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On Deformable Implicit Fractional Differential Equations in b-Metric Spaces

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Abstract. In this paper, we prove some existence and uniqueness results for some classes of deformable implicit fractional differential equations in b-Metric spaces with initial conditions. We base our arguments on some some fixed point theorems. Finally, we provide an example to illustrate our results.

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1 Introduction

Recently, fractional differential equations have been used in engineering, mathematics, physics, and other applied disciplines. The existence

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of solutions to the ordinary and fractional differential equations with various conditions has received much attention; see the monographs [1, 2, 3, 22, 23, 24] and the papers [21, 19, 13, 12, 8, 7]. Several results of implicit fractional differential equations have been recently provided, see [1, 20, 15], and the references therein.

In [10, 11], Czerwik introduced the concept of b-metric. Following these early studies, numerous problems with differential equations in b-metric spaces has been intensively researched; see [9, 5, 6, 14] as well as the included references.

In [25], Zulfeqarr *et al.* proposed the novel notion of deformable fractional derivative, employing the limit technique as in the usual derivative. It was termed "deformable" due to its inherent ability of continuously deforming function to derivative. Deformable derivatives can be thought of as fractional order derivatives.

The authors of [16] investigated further properties of the new concept of deformable derivative and used the results to study the following Cauchy problem with non-local condition:

$$\begin{aligned} \mathfrak{D}_0^\alpha x(t) &= f(t, x(t)), \quad t \in (0, T], \\ x(0) + g(x) &= x_0, \end{aligned}$$

where $\mathfrak{D}_0^\alpha$ is the deformable derivative of order $\alpha \in (0,1)$, and $g : \mathcal{C} \to \mathbb{R}$ is a continuous function. Their arguments are based on Krasnoselskii's fixed point theorem.

In [18], Meraj and Pandey studied the existence and uniqueness of mild solution for the following initial value problem:

$$\begin{aligned} \mathfrak{D}_0^\alpha x(t) &= Ax(t) + f(t, x(t)), \quad t \in J, \\ x(0) &= x_0, \end{aligned}$$

where $A : D(A) \subset X \to X$ is an infinitesimal generator of a C_0-semigroup $T(t)(t \geqslant 0)$ on a suitable space $X, x_0 \in X$, and $J = [0, b], b > 0$ is a constant. The results are obtained with the help of semigroup theory, Banach fixed point theorem, and Schauder fixed point theorem.

In [17], by using Weissinger's and Krasnoselskii's fixed point theorems, Mebrat and N'Guérékata studied the existence of solutions for the following problem:

$$\mathfrak{D}_0^\alpha \chi(\theta) = \Phi(\chi(\theta)) + \Psi(\theta, \chi(\theta)) + \int_0^\theta \Upsilon(\theta, s, \chi(s))ds, \ \theta \in \Theta,$$
$$\chi(0) = \chi_0,$$

where $\Psi : \Theta \times E \to E, \Upsilon : \Theta \times \Theta \times E \to E$ are continuous functions.

Motivated by the above-mentioned papers, first we study the existence and uniqueness of solutions for the implicit problem with nonlinear fractional differential equation involving the deformable fractional derivative:

$$\left(\mathfrak{D}_0^\alpha \chi\right)(\theta) = \Psi\left(\theta, \chi(\theta), \mathfrak{D}_0^\alpha \chi(\theta)\right); \ \theta \in \Theta := [0, \kappa], \tag{1}$$

with the initial condition

$$\chi(0) = \chi_0, \tag{2}$$

where $0 < \alpha < 1$, $\mathfrak{D}_0^\alpha$ is the deformable fractional derivative defined in [25], $\Psi : \Theta \times \mathbb{R}^2 \to \mathbb{R}$ is a given function to be specified later and $\chi_0 \in \mathbb{R}$.

Next, we discuss the existence of solutions for the following problem of deformable implicit nonlocal fractional differential equations:

$$\begin{cases} \left(\mathfrak{D}_0^\alpha \chi\right)(\theta) = \Psi\left(\theta, \chi(\theta), \mathfrak{D}_0^\alpha \chi(\theta)\right); \ \theta \in \Theta, \\ \chi(0) + \overline{\psi}(\chi) = \chi_0, \end{cases} \tag{3}$$

where $\overline{\psi} : C(\Theta, \mathbb{R}) \to \mathbb{R}$ is a continuous function.

2 Preliminaries

First, we give the definitions and the notations that we will use throughout this paper. We denote by $C(\Theta, \mathbb{R})$ the Banach space of all continuous functions from Θ into $\mathbb{R}$ with the following norm

$$\|\chi\|_\infty = \sup_{\theta \in \Theta}\{|\chi(\theta)|\}.$$

Consider the space $X_b^p(0,\kappa)$, $(b \in \mathbb{R},\ 1 \leq p \leq \infty)$ of those complex-valued Lebesgue measurable functions Ψ on $[0,\kappa]$ for which $\|\Psi\|_{X_b^p} < \infty$, where the norm is given by:

$$\|\Psi\|_{X_b^p} = \left(\int_0^\kappa |\theta^b \Psi(\theta)|^p \frac{d\theta}{\theta} \right)^{\frac{1}{p}}, \quad (1 \leq p < \infty, b \in \mathbb{R}).$$

Definition 2.1 (The deformable fractional derivative [16, 25])**.** Let $\Psi : [0,+\infty) \longrightarrow \mathbb{R}$ be a given function, then the non-conformable fractional derivative of Ψ of order α is defined by

$$\left(\mathfrak{D}_0^\alpha \Psi\right)(\theta) = \lim_{\varepsilon \to 0} \frac{(1+\varepsilon\beta)\Psi\left(\theta + \varepsilon\alpha\right) - \Psi(\theta)}{\varepsilon},$$

where $\alpha + \beta = 1$ and $\alpha \in (0,1]$. If the deformable fractional derivative of Ψ of order α exists, then we simply say that Ψ is α-differentiable.

Definition 2.2 (The α-fractional integral [16, 17])**.** For $\alpha \in (0,1]$ and a continuous function Ψ, let

$$\left(\mathcal{J}_{0^+}^\alpha \Psi\right)(\theta) = \frac{1}{\alpha} e^{\frac{-\beta}{\alpha}\theta} \int_0^\theta e^{\frac{\beta}{\alpha}\tau} \Psi(\tau) d\tau.$$

Lemma 2.3 ([16, 17])**.** *If $\alpha, \alpha_1 \in (0,1]$ such that $\alpha + \beta = 1$, Ψ and Φ are two α-differentiable functions at a point θ and m, n are two given numbers, then the improved conformable fractional derivative satisfies the following properties:*

- $\mathfrak{D}_0^\alpha(\lambda) = \beta\lambda$, *for any constant λ;*
- $\mathfrak{D}_0^\alpha(m\Psi + n\Phi) = m\mathfrak{D}_0^\alpha(\Psi) + n\mathfrak{D}_0^\alpha(\Phi)$*;*
- $\mathfrak{D}_0^\alpha(\Psi\Phi) = \Phi\mathfrak{D}_0^\alpha(\Psi) + \alpha\Psi\Phi'$*;*
- $\mathcal{J}_{0^+}^\alpha\ \mathcal{J}_{0^+}^{\alpha_1}\Psi = \mathcal{J}_{0^+}^{\alpha+\alpha_1}\Psi$*.*

Lemma 2.4 ([16, 17])**.** *If $\alpha \in (0,1]$, f is continuous function, then we have:*

- $\left(\mathcal{J}_{0^+}^\alpha\ \mathfrak{D}_0^\alpha(\Psi)\right)(\theta) = \Psi(\theta) - e^{\frac{-\beta}{\alpha}\theta}\Psi(0)$;
- $\mathfrak{D}_0^\alpha\left(\mathcal{J}_{0^+}^\alpha \Psi\right)(\theta) = \Psi(\theta)$.

Lemma 2.5. *Let $\Phi \in L^1(\Theta)$ and $0 < \alpha \leq 1$. Then the initial value problem*

$$\begin{cases} \left(\mathfrak{D}_0^{\alpha}\chi\right)(\theta) = \Phi(\theta); \ \theta \in \Theta := [0, \kappa], \\ \chi(0) = \chi_0, \end{cases} \tag{4}$$

has a unique solution defined by

$$\chi(\theta) = \chi_0 e^{\frac{-\beta}{\alpha}\theta} + \frac{1}{\alpha} e^{\frac{-\beta}{\alpha}\theta} \int_0^{\theta} e^{\frac{\beta}{\alpha}\tau} \Phi(\tau) d\tau. \tag{5}$$

Proof. Applying the α-fractional integral of order α to both sides the equation

$$\left(\mathfrak{D}_0^{\alpha}\chi\right)(\theta) = \Phi(\theta),$$

and by using Lemma 2.4 and if $\theta \in \Theta$, we get

$$\chi(\theta) - \chi(0) e^{\frac{-\beta}{\alpha}\theta} = \frac{1}{\alpha} e^{\frac{-\beta}{\alpha}\theta} \int_0^{\theta} e^{\frac{\beta}{\alpha}\tau} \Phi(\tau) d\tau. \tag{6}$$

From the initial conditions, we get

$$\chi(\theta) = \chi_0 e^{\frac{-\beta}{\alpha}\theta} + \frac{1}{\alpha} e^{\frac{-\beta}{\alpha}\theta} \int_0^{\theta} e^{\frac{\beta}{\alpha}\tau} \Phi(\tau) d\tau \tag{7}$$

Conversely, we can easily show by Lemma 2.3 that if χ verifies equation (5) then it satisfied the problem (4). □

Definition 2.6 ([5])**.** Let $\mathcal{H}$ be a set and $\varepsilon \geq 1$ be a given real number. A distance function $\delta : \mathcal{H} \times \mathcal{H} \to \mathbb{R}_+^*$ is called a b-metric if the following conditions hold for all $\xi_1, \xi_2, \xi_3 \in \mathcal{H}$:

(1) $\delta(\xi_1, \xi_2) = 0$ if and only if $\xi_1 = \xi_2$,

(2) $\delta(\xi_1, \xi_2) = \delta(\xi_2, \xi_1)$,

(3) $\delta(\xi_1, \xi_2) \leq \varepsilon[\delta(\xi_1, \xi_3) + \delta(\xi_3, \xi_2)]$.

Then, the pair $(\mathcal{H}, \delta, \varepsilon)$ is called a b-metric space with parameter ε.

Consider $\tilde{\Xi}$ the set of continuous and increasing function $\psi : \mathbb{R}_+^* \to \mathbb{R}_+^*$ satisfying the property: $\psi(\varepsilon\chi) \leq \varepsilon\psi(\chi) \leq \varepsilon\chi$, for $\varepsilon > 1$ and $\psi(0) = 0$. We denote by Ξ the family of all nondecreasing functions $\eta : \mathbb{R}_+^* \to [0, \frac{1}{\varepsilon^2})$ for some $\varepsilon \geq 1$.

Definition 2.7 ([5])**.** Let $(\mathcal{H}, \delta, \varepsilon)$ be a b-metric space, $\mathfrak{S} : \mathcal{H} \to \mathcal{H}$ is said to be a generalized ω-ψ-Geraghty mapping whenever there exists $\omega : \mathcal{H} \times \mathcal{H} \to \mathbb{R}_+^*$ such that

$$\omega(\xi_1, \xi_2)\psi(\varepsilon^3 d(\mathfrak{S}(\xi_1), \mathfrak{S}(\xi_2)) \leq \eta(\psi(\delta(\xi_1, \xi_2))\psi(\delta(\xi_1, \xi_2),$$

for $\xi_1, \xi_2 \in \mathcal{H}$, where $\eta \in \Xi$.

Definition 2.8 ([5])**.** Let $\mathcal{H}$ be a non empty set, $\mathfrak{S} : \mathcal{H} \to \mathcal{H}$ and $\omega : \mathcal{H} \times \mathcal{H} \to \mathbb{R}_+^*$ be given mappings. The operator $\mathfrak{S}$ is orbital ω-admissible if for $\chi \in \mathcal{H}$, we have

$$\omega(\chi, \mathfrak{S}(\chi)) \geq 1 \Rightarrow \omega(\mathfrak{S}(\chi), \mathfrak{S}^2(\chi)) \geq 1.$$

Theorem 2.9 ([5])**.** *Let $(\mathcal{H}, \delta)$ be a complete b-metric space and $\aleph : \mathcal{H} \to \mathcal{H}$ be a generalized ω-ψ-Geraghty mapping where*

(a) *$\aleph$ is ω-admissible;*

(b) *there exists $\chi_0 \in \mathcal{H}$ where $\omega(\chi_0, \aleph(\chi_0)) \geq 1$;*

(c) *If $(\chi_n)_{n\in N} \subset \mathcal{H}$ with $\chi_n \to \chi$ and $\omega(\chi_n, \chi_{n+1}) \geq 1$, then $\omega(\chi_n, \chi) \geq 1$,*

Then $\aleph$ admit a fixed point. Further, if

(d) *for all fixed points χ, χ' of $\aleph$, either*

$$\omega(\chi, \chi') \geq 1 \quad \text{or} \quad \omega(\chi', \chi) \geq 1,$$

Then $\aleph$ admit a unique fixed point.

3 Existence of Solutions for the First Problem

Let $(C(\Theta), \delta, 2)$ be the complete b-metric space with $\varepsilon = 2$, where $\delta : C(\Theta) \times C(\Theta) \to \mathbb{R}_+^*$, is given by:

$$\delta(\chi, \Im) = \|(\chi - \Im)^2\|_\infty := \sup_{\theta \in \Theta} |\chi(\theta) - \Im(\theta)|^2.$$

In this section, we establish some existence results for problem (1)-(2).

Definition 3.1. By a solution of problem (1)-(2), we mean a continuous function $\chi \in C(\Theta)$ given by

$$\chi(\theta) = \chi_0 e^{\frac{-\beta}{\alpha}\theta} + \frac{1}{\alpha} e^{\frac{-\beta}{\alpha}\theta} \int_0^\theta e^{\frac{\beta}{\alpha}\tau} \Phi(\tau) d\tau, \tag{8}$$

where $\Phi \in C(\Theta)$ such that $\Phi(\theta) = \Psi(\theta, \chi(\theta), \Phi(\theta))$.

The hypotheses:

(H_1) There exist $\gamma_1 : C(\Theta) \times C(\Theta) \to \mathbb{R}_+^*$ and $\gamma_2 : \Theta \to (0,1)$ where for $\chi, \Im, \chi_1, \Im_1 \in C(\Theta)$ and $\theta \in \Theta$, we have

$$|\Psi(\theta, \chi, \Im) - \Psi(\theta, \chi_1, \Im_1)| \leq \gamma_1(\chi, \Im)\|\chi - \Im\|_\infty + \gamma_2(\theta)\|\chi_1 - \Im_1\|_\infty$$

with

$$\left\| \frac{1}{\alpha} e^{\frac{-\beta}{\alpha}\theta} \int_0^\theta e^{\frac{\beta}{\alpha}\tau} \frac{\gamma_1(\chi, \Im)}{1 - \gamma_2{}^*} d\tau \right\|_\infty^2 \leq \psi(\|(\chi - \Im)^2\|_\infty).$$

(H_2) There exist $\psi \in \tilde{\Xi}$ and $\overline{\lambda}_0 \in C(\Theta)$ and a function $\gamma_3 : C(\Theta) \times C(\Theta) \to \mathbb{R}$, where

$$\gamma_3 \left(\overline{\lambda}_0(\theta), \frac{1}{\alpha} e^{\frac{-\beta}{\alpha}\theta} \int_0^\theta e^{\frac{\beta}{\alpha}\tau} \Phi(\tau) d\tau \right) \geq 0$$

where $\Phi \in C(\Theta)$ such that $\Phi(\theta) = \Psi(\theta, \overline{\lambda}_0(\theta), \Phi(\theta))$.

(H_3) For each $\theta \in \Theta$, and $\chi, \Im \in C(\Theta)$, we have:

$$\gamma_3(\chi(\theta), \Im(\theta)) \geq 0$$

implies

$$\gamma_3 \left(\frac{1}{\alpha} e^{\frac{-\beta}{\alpha}\theta} \int_0^\theta e^{\frac{\beta}{\alpha}\tau} \Phi(\tau) d\tau, \frac{1}{\alpha} e^{\frac{-\beta}{\alpha}\theta} \int_0^\theta e^{\frac{\beta}{\alpha}\tau} \Phi'(\tau) d\tau \right) \geq 0,$$

where $\Phi, \Phi' \in C(\Theta)$ such that

$$\Phi(\theta) = \Psi(\theta, \chi(\theta), \Phi(\theta))$$

and

$$\Phi'(\theta) = \Psi(\theta, \Im(\theta), \Phi'(\theta)).$$

(H_4) If $(\chi_n)_{n\in N} \subset C(\Theta)$ with $\chi_n \to \chi$ and $\gamma_3(\chi_n, \chi_{n+1}) \geq 1$, then

$$\gamma_3(\chi_n, \chi) \geq 1.$$

(H_5) For all fixed solutions χ, χ' of (1)-(2), either

$$\gamma_3(\chi(\theta), \chi'(\theta)) \geq 0,$$

or

$$\gamma_3(\chi'(\theta), \chi(\theta)) \geq 0.$$

Theorem 3.2. *Assume that the hypotheses (H_1)-(H_4) hold. Then the problem (1)-(2) has at least one solution. And, if (H_5) holds, then the solution is unique.*

Proof. Let $\mathcal{K}: \ C(\Theta) \to C(\Theta)$ be the operator given by:

$$(\mathcal{K}\chi)(\theta) = \chi_0 e^{\frac{-\beta}{\alpha}\theta} + \frac{1}{\alpha} e^{\frac{-\beta}{\alpha}\theta} \int_0^\theta e^{\frac{\beta}{\alpha}\tau} \Phi(\tau) d\tau, \tag{9}$$

where $\Phi \in C(\Theta)$ such that $\Phi(\theta) = \Psi(\theta, \chi_\theta, \Phi(\theta))$.
The function $\omega : C(\Theta) \times C(\Theta) \to \mathbb{R}_+^*$ is given by:

$$\begin{cases} \omega(\chi, \chi') = 1; & if \ \gamma_3(\chi(\theta), \chi'(\theta)) \geq 0, \ \theta \in \Theta, \\ \omega(\chi, \chi') = 0; & eles. \end{cases}$$

First, we demonstrate that $\mathcal{K}$ is a generalized ω-ψ-Geraghty operator: For any $\chi, \chi' \in C(\Theta)$.Then, for each $\theta \in \Theta$, we obtain

$$|(\mathcal{K}\chi)(\theta) - (\mathcal{K}\chi')(\theta)| \leq \frac{1}{\alpha} e^{\frac{-\beta}{\alpha}\theta} \int_0^\theta e^{\frac{\beta}{\alpha}\tau} |\Phi(\vartheta) - \Phi'(\vartheta)| d\vartheta,$$

where $\Phi, \ \Phi' \in C(\Theta)$ such that

$$\Phi(\theta) = \Psi(\theta, \chi(\theta), \Phi(\theta)) \ and \ \Phi'(\theta) = \Psi(\theta, \chi(\theta), \Phi'(\theta)).$$

From (H_1) we have

$$\|\Phi - \Phi'\|_\infty \leq \frac{\gamma_1(\chi, \chi')}{1 - {\gamma_2}^*} \|(\varkappa_2 - \varkappa_2')^2\|_\infty^{\frac{1}{2}},$$

where ${\gamma_2}^* = \sup_{\theta\in\Theta} |\gamma_2(\theta)|$.
Next, we have

$$|(\mathcal{K}\chi)(\theta) - (\mathcal{K}\chi')(\theta)| \leq \frac{1}{\alpha} e^{\frac{-\beta}{\alpha}\theta} \int_0^\theta e^{\frac{\beta}{\alpha}\tau} \frac{\gamma_1(\chi, \chi')}{1-{\gamma_2}^*} \|(\chi-\chi')^2\|_\infty^{\frac{1}{2}} d\vartheta.$$

Thus

$$\begin{aligned} &\omega(\varkappa_2, \varkappa_2')|(\mathcal{K}\varkappa_2)(\theta) - (\mathcal{K}\varkappa_2')(\theta)|^2 \\ &\quad \leq \|(\chi-\chi')^2\|_\infty \omega(\chi, \chi') \left\| \frac{1}{\alpha} e^{\frac{-\beta}{\alpha}\theta} \int_0^\theta e^{\frac{\beta}{\alpha}\tau} \frac{\gamma_1(\chi, \chi')}{1-{\gamma_2}^*} d\vartheta \right\|_\infty^2 \\ &\quad \leq \|(\chi-\chi')^2\|_\infty \psi(\|(\chi-\chi')^2\|_\infty). \end{aligned}$$

Hence

$$\omega(\chi, \chi')\psi(2^3 d(\mathcal{K}(\chi), \mathcal{K}(\chi')) \leq \eta(\psi(\delta(\chi, \chi'))\psi(\delta(\chi, \chi')),$$

where $\eta \in \Xi$, $\psi \in \tilde{\Xi}$, with $\eta(\theta) = \frac{1}{8}\theta$, and $\psi(\theta) = \theta$. So, $\mathcal{K}$ is generalized ω-ψ-Geraghty operator.

Let $\chi, \chi' \in C(\Theta)$ such that

$$\omega(\chi, z\chi') \geq 1.$$

Thus, for each $\theta \in \Theta$, we have

$$\gamma_3(\chi(\theta), \chi'(\theta)) \geq 0.$$

This implies from (H_3) that

$$\gamma_3(\mathcal{K}\chi(\theta), \mathcal{K}\chi'(\theta)) \geq 0,$$

which gives

$$\omega(\mathcal{K}(\chi), \mathcal{K}(\chi')) \geq 1.$$

Hence, $\mathcal{K}$ is a ω-admissible.

Now, by (H_2), there exist $\overline{\lambda}_0 \in C(\Theta)$ such that

$$\omega(\overline{\lambda}_0, \aleph(\overline{\lambda}_0)) \geq 1.$$

Thus, by (H_4), if $\left(\overline{\lambda}_n\right)_{n\in N} \subset \mathcal{H}$ with $\overline{\lambda}_n \to \overline{\lambda}$ and $\omega(\overline{\lambda}_n, \overline{\lambda}_{n+1}) \geq 1$, then

$$\omega(\overline{\lambda}_n, \overline{\lambda}) \geq 1.$$

By Theorem 2.9, we conclude that $\mathcal{K}$ has a fixed point χ which is a solution of (1)-(2).

Further, (H_5) implies that if χ and χ' are fixed points of $\mathcal{K}$, then either

$$\gamma_3(\chi, \chi') \geq 0 \quad or \quad \gamma_3(\chi', \chi) \geq 0.$$

Thus

$$\omega(\chi, \chi') \geq 1 \quad or \quad \omega(\chi', \chi) \geq 1,$$

Then, (1)-(2) has a unique solution. □

4 Existence of Solutions for the Second Problem

In this section, we establish some existence results for problem (3).

Let us introduce the following hypotheses:

(H_6) There exist constant $\varsigma > 0$ such that

$$|\overline{\psi}(\chi) - \overline{\psi}(\Im)| \leq \varsigma \|\chi - \Im\|_\infty,$$

for each $\chi, \Im \in C(\Theta)$.

(H_7) There exist $\overline{M} : C(\Theta) \times C(\Theta) \to \mathbb{R}_+^*$ and $\overline{N} : \Theta \to (0, 1)$ such that for each $\chi, \Im, \chi_1, \Im_1 \in C(\Theta)$ and $\theta \in \Theta$

$$|\Psi(\theta, \chi, \Im) - \Psi(\theta, \chi_1, \Im_1)| \leq \overline{M}(\chi, \Im)\|\chi - \Im\|_\infty + \overline{N}(\theta)\|\chi_1 - \Im_1\|_\infty$$

with

$$\left\| \varsigma + \frac{1}{\alpha} e^{\frac{-\beta}{\alpha}\theta} \int_0^\theta e^{\frac{\beta}{\alpha}\tau} \frac{\overline{M}(\chi, \Im)}{1 - \overline{N}^*} d\tau \right\|_\infty^2 \leq \psi(\|(\chi - \Im)^2\|_\infty).$$

(H_8) There exist $\psi \in \tilde{\Xi}$ and $\overline{\lambda}_0 \in C(\Theta)$ and a function $\gamma_3 : C(\Theta) \times C(\Theta) \to \mathbb{R}$, such that

$$\gamma_3 \left(\overline{\lambda}_0(\theta), -\overline{\psi}(\overline{\lambda}_0) + e^{\frac{-\beta}{\alpha}\theta} + \frac{1}{\alpha} e^{\frac{-\beta}{\alpha}\theta} \int_0^\theta e^{\frac{\beta}{\alpha}\tau} \Phi(\tau) d\tau \right) \geq 0$$

where $\Phi \in C(\Theta)$ such that $\Phi(\theta) = \Psi(\theta, \overline{\lambda}_0(\theta), \Phi(\theta))$.

(H_9) For each $\theta \in \Theta$, and $\chi, \Im \in C(\Theta)$, we have:

$$\gamma_3(\chi(\theta), \Im(\theta)) \geq 0$$

implies

$$\gamma_3 \left(-\overline{\psi}(\chi) + \frac{1}{\alpha} e^{\frac{-\beta}{\alpha}\theta} \int_0^\theta e^{\frac{\beta}{\alpha}\tau} \Phi(\tau) d\tau, -\overline{\psi}(\Im) + \frac{1}{\alpha} e^{\frac{-\beta}{\alpha}\theta} \int_0^\theta e^{\frac{\beta}{\alpha}\tau} \Phi'(\tau) d\tau \right) \geq 0,$$

where $\Phi, \Phi' \in C(\Theta)$ such that

$$\Phi(\theta) = \Psi(\theta, \chi(\theta), \Phi(\theta))$$

and

$$\Phi'(\theta) = \Psi(\theta, \Im(\theta), \Phi'(\theta)).$$

Theorem 4.1. *Assume that the hypotheses (H_4) and (H_6)-(H_9) hold. Then the problem (3) has at least one solution. And, if (H_5) holds, then the solution is unique.*

Proof. Let $\mathcal{K}' : C(\Theta) \to C(\Theta)$ be the operator given by:

$$(\mathcal{K}'\chi)(\theta) = (\chi_0 - \overline{\psi}(\chi)) e^{\frac{-\beta}{\alpha}\theta} + \frac{1}{\alpha} e^{\frac{-\beta}{\alpha}\theta} \int_0^\theta e^{\frac{\beta}{\alpha}\tau} \Phi(\tau) d\tau, \qquad (10)$$

where $\Phi \in C(\Theta)$ such that $\Phi(\theta) = \Psi(\theta, \chi_\theta, \Phi(\theta))$.
Clearly, the fixed points of the operator $\mathcal{K}'$ are solution of the problem (3). By repeating the same process of Theorem 3.2, we can easily show all the conditions of Theorem 4.1 are satisfied by $\mathcal{K}'$. $\square$

5 An Example

Consider the following problem which is an example of our problem (1)-(2):

$$\begin{cases} \left(\mathfrak{D}_0^{\frac{1}{2}}\xi\right)(\theta) = \frac{1+\sin(|\xi(\theta)|)}{4(1+|\xi(\theta)|)} + \frac{1}{4\left(1+\left|\left(\mathfrak{D}_0^{\frac{1}{2}}\xi\right)(\theta)\right|\right)}, \\ \xi(0) = 0. \end{cases} \tag{11}$$

Set

$$\Psi(\theta, \xi, \Im) = \frac{1+\sin(|\xi|)}{4(1+|\xi|)} + \frac{1}{4(1+|\Im|)},$$

where $\theta \in \Theta := [0,1],\ \xi, \Im \in \mathbb{R}$.

Let $(C(\Theta), \delta, 2)$ be the complete b-metric space with $\varepsilon = 2$, such that $\delta : C(\Theta) \times C(\Theta) \to \mathbb{R}_+^*$, is defined by:

$$\delta(\xi, \Im) = \|(\xi - \Im)^2\|_\infty := \sup_{\theta\in\Theta} |\xi(\theta) - \Im(\theta)|^2.$$

For any $\xi, \bar{\xi} \in C(\Theta),\ \Im, \bar{\Im} \in \mathbb{R}$ and $\theta \in \Theta$. If $|\xi(\theta)| \le |\Im(\theta)|$, then

$$\begin{aligned}
&|\Psi(\theta, \xi, \bar{\xi}) - \Psi(\theta, \Im, \bar{\Im})| \\
&\le \left| \frac{1+\sin(|\xi(\theta)|)}{4(1+|\xi(\theta)|+|\bar{\xi}(\theta)|)} - \frac{1+\sin(|\Im(\theta)|)}{4(1+|\Im(\theta)|+|\bar{\Im}(\theta)|)} \right| \\
&\quad + \frac{|\bar{\xi}(\theta) - \bar{\Im}(\theta)|}{4} \\
&\le \frac{1}{4}||\xi(\theta)| - |\Im(\theta)|| + \frac{1}{4}|\sin(|\xi(\theta)|) - \sin(|\Im(\theta)|)| \\
&\quad + ||\xi(\theta)|\sin(|\Im(\theta)|) - |\Im(\theta)|\sin(|\xi(\theta)|)| \\
&\quad + \frac{|\bar{\xi}(\theta) - \bar{\Im}(\theta)|}{4} \\
&\le |\xi(\theta) - \Im(\theta)| + \frac{1}{4}|\sin(|\xi(\theta)|) - \sin(|\Im(\theta)|)| \\
&\quad + \frac{|\bar{\xi}(\theta) - \bar{\Im}(\theta)|}{4} \\
&\quad + ||\Im(\theta)|\sin(|\Im(\theta)|) - |\Im(\theta)|\sin(|\xi(\theta)|)| \\
&= |\xi(\theta)) - \Im(\theta)| + (1 + |\Im(\theta)|)\,|\sin(|\xi(\theta)|) - \sin(|\Im(\theta)|)|
\end{aligned}$$

$$
\begin{aligned}
&+\frac{|\bar{\xi}(\theta)-\bar{\Im}(\theta)|}{4}\\
&\leq |\xi(\theta)-\Im(\theta)|+\frac{1}{2}(1+|\Im(\theta)|)\\
&\quad\times\left|\sin\left(\frac{||\xi(\theta)|-|\Im(\theta)|}{2}\right)\right|\left|\cos\left(\frac{|\xi(\theta)|+|\Im(\theta)||}{2}\right)\right|\\
&\leq (2+\|\Im\|_\infty)\|\xi-\Im\|_\infty+\frac{\|\bar{\xi}-\bar{\Im}\|_\infty}{4}.
\end{aligned}
$$

The case when $|\Im(\theta)|\leq|\xi(\theta)|$, we get

$$
|\Psi(\theta,\xi,\bar{\xi})-\Psi(\theta,\Im,\bar{\Im})|\leq(2+\|\xi\|_\infty)\|\xi-\Im\|_\infty+\frac{\|\bar{\xi}-\bar{\Im}\|_\infty}{4}.
$$

Hence

$$
|\Psi(\theta,\xi,\bar{\xi})-\Psi(\theta,\Im,\bar{\Im})|\leq\min\{2+\|\xi\|_\infty,2+\|\Im\|_\infty\}\|\xi-\Im\|_\infty+\frac{\|\bar{\xi}-\bar{\Im}\|_\infty}{4},
$$

Thus, hypothesis (H_1) is satisfied with

$$
\gamma_1(\xi,\Im)=\min\{2+\|\xi\|_\infty,2+\|\Im\|_\infty\}
$$

and

$$
\gamma_2(\theta)=\frac{1}{4}.
$$

Let $\lambda(\theta)=\frac{1}{8}\theta$, $\phi(\theta)=\theta$, $\varpi: C(\Theta)\times C(\Theta)\to\mathbb{R}_+^*$ with

$$
\begin{cases}
\varpi(\xi,\Im)=1;\ if\ \ \delta(\xi(\theta),\Im(\theta))\geq 0,\ \ \theta\in\Theta,\\
\varpi(\xi,\Im)=0;\ else,
\end{cases}
$$

and $\delta: C(\Theta)\times C(\Theta)\to\mathbb{R}$ with $\delta(\xi,\Im)=\|\xi-\Im\|_\infty$.
Hypothesis (H_2) is satisfied with $\overline{\mu}_0(\theta)=\xi_0$. Also, (H_3) holds from the definition of the function δ. Since all requirements of Theorem 3.2 are verified, then we conclude the existence and the uniqueness of solutions for problem (11).

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