m}{2}L_3\right]h(X_4,X_2) = 0.$$

It is clear from the last equation that either

$$h(X_4, X_2) = 0,$$

or

$$L_3 = \frac{2}{4m-(c+1)}.$$

This completes the proof. □

Proposition 3.9. *Let M be the invariant submanifold of the $(2m+1)-$dimensional Lorentz-Sasakian space form $\tilde{M}(c)$. Then M is W_7-Ricci generalized semiparallel submanifold if and only if M is totally geodesic.*

Let us now examine the case of W_7 $2-$Ricci generalized pseudoparallel for the submanifold M of the $(2m+1)-$dimensional Lorentz-Sasakian space form $\tilde{M}(c)$.

Definition 3.10. Let M be the invariant submanifold of the $(2m+1)-$dimensional Lorentz-Sasakian space form $\tilde{M}(c)$. If $W_7 \cdot \tilde{\bigtriangledown} h$ and $Q\left(S, \tilde{\bigtriangledown} h\right)$ are linearly dependent, M is called W_7 $2-$ Ricci generalized pseudoparallel submanifold.

Equivalent to this definition, it can be said that there is a function L_4 on the set $M_4 = \left\{X_1 \in M | \tilde{\bigtriangledown} h(X_1) \neq S(X_1)\right\}$ such that

$$W_7 \cdot \tilde{\bigtriangledown} h = L_4 Q\left(S, \tilde{\bigtriangledown} h\right).$$

If $L_4 = 0$ specifically, M is called a W_7 $2-$Ricci generalized semiparallel submanifold.

Theorem 3.11. *Let M be the invariant submanifold of the $(2m+1)-$dimensional Lorentz-Sasakian space form $\tilde{M}(c)$. If M is $W_7 - 2$ Ricci generalized pseudoparallel submanifold, then M is either a total geodesic or $L_4 = \dfrac{2}{4m-(c+1)}$ provided $4m \neq (c+1)$.*

Proof. Let's assume that M is a W_7 $2-$ Ricci generalized pseudoparallel submanifold. So, we can write

$$\left(W_7(X_1, X_2) \cdot \tilde{\bigtriangledown} h\right)(X_4, X_5, X_3) = L_4 Q\left(S, \tilde{\bigtriangledown} h\right)(X_4, X_5, X_3; X_1, X_2), \quad (30)$$

for all $X_1, X_2, X_4, X_5, X_3 \in \Gamma(TM)$. If we choose $X_1 = X_5 = \xi$ in (30), we can write

$$\begin{aligned} &R^{\perp}(\xi, X_2)\left(\tilde{\nabla}_{X_4}h\right)(\xi, X_3) - \left(\tilde{\nabla}_{W_7(\xi,X_2)X_4}h\right)(\xi, X_3) \\ &-\left(\tilde{\nabla}_{X_4}h\right)(W_7(\xi, X_2)\xi, X_3) - \left(\tilde{\nabla}_{X_4}h\right)(\xi, W_7(\xi, X_2)X_3) \\ &= -L_4\left\{\left(\tilde{\nabla}_{(\xi\wedge_S X_2)X_4}h\right)(\xi, X_3) + \left(\tilde{\nabla}_{X_4}h\right)((\xi\wedge_S X_2)\xi, X_3)\right. \\ &\left.+\left(\tilde{\nabla}_{X_4}h\right)(\xi, (\xi\wedge_S X_2)X_3)\right\}. \end{aligned} \tag{31}$$

Let's calculate all the expressions in (31). So, we can write from $(1), (6), (8), (12), (13), (14)$,

$$\begin{aligned} &R^{\perp}(\xi, X_2)\left(\tilde{\nabla}_{X_4}h\right)(\xi, X_3) = W_7^{\perp}(\xi, X_2)\left\{\nabla^{\perp}_{X_4}h(\xi, X_3)\right. \\ &\left.-h(\nabla_{X_4}X_3, \xi) - h(X_3, \nabla_{X_4}\xi)\right\} \\ &= R^{\perp}(\xi, X_2)\phi h(X_3, X_4), \end{aligned} \tag{32}$$

$$\begin{aligned} &\left(\tilde{\nabla}_{W_7(\xi,X_2)X_4}h\right)(\xi, X_3) = \nabla^{\perp}_{W_7(\xi,X_2)X_4}h(\xi, X_3) - h\left(\nabla_{W_7(\xi,X_2)X_4}\xi, X_3\right) \\ &-h\left(\xi, \nabla_{W_7(\xi,X_2)X_4}X_3\right) \\ &= -\eta(X_4)\phi h(X_2, X_3), \end{aligned} \tag{33}$$

$$\begin{aligned} &\left(\tilde{\nabla}_{X_4}h\right)(W_7(\xi, X_2)\xi, X_3) = \left(\tilde{\nabla}_{X_4}h\right)(\eta(X_2)\xi - X_2, X_3) \\ &= \nabla^{\perp}_{X_4}h(\eta(X_2)\xi, X_3) - h(\nabla_{X_4}\eta(X_2)\xi, X_3) \\ &-h(\eta(X_2)\xi, \nabla_{X_4}X_3) - \left(\tilde{\nabla}_{X_4}h\right)(X_2, X_3) \\ &= \eta(X_2)\phi h(X_4, X_3) - \left(\tilde{\nabla}_{X_4}h\right)(X_2, X_3), \end{aligned} \tag{34}$$

$$\left(\tilde{\nabla}_{X_4}h\right)(\xi, W_7(\xi, X_2)X_3) = \nabla^{\perp}_{X_4}h(\xi, W_7(\xi, X_2)X_3)$$
$$-h(\nabla_{X_4}\xi, W_7(\xi, X_2)X_3) - h(\xi, \nabla_{X_4}W_7(\xi, X_2)X_3) \tag{35}$$
$$= -\eta(X_3)\phi h(X_4, X_2)$$

$$\left(\tilde{\nabla}_{(\xi\wedge_S X_2)X_4}h\right)(\xi, X_3) = \nabla^{\perp}_{(\xi\wedge_S X_2)X_4}h(\xi, X_3) - h\left(\nabla_{(\xi\wedge_S X_2)X_4}\xi, X_3\right)$$
$$-h\left(\xi, \nabla_{(\xi\wedge_S X_2)X_4}X_3\right)$$
$$= \tfrac{(c+1)-4m}{2}\eta(X_4)\phi h(X_2, X_3), \tag{36}$$

$$\left(\tilde{\nabla}_{X_4}h\right)((\xi\wedge_S X_2)\xi, X_3) = \left(\tilde{\nabla}_{X_4}h\right)(S(X_2,\xi)\xi - S(\xi,\xi)X_2, X_3)$$
$$= \tfrac{(c+1)-4m}{2}\left\{\left(\tilde{\nabla}_{X_4}h\right)(-\eta(X_2)\xi + X_2, X_3)\right\}$$
$$= \tfrac{(c+1)-4m}{2}\left\{-\nabla^{\perp}_{X_4}h(\eta(X_2)\xi, X_3) + h(\nabla_{X_4}\eta(X_2)\xi, X_3)\right.$$
$$\left.+h(\eta(X_2)\xi, \nabla_{X_4}X_3) + \left(\tilde{\nabla}_{X_4}h\right)(X_2, X_3)\right\}$$
$$= \tfrac{(c+1)-4m}{2}\left\{\left(\tilde{\nabla}_{X_4}h\right)(X_2, X_3) - \eta(X_2)\phi h(X_4, X_3)\right\}, \tag{37}$$

$$\left(\tilde{\nabla}_{X_4}h\right)(\xi, (\xi\wedge_S X_2)X_3) = \left(\tilde{\nabla}_{X_4}h\right)(\xi, S(X_2, X_3)\xi - S(\xi, X_3)X_2)$$
$$= \left(\tilde{\nabla}_{X_4}h\right)(\xi, S(X_2, X_3)\xi) + \tfrac{(c+1)-4m}{2}\left(\tilde{\nabla}_{X_4}h\right)(\xi, \eta(X_3), X_2)$$
$$= \tfrac{(c+1)-4m}{2}\eta(X_3)\phi h(X_4, X_2). \tag{38}$$

If we substitute (32) , (33) , (34) , (35) , (36) , (37) , (38) for (31) , we obtain

$$\begin{aligned}
&R^{\perp}(\xi, X_2)\,\phi h(X_3, X_4) + \eta(X_4)\,\phi h(X_2, X_3) - \eta(X_2)\,\phi h(X_4, X_3) \\
&+\eta(X_3)\,\phi h(X_4, X_2) + \left(\tilde{\nabla}_{X_4} h\right)(X_2, X_3) \\
&= -L_4\left\{\tfrac{(c+1)-4m}{2}\eta(X_4)\,\phi h(X_2, X_3)\right. \\
&-\tfrac{(c+1)-4m}{2}\eta(X_2)\,\phi h(X_4, X_3) + \tfrac{(c+1)-4m}{2}\eta(X_3)\,\phi h(X_4, X_2) \\
&\left.+\tfrac{(c+1)-4m}{2}\left(\tilde{\nabla}_{X_4} h\right)(X_2, X_3)\right\}.
\end{aligned} \tag{39}$$

If we choose $X_3 = \xi$ in (39) , we get

$$\begin{aligned}
&\left(\tilde{\nabla}_{X_4} h\right)(X_2, \xi) + \phi h(X_4, X_2) = -\tfrac{(c+1)-4m}{2} L_4\left\{\left(\tilde{\nabla}_{X_4} h\right)(X_2, \xi)\right. \\
&\left.+\phi h(X_4, X_2)\right\}.
\end{aligned} \tag{40}$$

On the other hand, it is clear that

$$\left(\tilde{\nabla}_{X_4} h\right)(\xi, X_2) = \phi h(X_2, X_4). \tag{41}$$

If (41) is written instead of (40) , we obtain

$$2\phi h(X_2, X_4) = [4m - (c+1)]\, L_4 \phi h(X_2, X_4).$$

It is clear from the last equality

$$h(X_2, X_4) = 0 \text{ or } L_4 = \frac{2}{4m - (c+1)}.$$

This completes the proof. □

Proposition 3.12. *Let M be the invariant submanifold of the $(2m+1)-$dimensional Lorentz-Sasakian space form $\tilde{M}(c)$. Then M is W_7 $2-$Ricci generalized semiparallel submanifold if and only if M is totally geodesic.*

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