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Original Research Paper

Dual Solutions for the Problem of Mixed Convection Flow Through a Porous Medium Using an Iterative Finite Difference Method

A.S. Hashemi
Yazd Universuty

M. Heydari*
Yazd Universuty

G.B. Loghmani
Yazd Universuty

D.D. Ganji
Babol Noshirvani University of Technology

Abstract. The aim of this article is to approximate the multiple solutions of the problem of mixed convection in a porous medium on the half-line utilizing the quasilinearization method (QLM) combined with the finite difference method (FDM). For this purpose, at first, we transform the governing nonlinear differential equation to a sequence of linear differential equations via the quasilinearization approach. Then, we provide a sequence of linear algebraic systems by applying the FDM at each iteration to find the approximate solutions of the obtained linear differential equations. Moreover, we present a beneficial scheme to obtain appropriate initial guesses in order to compute both solutions of the problem. The convergence analysis is considered in detail and

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*Corresponding Author

some numerical results are reported to demonstrate the validity of the proposed iterative method.

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1 Introduction

Many problems in the various fields of science and engineering can be formulated by linear and nonlinear differential equations [19, 12, 13, 11]. Generally, finding the exact or numerical solutions for them is very indispensable in the comprehension and interpretation of these phenomena. The study of methods (analytical, semi-analytical and numerical) to find the multiple solutions of nonlinear boundary value problems (NBVPs) has been one of the interesting topics for researchers in recent years. Here, we mention a number of these research works as follows: In a pioneer work, Abbasbandy and Shivanian [2] provided an effective HAM based-method, so-called predictor HAM (PHAM), to find multiple solutions of NBVPs. Also, they applied this method to solve some nonlinear models in science and engineering [3, 27, 24, 21]. Freidoonimehr and Rashidi [10] employed the predictor HAM for solving the problem of MHD Jeffery-Hamel and calculated all branches of the solutions. Another research study on the existence of multiple solutions of second and third order BVPs has been presented by Shivanian and Abdolrazaghi [22, 23]. Shivanian et al. [20] applied the Chebyshev collocation method for computing all solutions of a class of nonlinear reactive transport models. They also improved the accuracy of the obtained solutions by using a discrete least square method. Wazwaz et al. [29, 28] have successfully employed the Adomian decomposition method and variational iteration method to calculate dual solutions for nonlinear BVPs such as the heat transfer model, diffusion and reaction model, first and second reactive transport models. Ben-Romdhane et al. [26, 7, 8] introduced an effective approach with a low computational cost for solving nonlinear BVPs. This method is based on a combination of the QLM and standard FDM. They also extended the proposed method to obtain dual solutions for the Bratu problem [9]. Recently, Karamollahi et al. [14, 15] developed a Hermite interpolation-based method to approximate multiple solutions of the Bratu equation and the nonlinear problem of heat transfer in a straight fin. In this work, we consider the problem of mixed convection flow through a porous medium. The governing differential equations of this problem are as follows [17]:

$$f'(\eta) = 1 + \Big(\frac{Ra_x}{Pe_x}\Big)\theta(\eta), \tag{1}$$

$$\begin{aligned} \theta''(\eta) + \Big(\frac{1}{2} - nI\Big) f(\eta)\theta'(\eta) - nIf'(\eta)\theta(\eta) \\ = xI\Big(f'(\eta)\frac{\partial\theta(\eta)}{\partial x} - \theta'(\eta)\frac{\partial f(\eta)}{\partial x}\Big), \end{aligned} \tag{2}$$

with the boundary conditions

$$\begin{aligned} f(0) = 0, \quad \theta(0) = 1, \\ f'(\infty) = 1, \quad \theta(\infty) = 0, \end{aligned}$$

where f is non-dimensional stream function, θ is non-dimensional temperature profile, $nI = \frac{\lambda}{1+3\lambda}$ is lumped parameter and η is similarity variable. As mentioned in [2], by taking $\frac{Ra_x}{Pe_x} = b$ in (1)-(2) where b is a constant, one can obtain the following system of ordinary differential equations

$$f'(\eta) = 1 + b\theta(\eta), \tag{3}$$

$$\theta''(\eta) + \Big(\frac{1+\lambda}{2(1+3\lambda)}\Big) f(\eta)\theta'(\eta) - \Big(\frac{\lambda}{1+3\lambda}\Big) f'(\eta)\theta(\eta) = 0. \tag{4}$$

Now, by taking $\theta = b^{-1}(f' - 1)$ and $\lambda = -1$ in (4), the problem of mixed convection flow can be modeled by the following nonlinear BVP as [17]:

$$2f'''(\eta) + f'(\eta) - (f'(\eta))^2 = 0. \tag{5}$$

Also, the boundary conditions can be reduced as:

$$f(0) = 0, \quad f'(0) = 1 + b, \quad f'(\infty) = 1. \tag{6}$$

As stated in [17], by solving the BVP (5) and using boundary conditions (6), one can obtain two solutions for f' as follows:

$$f'(\eta) = -\frac{1}{2} + \frac{3}{2}\tanh^2\left[\frac{\eta}{2\sqrt{2}} \pm \frac{1}{2}\ln\left(\frac{\sqrt{3}+\sqrt{3+2b}}{\sqrt{3}-\sqrt{3+2b}}\right)\right], \tag{7}$$

when $b \in [-\frac{3}{2}, 0)$. Moreover, substituting (7) into (3) leads to the corresponding two solutions for temperature profile as [1]:

$$\theta(\eta) = -\frac{3}{2b}\cosh^{-2}\left[\frac{\eta}{2\sqrt{2}} \pm \frac{1}{2}\ln\left(\frac{\sqrt{3}+\sqrt{3+2b}}{\sqrt{3}-\sqrt{3+2b}}\right)\right]. \tag{8}$$

Abbasbandy and Sivanian [1] applied pseudo-spectral collocation method based on Chebyshev polynomials to find multiple solutions of equation (5) numerically. Also, Abbasbandy et al. [4] utilized a new approximate analytical method based on padé-approximation and homotopy-padé approuch to solve this problem.
The quasilinearization scheme was first introduced by Bellman and Kalaba [6]. By applying this technique, the nonlinear problems are reduced to an iterative scheme of linear equations. The purpose of this article is to implement a numerical method based on the QLM and FDM to find two solutions of nonlinear BVP (5). For this goal, we follow [9] and obtain a sequence of approximate solutions. Moreover, we investigate the convergence analysis of the proposed method.
The remains of this paper has been organised as follows. In Section 2, we review the quasilinearization method and organize the iterative finite difference method to solve the governing differential equation. In order to prove the convergence of the proposed method, some theorems are presented in Section 3. In Section 4, the choice of initial approximation for iterative scheme is discussed. Some numerical results are provided in Section 5. Finally, the article concludes with a conclusion.

2 Iterative Finite Difference Method

In this section, we introduce the idea of the iterative finite difference method (IFDM) and utilize it to solve nonlinear BVP (5) with boundary conditions (6). To do this, suppose that $u = f'$, $g(u) = (u^2 - u)/2$ and $L \in \mathbb{R}^+$ is sufficiently large such that we can substitute the semi-infinite interval $[0, +\infty)$ with $[0, L]$. Then, the relations (5) and (6) can be rewritten as follows:

$$u''(\eta) = g(u(\eta)), \tag{9}$$

$$u(0) = 1 + b, \quad u(L) = 1. \tag{10}$$

Let us consider u_0 as an initial approximation of the exact solution u. This approximation is usually chosen such that the boundary conditions (10) are satisfied. Furthermore, we suppose that $g(u)$ is a differentiable function. Now, using the QLM [6] for nonlinear BVP (9)-(10), one can get the following recurrence relation:

$$u''_{n+1}(\eta) - g'(u_n(\eta))u_{n+1}(\eta) = g(u_n(\eta)) - g'(u_n(\eta))u_n(\eta), \tag{11}$$
$$u_{n+1}(0) = 1 + b, \quad u_{n+1}(L) = 1, \quad n = 0, 1, 2, \dots. \tag{12}$$

The main characteristic of this iterative relation is that u_{n+1} can be obtained by solving the linear BVP (11)-(12) in each iteration. In Section 3, we will prove that the sequence of approximate solutions $\{u_n\}$ is quadratically convergent to the exact solution of the problem (9)-(10).
As we know, except for some special cases, it is generally not possible to determine the exact analytical solution for the linear differential equations. So, semi-analytical or numerical schemes can be applied for the approximate solution of these problems.
Here, we employ the standard FDM for solving linear BVP (11)-(12) in each iteration. For this purpose, suppose that $M \in \mathbb{N}$,

$$\Omega_M = \{\eta_i : \eta_i = ih, i = 0, 1, \dots, M\},$$

is a uniform partition over the interval $[0, L]$ with step-size $h = \frac{L}{M}$ and

$$V^{[n]} = \{v_i^{[n]} : i = 0, 1, \dots, M\}, \quad n = 1, 2, \dots,$$

is a grid function on Ω_M in which $v_i^{[n]}$ is a numerical approximation of the exact value $u_n(\eta_i)$. Employing the standard finite difference approximation of the second derivative

$$u''_{n+1}(\eta_i) \simeq \frac{1}{h^2}\Big[v_{i+1}^{[n+1]} - 2v_i^{[n+1]} + v_{i-1}^{[n+1]}\Big],$$

and substituting it in (11)-(12), we have the following discrete form

$$\begin{cases} \frac{1}{h^2}\Big[v_{i+1}^{[n+1]} - 2v_i^{[n+1]} + v_{i-1}^{[n+1]}\Big] - g'(v_i^{[n]})v_i^{[n+1]} = g(v_i^{[n]}) - g'(v_i^{[n]})v_i^{[n]}, \\ v_0^{[n+1]} = 1 + b, \quad v_M^{[n+1]} = 1, \quad i = 1, 2, \dots, M-1, \end{cases}$$

or equivalently,

$$\begin{cases} \frac{1}{h^2}\left[v_{i+1}^{[n+1]} - (2 + h^2 g'(v_i^{[n]}))v_i^{[n+1]} + v_{i-1}^{[n+1]}\right] = -\frac{1}{2}(v_i^{[n]})^2, \\ v_0^{[n+1]} = 1 + b, \quad v_M^{[n+1]} = 1, \end{cases} \tag{13}$$

where $n = 0, 1, 2, \ldots$ and $v_i^{[0]} = u_0(\eta_i),\ i = 0, 1, \ldots, M$. To show the solvability of the linear system of algebraic equations, we rewrite (13) in the following matrix form

$$A^{[n]}\mathcal{V}^{[n+1]} = B^{[n]}, \quad n = 0, 1, 2, \ldots, \tag{14}$$

where

$$A^{[n]} = (a_{i,j}^{[n]}), \quad a_{i,j}^{[n]} = \begin{cases} h^{-2}, & j = i+1 \text{ and } j = i-1, \\ -(2h^{-2} + g'(v_i^{[n]})), & i = j, \\ 0, & \text{otherwise}, \end{cases}$$

for $i, j = 1, 2, \ldots, M-1$ and,

$$\tilde{B}^{[n]} = \begin{pmatrix} -\frac{1}{2}(v_1^{[n]})^2 - \frac{1}{h^2}(1+b) \\ -\frac{1}{2}(v_2^{[n]})^2 \\ \vdots \\ -\frac{1}{2}(v_{M-2}^{[n]})^2 \\ -\frac{1}{2}(v_{M-1}^{[n]})^2 - \frac{1}{h^2} \end{pmatrix}, \quad \mathcal{V}^{[n+1]} = \begin{pmatrix} v_1^{[n+1]} \\ v_2^{[n+1]} \\ \vdots \\ v_{M-2}^{[n+1]} \\ v_{M-1}^{[n+1]} \end{pmatrix}.$$

In what follow, we prove that for sufficiently small values of h, the system (14) has a unique solution.

Lemma 2.1. *For sufficiently small values of h, the linear system (14) has a unique solution.*

Proof. Multiplying the system (14) by $-h^2$, we have

$$\tilde{A}^{[n]}\mathcal{V}^{[n+1]} = \tilde{B}^{[n]}, \quad n = 0, 1, 2, \ldots,$$

where

$$\tilde{A}^{[n]} = (\tilde{a}_{i,j}^{[n]}), \quad \tilde{a}_{i,j}^{[n]} = \begin{cases} -1, & j = i+1 \text{ and } j = i-1, \\ (2 + h^2 g'(v_i^{[n]})), & i = j, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$\tilde{B}^{[n]} = \begin{pmatrix} \frac{h^2}{2}(v_1^{[n]})^2 + (1+b) \\ \frac{h^2}{2}(v_2^{[n]})^2 \\ \vdots \\ \frac{h^2}{2}(v_{M-2}^{[n]})^2 \\ \frac{h^2}{2}(v_{M-1}^{[n]})^2 + 1 \end{pmatrix}.$$

Thus, the symmetric tridiagonal matrix $\tilde{A}^{[n]} = -h^2 A^{[n]}$ tends to

$$D = \begin{pmatrix} 2 & -1 & 0 & 0 & ... & 0 \\ -1 & 2 & -1 & 0 & ... & 0 \\ 0 & -1 & 2 & -1 & ... & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & ... & 0 & -1 & 2 & -1 \\ 0 & ... & 0 & 0 & -1 & 2 \end{pmatrix}_{(M-1)\times(M-1)},$$

when $h \to 0$. On the other hand, the matrix D is positive definite and therefore the system (14) has a unique solution when the step-size h is small enough. $\square$

3 Convergence Analysis

In this section, we investigate the convergence analysis of the IFDM for the BVP (9)-(10). To this goal, at first, we prove the quadratic convergence of the sequence $\{u_n\}$ by inspiring [18].
Consider the linear BVP

$$z''(\eta) + p(\eta)z(\eta) = q(\eta), \tag{15}$$
$$z(0) = 0, \quad z(L) = 0. \tag{16}$$

where $p(\eta)$ and $q(\eta)$ are continuous on $[0, L]$. Also, assume that $z_1(\eta)$ and $z_2(\eta)$ are two linear independent solutions of the corresponding homogeneous differential equation (15). Then, one can obtain the solution of the BVP (15)-(16) as follows [5]:

$$z(\eta) = \int_0^L G(\eta, s)q(s)ds, \tag{17}$$

where

$$G(\eta,s)=\begin{cases} z_1(\eta)z_2(s), & 0\le s\le \eta,\\ z_2(\eta)z_1(s), & \eta\le s\le L,\end{cases}$$

is the Green's function.

Theorem 3.1. *Let $\{u_n\}$ be the sequence of continuous approximate solutions generated by (11)-(12) and $\tilde{e}_n=u_n-u_{n-1}$. Then the sequence $\{u_n\}$ converges quadratically.*

Proof. Using recurrence relations (11)-(12) and subtracting two consecutive iterations, one can get

$$\begin{aligned} &\tilde{e}''_{n+1}(\eta)-g'(u_n(\eta))\tilde{e}_{n+1}(\eta)\\ &=g(u_n(\eta))-g(u_{n-1}(\eta))-g'(u_{n-1}(\eta))\tilde{e}_n(\eta), \qquad (18)\\ &\tilde{e}_{n+1}(0)=0,\quad \tilde{e}_{n+1}(L)=0. \qquad (19)\end{aligned}$$

Imposing $g(u)=(u^2-u)/2$ on the right-hand side of (18), we deduce that

$$\begin{cases} \tilde{e}''_{n+1}(\eta)-g'(u_n(\eta))\tilde{e}_{n+1}(\eta)=\frac{1}{2}(\tilde{e}_n(\eta))^2,\\ \tilde{e}_{n+1}(0)=0,\quad \tilde{e}_{n+1}(L)=0.\end{cases} \qquad (20)$$

Now, from (15)-(17), we can get

$$\tilde{e}_{n+1}(\eta)=\frac{1}{2}\int_0^L G_{n+1}(\eta,s)(\tilde{e}_n(s))^2ds, \qquad (21)$$

where

$$G_{n+1}(\eta,s)=\begin{cases} \tilde{e}_{n+1,1}(\eta)\tilde{e}_{n+1,2}(s), & 0\le s\le \eta,\\ \tilde{e}_{n+1,2}(\eta)\tilde{e}_{n+1,1}(s), & \eta\le s\le L,\end{cases}$$

and, $\tilde{e}_{n+1,1}(\eta)$ and $\tilde{e}_{n+1,2}(\eta)$ are two linear independent solutions of the corresponding homogeneous differential equation (20). Equation (21) implies

$$|\tilde{e}_{n+1}(\eta)|\le \frac{1}{2}\int_0^L |G_{n+1}(\eta,s)||\tilde{e}_n(s)|^2ds\le J\|\tilde{e}_n\|^2,$$

in which

$$\|\tilde{e}_n\| = \max_{\eta\in[0,L]} |\tilde{e}_n(\eta)|, \;\; J = \max_{n\in\{0\}\cup\mathbb{N}} \Big(\max_{\eta\in[0,L]} \Big(\frac{1}{2}\int_0^L |G_{n+1}(\eta,s)|ds \Big)\Big).$$

Thus, we have

$$\|\tilde{e}_{n+1}\| \leq J\|\tilde{e}_n\|^2, \tag{22}$$

and the convergence is quadratic. □

Theorem 3.2. *Let $\{u_n\}$ be the sequence of continuous approximate solutions generated by (11)-(12). If there exists a positive integer m_0 such that $\omega = J\|\tilde{e}_{m_0}\| < 1$, then the sequence $\{u_n\}$ converges uniformly to a continuous function v on the interval $[0, L]$.*

Proof. From (22) and using induction on n, one can get

$$\|\tilde{e}_{n+1}\| \leq J^{-1}(J\|\tilde{e}_{m+1}\|)^{2^{n-m}} \leq J^{-1}(J\|\tilde{e}_{m_0}\|)^{2^{n-m_0+1}},$$

for any $m \in \{m_0-1, m_0, \ldots, n-1\}$. Now, for any $n > m > m_0$, we have

$$\begin{aligned}
\|u_n - u_m\| &\leq \|\tilde{e}_n\| + \|\tilde{e}_{n-1}\| + \ldots + \|\tilde{e}_{m+1}\| \\
&\leq J^{-1}(\omega^{2^{n-m_0}} + \omega^{2^{n-m_0-1}} + \ldots + \omega^{2^{m-m_0+1}}) \\
&= J^{-1}\omega^{2^{m-m_0+1}} \sum_{k=0}^{n-m} \omega^{(2^k-1)2^{m-m_0+1}} \\
&\overset{0<\omega<1}{\leq} J^{-1}\omega^{2^{m-m_0+1}} \sum_{k=0}^{n-m} \omega^{(2^k-1)} \\
&\leq J^{-1}\omega^{2^{m-m_0+1}} \sum_{k=0}^{n-m} \omega^{k} \\
&= (J(1-\omega))^{-1}(1-\omega^{n-m+1})\omega^{2^{m-m_0+1}}.
\end{aligned}$$

Therefore

$$\lim_{n,m\to\infty} \|u_n - u_m\| = 0,$$

and the Cauchy sequence $\{u_n\}$ converges to a continuous function v in the Banach space $C[0, L]$. □

Corollary 3.3. *Let $\{u_n\}$ be the sequence of continuous approximate solutions generated by (11)-(12), u be the exact solution of the problem* (9)-(10) *and suppose $e_n = u - u_n$. If there exists a positive integer $\tilde{m}_0$ such that $\tilde{\omega} = J\|e_{\tilde{m}_0}\| < 1$, then $u = v$.*

Proof. Using $g(u) = (u^2 - u)/2$, we first rewrite the nonlinear BVP (9)-(10) as follows:

$$\begin{aligned} &u''(\eta) - g'(u_n(\eta))u(\eta) \\ &= g(u_n(\eta)) - g'(u_n(\eta))u_n(\eta) + \frac{1}{2}(u(\eta) - u_n(\eta))^2, \end{aligned} \tag{23}$$

$$u(0) = 1 + b, \quad u(L) = 1. \tag{24}$$

Subtracting (23)-(24) from (11)-(12), one can obtain

$$\begin{aligned} &e''_{n+1}(\eta) - g'(u_n(\eta))e_{n+1}(\eta) = \frac{1}{2}(e_n(\eta))^2, \\ &e_{n+1}(0) = 0, \quad e_{n+1}(L) = 0. \end{aligned}$$

Now, similar to (20)-(21), we have

$$e_{n+1}(\eta) = \frac{1}{2}\int_0^L G_{n+1}(\eta, s)(e_n(s))^2 ds,$$

and therefore

$$\|e_{n+1}\| \leq J\|e_n\|^2.$$

Furthermore, we can conclude that

$$\|e_n\| \leq J^{-1}\tilde{\omega}^{2^{n-\tilde{m}_0}}, \tag{25}$$

for every $n \geq \tilde{m}_0$. Now, from (25) and Theorem 3.2, we obtain that

$$\lim_{n\to\infty} \|e_n\| = \|u - v\| = 0.$$

□

In what follows, we investigate the convergence analysis of the IFDM. To do this, we employ the discrete inner product and norm

$$\langle w, v\rangle = \sum_{j=1}^{M-1} w_j v_j, \quad \|w\|_2 = \Big(\sum_{j=1}^{M-1} |w_j|^2\Big)^{\frac{1}{2}}, \quad \forall w, v \in W_h, \tag{26}$$

where $W_h = \{w : w = (w_0, w_1, \ldots, w_M), w_0 = w_M = 0\}$. Also, we consider the eigenvalues $\lambda_j,\ j = 1, 2, \ldots, M-1$ and the corresponding eigenvectors $\mu_j,\ j = 1, 2, \ldots, M-1$ of the matrix D [25] as follows:

$$\lambda_j = 2 - 2\cos\left(\frac{j\pi}{M}\right) = 4\sin^2\left(\frac{j\pi}{2M}\right), \quad \mu_j = \begin{pmatrix} \sin\left(\frac{j\pi}{M}\right) \\ \sin\left(\frac{2j\pi}{M}\right) \\ \vdots \\ \sin\left(\frac{(M-2)\pi}{M}\right) \\ \sin\left(\frac{(M-1)\pi}{M}\right) \end{pmatrix},$$

for $j = 1, 2, \ldots, M-1$. We note that the set $\{\mu_j\}$ is an orthogonal basis for $\mathbb{R}^{M-1}$. Furthermore, for sufficiently small $h = \frac{L}{M}$, one can get

$$\min_{1\le j\le M-1} \lambda_j = 4\sin^2\left(\frac{\pi}{2M}\right) = 4\sin^2\left(\frac{h\pi}{2L}\right) \simeq \left(\frac{\pi}{L}\right)^2 h^2. \tag{27}$$

Theorem 3.4. *Let $u, u_n \in C^2[0, L]$ be the exact solutions of (9)-(10) and (11)-(12), respectively, and $v_i^{[n]}$ be the numerical approximation of $u_n(\eta_i)$. Then the approximate values $v_i^{[n]}$ converge to the exact values $u(\eta_i)$ as $n \to \infty$ and $h \to 0$.*

Proof. Using $g(u) = (u^2 - u)/2$ and the centered-difference formula for $u''_{n+1}(\eta_i)$ in (11)-(12), one can obtain the following discrete problem at the grade points Ω_M:

$$\begin{cases} \frac{1}{h^2}\Big[u_{n+1}(\eta_{i+1}) - 2u_{n+1}(\eta_i) + u_{n+1}(\eta_{i-1})\Big] \\ -\Big(u_n(\eta_i) - \frac{1}{2}\Big)u_{n+1}(\eta_i) = -\frac{1}{2}(u_n(\eta_i))^2 - T_i^{[n+1]}, \\ u_{n+1}(\eta_0) = 1 + b, \quad u_{n+1}(\eta_M) = 1, \end{cases} \tag{28}$$

where $T_i^{[n+1]} = -\frac{h^2}{12}u_{n+1}^{(4)}(\xi_{n+1,i})$ is the truncation error. Let $e_i^{[n]} = u_n(\eta_i) - v_i^{[n]}$ and u_0 be an initial approximation of the exact solution u. We define

$$\mathcal{E}^{[n]} = \begin{pmatrix} e_1^{[n]} \\ e_2^{[n]} \\ \vdots \\ e_{M-1}^{[n]} \end{pmatrix}, \ \mathcal{T}^{[n]} = \begin{pmatrix} T_1^{[n]} \\ T_2^{[n]} \\ \vdots \\ T_{M-1}^{[n]} \end{pmatrix}, \quad n = 0, 1, 2, \ldots.$$

According to the first step of the IFDM, we have $e_i^{[0]} = u_0(\eta_i) - v_i^{[0]} = 0$ and therefore $\mathcal{E}^{[0]} = 0$. As a result, we only have the approximation error to compute u_1 by the finite difference method.
For $n = 0$, subtracting (28) form (13), we arrive at

$$\begin{cases} \frac{1}{h^2}\Big[e_{i+1}^{[1]} - 2e_i^{[1]} + e_{i-1}^{[1]}\Big] - \Big(u_0(\eta_i) - \frac{1}{2}\Big)e_i^{[1]} = -T_i^{[1]}, \\ e_0^{[1]} = 0, \quad e_M^{[1]} = 0, \end{cases}$$

for $i = 1, 2, \ldots, M-1$ or equivalently,

$$\hat{A}^{[0]}\mathcal{E}^{[1]} = h^2\mathcal{T}^{[1]}, \tag{29}$$

where

$$\hat{A}^{[0]} = (\hat{a}_{i,j}^{[0]}), \quad \hat{a}_{i,j}^{[0]} = \begin{cases} -1, & j = i+1 \text{ and } j = i-1, \\ 2 + h^2\Big(u_0(\eta_i) - \frac{1}{2}\Big), & i = j, \\ 0, & \text{otherwise}, \end{cases}$$

Following [16], since the set of eigenvectors of the matrix D forms an orthogonal basis for $\mathbb{R}^{M-1}$, one can obtain

$$\mathcal{E}^{[1]} = \sum_{j=1}^{M-1} \alpha_j \mu_j, \quad \|\mathcal{E}^{[1]}\|_2^2 = \sum_{j=1}^{M-1} |\alpha_j|^2 \|\mu_j\|_2^2. \tag{30}$$

Taking the inner product (26) on both sides of (29) with $\mathcal{E}^{[1]}$, yields

$$\Big\langle \hat{A}^{[0]}\mathcal{E}^{[1]}, \mathcal{E}^{[1]} \Big\rangle = \Big\langle h^2\mathcal{T}^{[1]}, \mathcal{E}^{[1]} \Big\rangle \le h^2\|\mathcal{T}^{[1]}\|_2\|\mathcal{E}^{[1]}\|_2. \tag{31}$$

On the other hand, from (27) and (30), we have

$$\begin{aligned} \Big\langle \hat{A}^{[0]}\mathcal{E}^{[1]}, \mathcal{E}^{[1]} \Big\rangle &= \Big\langle \sum_{j=1}^{M-1} \alpha_j \hat{A}^{[0]}\mu_j, \sum_{j=1}^{M-1} \alpha_j\mu_j \Big\rangle \xrightarrow{h\to 0} \Big\langle \sum_{j=1}^{M-1} \alpha_j D\mu_j, \sum_{j=1}^{M-1} \alpha_j\mu_j \Big\rangle \\ &= \Big\langle \sum_{j=1}^{M-1} \alpha_j\lambda_j\mu_j, \sum_{j=1}^{M-1} \alpha_j\mu_j \Big\rangle \overset{\lambda_j \ge 0}{=} \sum_{j=1}^{M-1} \lambda_j |\alpha_j|^2 \|\mu_j\|_2^2 \\ &\ge \lambda_1 \sum_{j=1}^{M-1} |\alpha_j|^2 \|\mu_j\|_2^2 \ge \tilde{C}\Big(\frac{\pi}{L}\Big)^2 h^2 \|\mathcal{E}^{[1]}\|_2^2, \end{aligned} \tag{32}$$

where $\tilde{C}$ is a positive constant. Thus, from (31) and (32), we conclude that

$$\|\mathcal{E}^{[1]}\|_2 \leq C_1 \|\mathcal{T}^{[1]}\|_2 = O(h^2),$$

where $C_1 = \frac{1}{\tilde{C}}\Big(\frac{L}{\pi}\Big)^2$.

Now for $n = 1$, inserting $v_i^{[1]} = u_1(\eta_i) - e_i^{[1]}$ into (13) and subtracting (28) form it, we obtain

$$\begin{cases} \frac{1}{h^2}\Big[e_{i+1}^{[2]} - 2e_i^{[2]} + e_{i-1}^{[2]}\Big] - \Big(u_1(\eta_i) - e_i^{[1]} - \frac{1}{2}\Big)e_i^{[2]} = -T_i^{[2]} \\ +\Big(u_2(\eta_i) - u_1(\eta_i) + \frac{1}{2}e_i^{[1]}\Big)e_i^{[1]}, \quad i = 1, 2, \dots, M-1, \\ e_0^{[2]} = 0, \quad e_M^{[2]} = 0, \end{cases}$$

or equivalently,

$$\hat{A}^{[1]}\mathcal{E}^{[2]} = h^2(\mathcal{T}^{[2]} - \mathcal{K}^{[2]}\mathcal{E}^{[1]}), \tag{33}$$

in which

$$\hat{A}^{[1]} = (\hat{a}_{i,j}^{[1]}),$$

where

$$\hat{a}_{i,j}^{[1]} = \begin{cases} -1, & j = i+1 \text{ and } j = i-1, \\ 2 + h^2\Big(u_1(\eta_i) - e_i^{[1]} - \frac{1}{2}\Big), & i = j, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$\mathcal{K}^{[2]} = diag(k_1^{[2]}, k_2^{[2]}, \dots, k_{M-1}^{[2]}), \ k_i^{[2]} = u_2(\eta_i) - u_1(\eta_i) + \frac{1}{2}e_i^{[1]},$$

for $i = 1, 2, \dots, M-1$. Similarly, taking the inner product (26) on both sides of (33) with $\mathcal{E}^{[2]}$, when $h \to 0$ we arrive at

$$\|\mathcal{E}^{[2]}\|_2 \leq C_2(\|\mathcal{T}^{[2]}\|_2 + \|\mathcal{E}^{[1]}\|_2) \leq C_2(\|\mathcal{T}^{[2]}\|_2 + \|\mathcal{T}^{[1]}\|_2),$$

where $C_2 = \max\{C_1, C_1^2\|\mathcal{K}^{[2]}\|_2\}$. Continuing this process for a sufficiently large n ($n > max\{m_0, \tilde{m}_0\}$), one can get

$$\|\mathcal{E}^{[n-1]}\|_2 \leq C_{n-1}\sum_{k=1}^{n-1}\|\mathcal{T}^{[k]}\|_2 = O(h^2), \tag{34}$$

where $C_{n-1} = \max\{C_1, C_1 C_{n-2} \|\mathcal{K}^{[n-1]}\|_2\}$.
Now, from Theorem 3.2, we have $\lim_{n\to\infty} \tilde{e}_n = 0$, i.e., for every $\varepsilon > 0$ there exists a positive integer N such that

$$|u_n(\eta) - u_{n-1}(\eta)| < \varepsilon, \quad \forall n > N.$$

Thus, for $\varepsilon = \sqrt{\frac{h}{L}}$ and for every $n > max\{N, m_0, \tilde{m}_0\}$, one can get

$$\hat{A}^{[n-1]} \mathcal{E}^{[n]} = h^2 (\mathcal{T}^{[n]} - \mathcal{K}^{[n]} \mathcal{E}^{[n-1]}), \tag{35}$$

in which

$$\hat{A}^{[n-1]} = (\hat{a}_{i,j}^{[n-1]}),$$

where

$$\hat{a}_{i,j}^{[n-1]} = \begin{cases} -1, & j = i+1 \text{ and } j = i-1, \\ 2 + h^2 \Big(u_{n-1}(\eta_i) - e_i^{[n-1]} - \frac{1}{2} \Big), & i = j, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\mathcal{K}^{[n]} = diag(k_1^{[n]}, k_2^{[n]}, \dots, k_{M-1}^{[n]}), \; k_i^{[n]} = u_n(\eta_i) - u_{n-1}(\eta_i) + \frac{1}{2} e_i^{[n-1]},$$

for $i = 1, 2, \dots, M-1$. Using the Frobenius norm $\|\mathcal{K}^{[n]}\|_F$ and Minkowski inequality, we conclude that

$$\begin{aligned} \|\mathcal{K}^{[n]}\|_2 \le \|\mathcal{K}^{[n]}\|_F &= \Big(\sum_{i=1}^{M-1} |u_n(\eta_i) - u_{n-1}(\eta_i) + \frac{1}{2} e_i^{[n-1]}|^2 \Big)^{\frac{1}{2}} \\ &\le \Big(\sum_{i=1}^{M-1} (|u_n(\eta_i) - u_{n-1}(\eta_i)| + \frac{1}{2} |e_i^{[n-1]}|)^2 \Big)^{\frac{1}{2}} \\ &\le \Big(\sum_{i=1}^{M-1} |u_n(\eta_i) - u_{n-1}(\eta_i)|^2 \Big)^{\frac{1}{2}} + \frac{1}{2} \Big(\sum_{i=1}^{M-1} |e_i^{[n-1]}|^2 \Big)^{\frac{1}{2}} \\ &\le \varepsilon \sqrt{M-1} + \frac{1}{2} \|\mathcal{E}^{[n-1]}\|_2 \le \sqrt{M} \sqrt{\frac{h}{L}} + \frac{1}{2} \|\mathcal{E}^{[n-1]}\|_2 \\ &= 1 + \frac{1}{2} \|\mathcal{E}^{[n-1]}\|_2 \end{aligned}$$

Taking the inner product (26) on both sides of (35) with $\mathcal{E}^{[n]}$ and employing (34), for all $n > max\{N, m_0, \tilde{m}_0\}$ we have

$$\|\mathcal{E}^{[n]}\|_2 \leq C_n(\|\mathcal{T}^{[n]}\|_2 + \frac{1}{2}\|\mathcal{E}^{[n-1]}\|_2^2 + \|\mathcal{E}^{[n-1]}\|_2) = O(h^2),$$

where $C_n = C_1$. Therefore,

$$\|\mathcal{E}^{[n]}\|_2 \to 0 \quad \text{as} \quad h \to 0 \quad \text{and} \quad n \to \infty. \tag{36}$$

Now, from Corollary 3.3 and (36), we conclude that

$$|u(\eta_i) - v_i^{[n]}| \leq |e_n(\eta_i)| + |e_i^{[n]}| \to 0 \quad \text{as} \quad h \to 0 \quad \text{and} \quad n \to \infty.$$

□

4 The Initial Approximation $u_0(\eta)$

The important argument in any iterative approach is to choose an appropriate initial approximation so that the method is convergent. According to Theorem 3.2, if we determine the initial approximation u_0 such that the boundary conditions in (12) are satisfied, then by considering the continuity of u and its derivatives, $\tilde{e}_1$ is small near the boundaries $\eta = 0$ and $\eta = L$. Therefore, we expect that the convergence is started in the small intervals near the boundaries and extended over the whole interval $[0, L]$ in subsequent iterations. As mentioned in [17], the BVP (5)-(6) has a unique solution for $b \in (0, +\infty)$, and two solutions for $b \in [-\frac{3}{2}, 0)$. In this section, we intend to provide the appropriate initial approximations to identify unique and dual solutions for various values of parameter b. For $b \in (0, +\infty)$, the mixed convection flow equation has a unique solution in the following form

$$u(\eta) = 1 + \frac{3}{2\sinh^2\left(\frac{\eta}{2\sqrt{2}} + \ln\left(\sqrt{\frac{3}{2b}} + \sqrt{1 + \frac{3}{2b}}\right)\right)},$$

where $u = f'$. In this case, utilizing the boundary conditions $u(0) = 1+b$ and $u(\infty) = 1$ and asymptotic behavior of the solution at infinity, we propose the initial approximation $u_0 = 1 + be^{-\eta}$ for the IFDM.

For $b \in [-\frac{3}{2}, 0)$, we require two suitable initial approximations to find dual solutions. To achieve this aim, we first compute the value $u''(0)$ using (9) and the boundary conditions $u(0) = 1 + b$ and $u(\infty) = 1$ as follows:

$$u''(0) = \frac{b(1+b)}{2}.$$

Multiplying (9) by u' and integrating over η yields

$$u'(\eta)^2 = -\frac{1}{2}u^2(\eta) + \frac{1}{3}u^3(\eta) + \gamma, \tag{37}$$

where γ is a real constant. Using (37), boundary condition $u(\infty) = 1$ and the fact that $u'(\infty) = 0$ (because of the asymptotic behavior of u as η tends to infinity), we obtain $\gamma = \frac{1}{6}$ and therefore

$$u'(0) = \pm b\sqrt{\frac{2b+3}{6}}.$$

Now, we consider the initial approximation u_0 in the following form

$$u_0(\eta) = c_1 + (c_2 + c_3\eta + c_4\eta^2)e^{-\eta},$$

where the unknown constants c_1, c_2, c_3 and c_4 are determined such that

$$u_0(0) = 1 + b, \quad u_0'(0) = \pm b\sqrt{\frac{2b+3}{6}} \quad u_0''(0) = \frac{b(1+b)}{2}, \quad u_0(\infty) = 1.$$

Consequently, we obtain two initial approximations u_0^f and u_0^s for the first and second solutions of the BVP (9)-(10), respectively, as follows:

$$u_0^f(\eta) = 1 + \left(\frac{1}{12}\,\eta^2 b\left(9 + 3\,b + 2\,\sqrt{12\,b + 18}\right) + \frac{1}{6}\,\eta\,b\left(6 + \sqrt{12\,b + 18}\right) + b\right)\mathrm{e}^{-\eta},$$

and

$$u_0^s(\eta) = 1 + \left(\frac{1}{12}\,\eta^2 b\left(9 + 3\,b - 2\,\sqrt{12\,b + 18}\right) + \frac{1}{6}\,\eta\,b\left(6 - \sqrt{12\,b + 18}\right) + b\right)\mathrm{e}^{-\eta}.$$

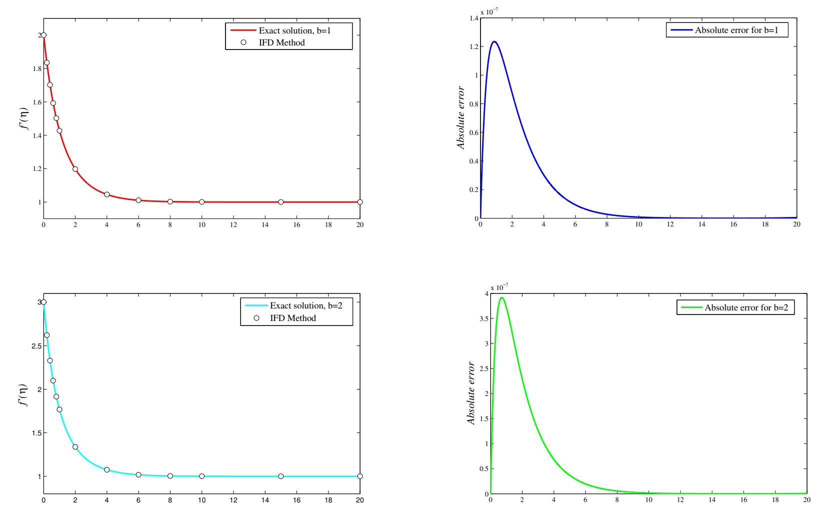

Figure 1: Plots of $f'(\eta)$ and the absolute error for $b = 1$ and $b = 2$.

5 Numerical Results

In this section, we present some numerical results based on the IFDM on the interval $[0, 20]$ with the iteration number $n = 10$ and the mesh number $M = 10000$, and compare them with those obtained by the Chebyshev pseudo-spectral collocation method [1]. Also, we examine different values of the parameter b to illustrate the capability and efficiency of the proposed scheme.
As mentioned in section 4, for $b = 1$ and $b = 2$, we have the unique solutions and we use the initial approximations $u_0 = 1 + e^{-\eta}$ and $u_0 = 1 + 2e^{-\eta}$, respectively, for the IFDM. Fig. 1 shows the plots of the exact solution $f' = u$ versus the numerical solution by the IFDM and the absolute error. Also, the absolute error of the approximate solution for some points is given in Table 1. Fig. 1 and Table 1 indicate that the proposed method has good accuracy when the problem has a unique solution. For $b = -1$ and $b = -1.4$, we have two solutions and

Table 1: The absolute error for $b = 1$ and $b = 2$.

η_i	$b=1$ $\lvert f'(\eta_i) - v_i^{[10]}\rvert$	$b=2$ $\lvert f'(\eta_i) - v_i^{[10]}\rvert$
0	0	0
0.2	$6.43518e-08$	$6.43518e-08$
0.4	$9.97347e-08$	$9.97346e-08$
0.6	$1.16922e-07$	$1.16922e-07$
0.8	$1.23068e-07$	$1.23068e-07$
1	$1.22569e-07$	$1.22569e-07$
2	$8.75760e-08$	$8.75752e-08$
4	$3.02811e-08$	$3.02796e-08$
6	$9.36447e-09$	$9.36355e-09$
8	$2.76411e-09$	$2.76401e-09$
10	$7.90983e-10$	$7.91167e-10$
20	$4.63728e-10$	$4.63441e-10$

we employ the following initial approximations for the IFDM:

$$\text{For } b=-1: \quad \begin{cases} u_0^f(\eta) = 1 + \left(-0.9082482908\,\eta^2 - 1.408248290\,\eta - 1\right)\mathrm{e}^{-\eta}, \\ u_0^s(\eta) = 1 + \left(-0.0917517095\,\eta^2 - 0.5917517095\,\eta - 1\right)\mathrm{e}^{-\eta}, \end{cases}$$

$$\text{For } b=-1.4: \quad \begin{cases} u_0^f(\eta) = 1 + \left(-0.8156038602\,\eta^2 - 1.655603860\,\eta - 1.4\right)\mathrm{e}^{-\eta}, \\ u_0^s(\eta) = 1 + \left(-0.3043961398\,\eta^2 - 1.144396140\,\eta - 1.4\right)\mathrm{e}^{-\eta}. \end{cases}$$

Fig. 2 shows the plots of the exact dual solutions $f' = u$ and θ versus the numerical solutions obtained by the proposed iterative method. A comparison between the IFDM and the numerical results using the Chebyshev pseudo-spectral collocation method [1] is reported in Tables 2 and 3. Also, the absolute errors obtained by the IFDM for various values of n are drawn in Fig. 3.

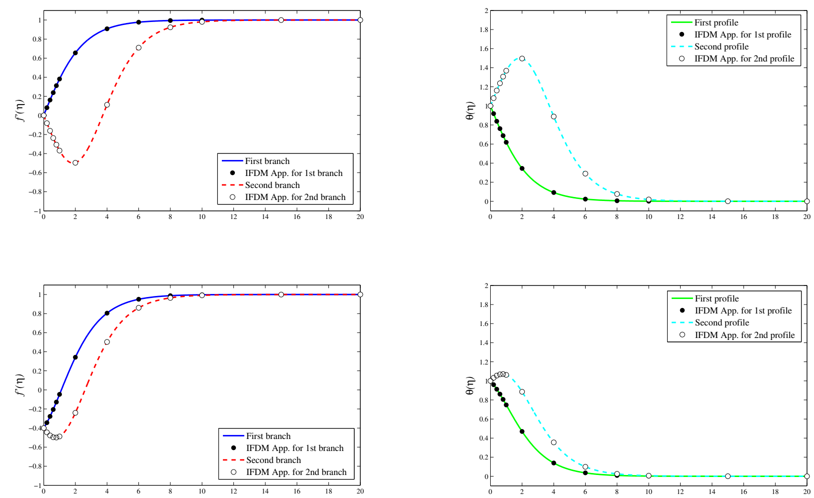

Figure 2: Plots of $f'(\eta)$ and temperature profile $\theta(\eta)$ for $b = -1$ and $b = -1.4$.

Table 2: The absolute error for the approximate solution of the first branch.

x_i	$b = -1$ IFDM	$b = -1$ Error in [1]	$b = -1.4$ IFDM	$b = -1.4$ Error in [1]
0	0	$5.81443e-10$	0	$7.25763e-08$
0.2	$1.91893e-08$	$6.28075e-10$	$4.08208e-08$	$4.91646e-10$
0.4	$3.46300e-08$	$2.25942e-08$	$8.29014e-08$	$3.73690e-08$
0.6	$4.59201e-08$	$3.07356e-08$	$1.22825e-07$	$4.70048e-08$
0.8	$5.32750e-08$	$1.23594e-08$	$1.58200e-07$	$2.35434e-08$
1	$5.71443e-08$	$7.76342e-09$	$1.87325e-07$	$1.34314e-08$
2	$4.47367e-08$	$7.59453e-08$	$2.27173e-07$	$1.71527e-07$
4	$5.33974e-09$	$2.76404e-07$	$8.62272e-08$	$5.74120e-07$
6	$2.27413e-09$	$8.54071e-07$	$1.68211e-08$	$1.77820e-06$
8	$1.55844e-09$	$1.51438e-06$	$2.09727e-09$	$3.17955e-06$
10	$6.29254e-10$	$9.37897e-07$	$2.92667e-11$	$1.99634e-06$
20	$9.81601e-10$	$1.49243e-06$	$2.16003e-09$	$3.19112e-06$

Table 3: The absolute error for the approximate solution of the second branch.

x_i	$b=-1$ IFDM	$b=-1$ Error in [1]	$b=-1.4$ IFDM	$b=-1.4$ Error in [1]
0	0	$7.88040e-07$	0	$6.29726e-07$
0.2	$2.25312e-08$	$1.34397e-07$	$3.70388e-08$	$2.24812e-09$
0.4	$4.76328e-08$	$2.85907e-06$	$6.76984e-08$	$5.81157e-08$
0.6	$7.33340e-08$	$5.11548e-06$	$8.89501e-08$	$9.03898e-08$
0.8	$9.74124e-08$	$2.17614e-06$	$9.91239e-08$	$1.18395e-08$
1	$1.17285e-07$	$4.75643e-07$	$9.76418e-08$	$2.53955e-08$
2	$7.50363e-08$	$1.32949e-06$	$3.34736e-08$	$2.10928e-08$
4	$3.42757e-07$	$1.05565e-05$	$1.42619e-07$	$7.05288e-07$
6	$1.83366e-07$	$2.24473e-05$	$4.11377e-08$	$2.59616e-06$
8	$4.45827e-08$	$1.58856e-05$	$5.66369e-09$	$3.17955e-06$
10	$8.20533e-09$	$1.67834e-06$	$9.83049e-11$	$4.79355e-06$
20	$1.36873e-08$	$4.61258e-06$	$6.21635e-09$	$7.84689e-06$

Table 4: The absolute error for the temperature profile $\theta(\eta)$.

x_i	$b=-1$ First profile	$b=-1$ Second profile	$b=-1.4$ First profile	$b=-1.4$ Second profile
0	0	0	0	0
0.2	$1.91893e-08$	$2.25309e-08$	$2.91577e-08$	$2.64563e-08$
0.4	$3.46299e-08$	$4.76321e-08$	$5.92153e-08$	$4.83560e-08$
0.6	$4.59200e-08$	$7.33330e-08$	$8.77326e-08$	$6.35358e-08$
0.8	$5.32749e-08$	$9.74111e-08$	$1.13000e-07$	$7.08028e-08$
1	$5.71442e-08$	$1.17283e-07$	$1.33803e-07$	$6.97441e-08$
2	$4.47365e-08$	$7.50352e-08$	$1.62266e-07$	$2.39097e-08$
4	$5.34018e-09$	$3.42754e-07$	$6.15908e-08$	$1.01871e-07$
6	$2.27391e-09$	$1.83362e-07$	$1.20151e-08$	$2.93840e-08$
8	$1.55855e-09$	$4.45802e-08$	$1.49805e-09$	$4.04549e-09$
10	$6.29198e-10$	$8.20300e-09$	$2.09047e-11$	$7.02177e-11$
20	$9.81808e-10$	$1.36881e-08$	$1.54288e-09$	$4.44025e-09$

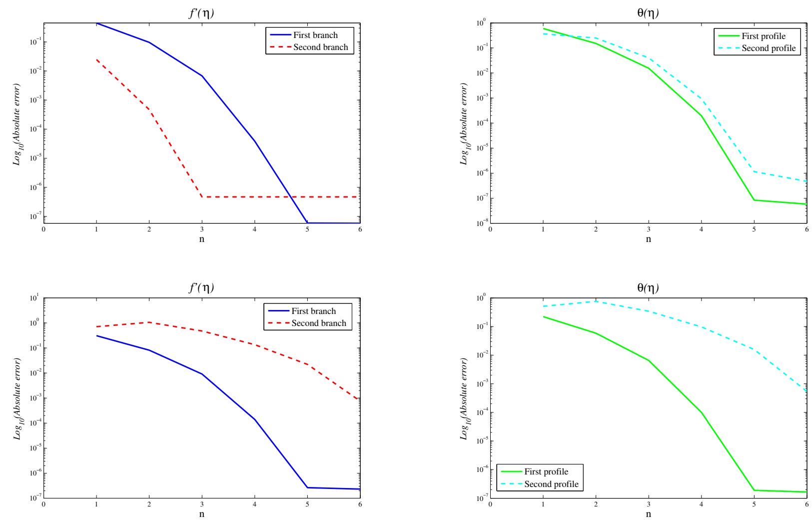

Figure 3: Plots of absolute errors for different values of n for $b = -1$ and $b = -1.4$.

6 Conclusion

To solve the nonlinear BVP with multiple solutions, it is important to use a method that calculates all the solutions numerically or analytically. In this regard, the QLM and the standard FDM have been combined to solve the problem of mixed convection in a porous medium on a semi-infinite interval which admits the unique and dual solutions for different values of embedding parameter b. Convergence analysis of the method was studied in detail. The efficiency and accuracy of the proposed method were measured by comparing the computed results with other techniques. The main feature of this iterative discrete method is that, by choosing suitable initial guesses, all branches of the solutions can be found.

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Azam Sadat Hashemi
Department of Mathematical Sciences
Ph.D. Student
Yazd University
Yazd, Iran
E-mail: as.hashemi@stu.yazd.ac.ir

Mohammad Heydari
Department of Mathematical Sciences
Associate Professor of Mathematics
Yazd University
Yazd, Iran
m.heydari@yazd.ac.ir

Ghasem Barid Loghmani
Department of Mathematical Sciences
Professor of Mathematics
Yazd University
Yazd, Iran
E-mail: loghmani@yazd.ac.ir

Davood Domiri Ganji
Department of Mechanical Engineering
Professor of Mechanical Engineering
Babol University of Technology
Babool, Iran
E-mail: mirgang@nit.ac.ir