Journal of Mathematical Extension
Vol. 17, No. 2, (2023) (5)1-37
URL: https://doi.org/10.30495/JME.2023.2424
ISSN: 1735-8299
Original Research Paper

Generalized Almost Simulative $\hat{Z}^{\Theta}_{\Psi^*}-$Contraction Mappings in Modular $b-$Metric Spaces

M. Öztürk
Sakarya University

F. Golkarmanesh
Islamic Azad University

A. Büyükkaya
Karadeniz Technical University

V. Parvaneh*
Islamic Azad University

Abstract. This article aims to characterize a new class of simulation functions by extending the class of $\mathcal{Z}$ functions demonstrated by Cho in [16] and, via this novel notion, to establish some common fixed point theorems in modular $b-$metric spaces. Furthermore, some corollaries have been procured that expand the existing literature. An example and an application to the nonlinear integral equations have been furnished to state the applicability and effectiveness of the outcomes.

AMS Subject Classification: 47H10; 54H25.
Keywords and Phrases: simulation functions, modular $b-$metric space, almost contraction.

Received: June 2022; Accepted: February 2023
*Corresponding Author

1 Introduction and Preliminaries

In the course of the study, the symbols $\mathbb{N}$ and $\mathbb{R}_+$ state in order of the set of all positive natural numbers and the set of all non-negative real numbers.

Let $\mathcal{J}, \mathcal{K} : \mathcal{L} \to \mathcal{L}$ be self-mappings, where $\mathcal{L}$ is a non-void set. In this case, the following sets stand for the set of fixed points of $\mathcal{J}$ and the set of common fixed points of the mappings $\mathcal{J}$ and $\mathcal{K}$, respectively:

$$Fix\,(\mathcal{J}) = \{\xi \in \mathcal{Q} : \ \mathcal{J}\xi = \xi\}\,;$$
$$C_{Fix}\,(\mathcal{J}, \mathcal{K}) = \{\xi \in \mathcal{Q} : \ \mathcal{J}\xi = \mathcal{K}\xi = \xi\}\,.$$

Metric fixed point theory is a favorite research field for the time being. The central constituent in this theory is the Banach contraction principle (BCP), asserted by Banach in 1922 [10]. Many studies on BCP are being done to evolve this theory. In order to find convenient circumstances on mappings that assure the existence and uniqueness of fixed points, researchers have made a great number of studies based on BCP, either by modifying the contraction conditions with some auxiliary functions, generalizing existing spaces, or using both.

Since the metric function, and hence the metric space structure, forms the basis of the metric fixed point theory, many researchers have extensively studied this function. A plethora of novel distance functions has begun to appear in the literature. The b-metric function, which first appeared in Bakhtin's study [11] and then in Czerwik's [17, 18], is one of them and can even be contemplated as one of the principal generalizations.

Definition 1.1. [17] Let $\mathcal{L}$ be a non-void set, and $\mathfrak{c} \geq 1$ be a real-valued constant. For all $\xi, \ell, \mathfrak{t} \in \mathcal{L}$, if the axioms

(σ_1) $\sigma(\xi, \ell) = 0 \Leftrightarrow \xi = \ell$,

(σ_2) $\sigma(\xi, \ell) = \sigma(\ell, \xi)$,

(σ_3) $\sigma(\xi, \ell) \leq \mathfrak{c}\,[\sigma(\xi, \mathfrak{t}) + \sigma(\mathfrak{t}, \ell)]$

are provided, then the function $\sigma : \mathcal{L} \times \mathcal{L} \to \mathbb{R}_+$ is entitled as a $b-$metric on $\mathcal{L}$. Moreover, the pair $(\mathcal{L}, \sigma)$ represents a $b-$metric space which is abbreviated with $b - \mathfrak{MS}$.

Observe that $b-$metric and ordinary metric coincide, provided that $\mathfrak{c} = 1$. Furthermore, except for the continuity, nearly all of the topological specifications of $b-\mathfrak{MS}$ overlap the metric ones. Thereby, the following lemma can be verified in a $b-$metric.

Lemma 1.2. [4] *Let $(\mathcal{L}, \sigma, \mathfrak{c} \geq 1)$ be a $b-\mathfrak{MS}$ and $\{\xi_{\mathfrak{z}}\}$ and $\{\ell_{\mathfrak{z}}\}$ be convergent to ξ and ℓ, respectively. Then*

$$\frac{1}{\mathfrak{c}^2}\sigma(\xi,\ell) \leq \liminf_{\mathfrak{z}\to\infty} \sigma(\xi_{\mathfrak{z}},\ell_{\mathfrak{z}}) \leq \limsup_{\mathfrak{z}\to\infty} \sigma(\xi_{\mathfrak{z}},\ell_{\mathfrak{z}}) \leq \mathfrak{c}^2\sigma(\xi,\ell).$$

Especially, if $\xi = \ell$, then $\lim_{\mathfrak{z}\to\infty} \sigma(\xi_{\mathfrak{z}},\ell_{\mathfrak{z}}) = 0$. Also, for $\mathfrak{t} \in \mathcal{L}$, we attain

$$\frac{1}{\mathfrak{c}}\sigma(\xi,\mathfrak{t}) \leq \liminf_{\mathfrak{z}\to\infty} \sigma(\xi_{\mathfrak{z}},\mathfrak{t}) \leq \limsup_{\mathfrak{z}\to\infty} \sigma(\xi_{\mathfrak{z}},\mathfrak{t}) \leq \mathfrak{c}\sigma(\xi,\mathfrak{t}).$$

To more figure out $b-\mathfrak{MS}$, see ([2], [8], [24]-[28]).

In 2010, Chistyakov put forward a novel metric function titled modular metric. In addition, various authors have introduced plenty of generalized metric structures to the literature by handling the metric modular concept.

Primarily, rest of the work, we will prefer the represent $\varpi_\hbar(\xi,\ell)$ instead of $\varpi(\hbar,\xi,\ell)$ for all $\hbar > 0$ and $\xi,\ell \in \mathcal{L}$, where $\varpi : (0,\infty)\times\mathcal{L}\times\mathcal{L} \to [0,\infty]$ is a function, and $\mathcal{L}$ is a non-void set.

Definition 1.3. [13, 14] Let $\varpi : (0,\infty)\times\mathcal{L}\times\mathcal{L} \to [0,\infty]$ be a function on non-void set $\mathcal{L}$. Thereupon if the circumstances

(ϖ_1) $\varpi_\hbar(\xi,\ell) = 0$ for all $\hbar > 0 \Leftrightarrow \xi = \ell$,

(ϖ_2) $\varpi_\hbar(\xi,\ell) = \varpi_\hbar(\ell,\xi)$ for all $\hbar > 0$,

(ϖ_3) $\varpi_{\hbar+\mu}(\xi,\ell) \leq \varpi_\hbar(\xi,\mathfrak{t}) + \varpi_\mu(\mathfrak{t},\ell)$ for all $\hbar,\mu > 0$,

are provided for all $\xi,\ell,\mathfrak{t} \in \mathcal{L}$, then, ϖ and $(\mathcal{L},\varpi)$ are termed as a modular metric and modular metric space (briefly $\mathfrak{MMS}$), respectively. Furthermore, if ϖ has the property ($\varpi_1{}'$) given below instead of (ϖ_1), then ϖ is entitled a pseudomodular (metric) on $\mathcal{L}$:

($\varpi_1{}'$) $\varpi_\hbar(\xi,\xi) = 0$ for all $\hbar > 0$.

Also, to investigate more detail about modular metric structure, see ([1], [6], [12], [13], [14], [15], [19], [22], [23], [31]).

By combining modular metric and $b-$metric structures, Ege and Alaca [20] have lately brought the concept of modular b-metric to the literature.

Definition 1.4. [20] Let $\mathfrak{c} \geq 1$ be a real-valued constant and $\mathcal{L}$ be a non-empty set. If, for all $\xi, \ell, \mathfrak{t} \in \mathcal{L}$, the conditions

(κ_1) $\kappa_\hbar(\xi, \ell) = 0$ for all $\hbar > 0 \Leftrightarrow \xi = \ell$,

(κ_2) $\kappa_\hbar(\xi, \ell) = \kappa_\hbar(\ell, \xi)$ for all $\hbar > 0$,

(κ_3) $\kappa_{\hbar+\mu}(\xi, \ell) \leq \mathfrak{c}\left[\kappa_\hbar(\xi, \mathfrak{t}) + \kappa_\mu(\mathfrak{t}, \ell)\right]$ for all $\hbar, \mu > 0$,

are fulfilled, the map $\kappa : (0, \infty) \times \mathcal{L} \times \mathcal{L} \to [0, \infty]$ is labeled as a modular $b-$metric, and the pair $(\mathcal{L}, \kappa)$ is a modular $b-$metric space which is abbreviated with $\mathfrak{M}_\mathfrak{b}\mathfrak{M}\mathfrak{S}$.

In the illustration of $\mathfrak{M}_\mathfrak{b}\mathfrak{M}\mathfrak{S}$, it is an inherent consequence of attaining the modular metric provided that $\mathfrak{c} = 1$. In what follows, there are some examples of $\mathfrak{M}_\mathfrak{b}\mathfrak{M}\mathfrak{S}$.

Example 1.5. [20] Taking into account the set

$$l_p = \left\{ (\xi_\mathfrak{z}) \subset \mathbb{R} : \sum_{\mathfrak{z}=1}^{\infty} |\xi_\mathfrak{z}|^p < \infty \right\}; \quad 0 < p < 1,$$

with $\hbar \in (0, \infty)$ and $\kappa_\hbar(\xi, \ell) = \frac{m(\xi,\ell)}{\hbar}$ such that

$$m(\xi, \ell) = \left(\sum_{\mathfrak{z}=1}^{\infty} |\xi_\mathfrak{z} - \ell_\mathfrak{z}|^p \right)^{\frac{1}{p}}, \quad \xi = \xi_\mathfrak{z}, \ell = \ell_\mathfrak{z} \in l_p.$$

Thereby, $(\mathcal{L}, \kappa)$ is an $\mathfrak{M}_\mathfrak{b}\mathfrak{M}\mathfrak{S}$.

Example 1.6. [32] Let $(\mathcal{L}, \varpi)$ be an $\mathfrak{M}\mathfrak{M}\mathfrak{S}$. Take $\kappa_\hbar(\xi, \ell) = (\varpi_\hbar(\xi, \ell))^s$, where $s \geq 1$. For $t \geq 0$, $\mathcal{J}(t) = t^s$ is a convex function, and considering Jensen's inequality, we attain

$$(\mathfrak{x} + \mathfrak{z})^s \leq 2^{s-1}(\mathfrak{x}^s + \mathfrak{z}^s)$$

for all $\mathfrak{x}, \mathfrak{z} \geq 0$. Thus, $(\mathcal{L}, \kappa)$ is an $\mathfrak{M}_\mathfrak{b}\mathfrak{M}\mathfrak{S}$ with $\mathfrak{c} = 2^{s-1}$.

The set $\mathcal{L}_\kappa = \left\{\ell \in \mathcal{L} : \ell \overset{\kappa}{\sim} \xi\right\}$ is mentioned as a modular set on $\mathcal{L}$ such that κ is a modular $b-$metric and $\overset{\kappa}{\sim}$ is a binary relation that is identified with $\xi \sim \ell \Leftrightarrow \lim\limits_{\hbar\to\infty} \kappa_\hbar(\xi, \ell) = 0$, where $\xi, \ell \in \mathcal{L}$.

Also, the sets

$$\mathcal{L}^*_\kappa = \{\xi \in \mathcal{L} : \exists\hbar = \hbar(\xi) > 0 \text{ such that } \kappa_\hbar(\xi, \xi_0) < \infty\} \ \ (\xi_0 \in \mathcal{L})$$

are an $\mathfrak{M}_\flat\mathfrak{M}\mathfrak{S}$ (around ξ_0).

As follows, some topological features of an $\mathfrak{M}_\flat\mathfrak{M}\mathfrak{S}$ are put forward.

Definition 1.7. [20] Let $(\mathcal{L}, \kappa)$ be an $\mathfrak{M}_\flat\mathfrak{M}\mathfrak{S}$ and $(\xi_\mathfrak{z})_{\mathfrak{z}\in\mathbb{N}}$ be a sequence in $\mathcal{L}^*_\kappa$.

(i) the sequence $(\xi_\mathfrak{z})_{\mathfrak{z}\in\mathbb{N}}$ is $\kappa-$convergent to $\xi \in \mathcal{L}^*_\kappa \Leftrightarrow \kappa_\hbar(\xi_\mathfrak{z}, \xi) \to 0$, as $\mathfrak{z} \to \infty$ for all $\hbar > 0$.

(ii) the sequence $(\xi_\mathfrak{z})_{\mathfrak{z}\in\mathbb{N}}$ in $\mathcal{L}^*_\kappa$ is a $\kappa-$Cauchy sequence provided that $\lim\limits_{\mathfrak{z},m\to\infty} \kappa_\hbar(\xi_\mathfrak{z}, \xi_m) = 0$ for all $\hbar > 0$.

(iii) if any $\kappa-$Cauchy sequence in $\mathcal{L}^*_\kappa$ $\kappa-$convergence to a point of $\mathcal{L}^*_\kappa$, then $\mathcal{L}^*_\kappa$ is termed as a $\kappa-$complete space.

Let $\vartheta : [0, \infty) \to [0, \infty)$ be a function that fulfills the subsequent features, and the set Ψ be the family of all such functions.

- ϑ is continuous and strictly increasing,
- $\vartheta(\mathfrak{a}) = 0 \ \Leftrightarrow \ \mathfrak{a} = 0$.

We will present the concept of simulation functions, which we denote as $\mathcal{S}\mathfrak{f}$.

Definition 1.8. Let $\Omega : [0, \infty) \times [0, \infty) \to \mathbb{R}$ be a mapping fulfilling the succeeding circumstances.

(Ω_1) $\Omega(0, 0) = 0$,

(Ω_2) $\Omega(\mathfrak{a}, \mathfrak{t}) < \mathfrak{t} - \mathfrak{a}$ for all $\mathfrak{a}, \mathfrak{t} > 0$,

$(\Omega_2{}')$ $\Omega(\mathfrak{a}, \mathfrak{t}) < \vartheta(\mathfrak{t}) - \vartheta(\mathfrak{a})$ for all $\mathfrak{a}, \mathfrak{t} > 0$ and for some $\vartheta \in \Psi$,

(Ω_2'') $\Omega(\mathfrak{a},\mathfrak{t}) < \vartheta(\mathfrak{t}) - \vartheta\left(\mathfrak{c}^{\lambda}\mathfrak{a}\right)$ for all $\mathfrak{a},\mathfrak{t} > 0$, for some $\vartheta \in \Psi$ and for a coefficient $\lambda \geq 1$.

(Ω_3) if $\{\mathfrak{a}_{\mathfrak{z}}\}$, $\{\mathfrak{t}_{\mathfrak{z}}\}$ are sequences in the interval $(0,\infty)$ such that $\lim_{\mathfrak{z}\to\infty} \mathfrak{a}_{\mathfrak{z}} = \lim_{\mathfrak{z}\to\infty} \mathfrak{t}_{\mathfrak{z}} > 0$, then

$$\limsup_{\mathfrak{z}\to\infty} \Omega(\mathfrak{a}_{\mathfrak{z}},\mathfrak{t}_{\mathfrak{z}}) < 0. \tag{1}$$

(Ω_3') if $\{\mathfrak{a}_{\mathfrak{z}}\}$, $\{\mathfrak{t}_{\mathfrak{z}}\}$ are sequences belong to $(0,\infty)$ such that $\lim_{\mathfrak{z}\to\infty} \mathfrak{a}_{\mathfrak{z}} = \lim_{\mathfrak{z}\to\infty} \mathfrak{t}_{\mathfrak{z}} > 0$ and $\mathfrak{a}_{\mathfrak{z}} \leq \mathfrak{t}_{\mathfrak{z}}$; thereby, the statement (1) is fulfilled.

Taking into account the function Ω_i, we say that

- $i = 1,2,3 \Rightarrow \mathcal{S}\mathfrak{f}$ with respect to Khojasteh et al., [29],
- $i = 2,3 \Rightarrow \mathcal{S}\mathfrak{f}$ w.r.t. Argoubi et al., [7],
- $i = 1,2,3' \Rightarrow \mathcal{S}\mathfrak{f}$ w.r.t. Roldan Lopez de Hierro et al., [33],
- $i = 2',3 \Rightarrow \Psi{-}\mathcal{S}\mathfrak{f}$ w.r.t. Joonaghany et al., [21],
- $i = 2'',3' \Rightarrow \Psi_{\mathfrak{c}}{-}\mathcal{S}\mathfrak{f}$ w.r.t. of Zoto et al., [35].

We furnish some instances of $\mathcal{S}\mathfrak{f}$.

Example 1.9. Let $\Omega_i : [0,\infty)\times[0,\infty) \to \mathbb{R}$, $\quad i = 1,2,3,4,5$ be functions.

(*i*) $\Omega_1(\mathfrak{a},\mathfrak{t}) = \vartheta(\mathfrak{t}) - \phi(\mathfrak{a})$ for all $\mathfrak{a},\mathfrak{t} \in [0,\infty)$, where $\vartheta,\phi : [0,\infty) \to [0,\infty)$ are two continuous functions such that $\vartheta(\mathfrak{a}) = \phi(\mathfrak{a}) = 0$ if and only if $\mathfrak{a} = 0$ and $\vartheta(\mathfrak{a}) < \mathfrak{a} \leq \phi(\mathfrak{a})$ for all $\mathfrak{a} > 0$.

(*ii*) $\Omega_2(\mathfrak{a},\mathfrak{t}) = \mathfrak{t} - \frac{\alpha(\mathfrak{a},\mathfrak{t})}{\beta(\mathfrak{a},\mathfrak{t})}$ for all $\mathfrak{a},\mathfrak{t} \in [0,\infty)$, where $\alpha,\beta : [0,\infty)^2 \to (0,\infty)$ are two continuous functions with respect to each variable such that $\alpha(\mathfrak{a},\mathfrak{t}) > \beta(\mathfrak{a},\mathfrak{t})$ for all $\mathfrak{a},\mathfrak{t} > 0$.

(*iii*) $\Omega_3(\mathfrak{a},\mathfrak{t}) = \phi(\mathfrak{t})\vartheta(\mathfrak{t}) - \vartheta(\mathfrak{a})$, where $\phi : [0,\infty) \to [0,\infty)$ is a function such that $\limsup_{\mathfrak{a}\to\mathfrak{t}} \phi(\mathfrak{a}) < 1$ for each $\mathfrak{t} > 0$.

(*iv*) $\Omega_4(\mathfrak{a},\mathfrak{t}) = \vartheta(\mathfrak{t}) - \phi(\mathfrak{t}) - \vartheta(\mathfrak{a})$, where $\phi : [0,\infty) \to [0,\infty)$ is a function such that $\liminf_{\mathfrak{a}\to\mathfrak{t}} \phi(\mathfrak{a}) > 0$ for each $\mathfrak{t} > 0$.

(v) $\Omega_5(\mathfrak{a},\mathfrak{t}) = \phi(\mathfrak{t}) - \vartheta(\mathfrak{c}^{\lambda}\mathfrak{a})$ for all $\mathfrak{a},\mathfrak{t} \in (0,\infty)$, where $\phi,\vartheta : [0,\infty) \to [0,\infty)$ are continuous functions and ϑ is an increasing function such that $\phi(\mathfrak{t}) < \vartheta(\mathfrak{t})$ for each $\mathfrak{t} > 0$ and a coefficient $\lambda \geq 1$.

Then Ω_i for $i = 1,2$ are $\mathcal{S}\mathfrak{f}$ given in [29]. Also, for $i = 3,4$, Ω_i are $\mathcal{S}\mathfrak{f}$ given in [21]. Lastly, Ω_5 is an $\mathcal{S}\mathfrak{f}$ presented in [35].

In the sequel, the family Ξ represents the set of all $\mathcal{S}\mathfrak{f}$ in the sense of Khojasteh et al. [29]. Moreover, by considering the axiom (Ω_2), we have $\Omega(\mathfrak{a},\mathfrak{a}) < 0$ for all $\mathfrak{a} > 0$.

Definition 1.10. [29] Let $\mathcal{J} : \mathcal{L} \to \mathcal{L}$ be a map on a metric space $(\mathcal{L}, m)$ and $\Omega \in \Xi$. If

$$\Omega(m(\mathcal{J}\xi, \mathcal{J}\ell), m(\xi,\ell)) \geq 0 \quad \text{for all } \xi,\ell \in \mathcal{L},$$

is fulfilled, then $\mathcal{J}$ is termed as a Ξ-contraction with respect to Ω.

Further, contemplating $\Omega \in \Xi$ as $\Omega(\mathfrak{a},\mathfrak{t}) = q\mathfrak{t} - \mathfrak{a}$ for all $\mathfrak{a},\mathfrak{t} \in [0,\infty)$, we achieve the Banach contraction.

Remark 1.11. Let $\mathcal{J}$ be a Ξ-contraction. Then, we have $\Omega(\mathfrak{a},\mathfrak{t}) < 0$ for all $\mathfrak{a} \geq \mathfrak{t} > 0$ w.r.t. $\Omega \in \Xi$ and accordingly, $m(\mathcal{J}\xi, \mathcal{J}\ell) < m(\xi,\ell)$. We acquire that every Ξ-contraction is a contractive mapping and, eventually, continuous.

In 2018, Cho [16] revised the specification of simulation functions and termed the $\mathcal{Z}$ simulation function (briefly $\mathcal{Z} - \mathcal{S}\mathfrak{f}$), as indicated below.

Definition 1.12. [16] Let $\zeta : [1,\infty) \times [1,\infty) \to \mathbb{R}$ be a mapping fulfilling the below statements.

(ζ_1) $\zeta(1,1) = 1$;

(ζ_2) $\zeta(\mathfrak{a},\mathfrak{t}) < \mathfrak{t}/\mathfrak{a}, \quad \forall\, \mathfrak{a},\mathfrak{t} > 1$;

(ζ_3) for all sequences $\{\mathfrak{a}_{\mathfrak{z}}\}, \{\mathfrak{t}_{\mathfrak{z}}\} \subset (1,\infty)$ with $\mathfrak{a}_{\mathfrak{z}} \leq \mathfrak{t}_{\mathfrak{z}}, \quad \forall\, \mathfrak{z} = 1,2,3,...$

$$\lim_{\mathfrak{z}\to\infty} \mathfrak{a}_{\mathfrak{z}} = \lim_{\mathfrak{z}\to\infty} \mathfrak{t}_{\mathfrak{z}} > 1 \quad \Rightarrow \quad \limsup_{\mathfrak{z}\to\infty} \zeta(\mathfrak{a}_{\mathfrak{z}},\mathfrak{t}_{\mathfrak{z}}) < 1.$$

Thereby, ζ is entitled as $\mathcal{Z}-\mathcal{S}\mathfrak{f}$, and the set $\mathcal{Z}$ denotes the family of all such mappings.

Also, note that $\zeta(\mathfrak{a},\mathfrak{a})<1\ ,\quad \forall\ \mathfrak{a}>1$.

Example 1.13. [16] The functions $\zeta_1,\zeta_2,\zeta_3:[1,\infty)\times[1,\infty)\to\mathbb{R}$ that identified below, are belong to $\mathcal{Z}$.

(1) $\zeta_1(\mathfrak{a},\mathfrak{t})=\mathfrak{t}^{\lambda}/\mathfrak{a}\ ,\quad \forall\ \mathfrak{a},\mathfrak{t}\geq 1$, where $\lambda\in(0,1)$;

(2) $\zeta_2(\mathfrak{a},\mathfrak{t})=\mathfrak{t}/\mathfrak{a}\phi(\mathfrak{t})\ ,\quad \forall\ \mathfrak{a},\mathfrak{t}\geq 1$, where ϕ is a non-decreasing and lower semi-continuous self-mapping on $[1,\infty)$ such that $\phi^{-1}(\{1\})=1$;

(3)
$$\zeta_3(\mathfrak{a},\mathfrak{t})=\begin{cases}1, & \text{if } (\mathfrak{a},\mathfrak{t})=(1,1),\\ \frac{\mathfrak{t}}{2\mathfrak{a}}, & \text{if } \mathfrak{t}<\mathfrak{a},\\ \frac{\mathfrak{t}^{\lambda}}{\mathfrak{a}}, & \text{otherwise,}\end{cases}$$
$\forall\ \mathfrak{a},\mathfrak{t}\geq 1$, where $\lambda\in(0,1)$.

In [26], Jleli and Samet introduced the class $\mathrm{T}=\{\gamma:(0,\infty)\to(1,\infty)\}$, in which the functions in this class satisfy the following features:

(γ_1) γ is non-decreasing;

(γ_2) for each sequence $\{\mathfrak{a}_{\mathfrak{z}}\}\subset(0,\infty)$, $\lim\limits_{\mathfrak{z}\to\infty}\gamma(\mathfrak{a}_{\mathfrak{z}})=1\Leftrightarrow\lim\limits_{\mathfrak{z}\to\infty}\mathfrak{a}_{\mathfrak{z}}=0^+$;

(γ_3) there exist $r\in(0,1)$ and $\mathfrak{s}\in(0,\infty]$ such that $\lim\limits_{\mathfrak{a}\to 0^+}\frac{\gamma(\mathfrak{a})}{\mathfrak{a}^r}=\mathfrak{s}$.

Additionally, they proved the following theorem in the framework of generalized metric spaces. Some researchers call this space Branciari metric space, while others call it rectangular metric space.

Theorem 1.14. *Let $\mathcal{J}:\mathcal{L}\to\mathcal{L}$ be a given map on a complete generalized metric space $(\mathcal{L},m)$. Presume that a function $\gamma\in\mathrm{T}$ and a constant $k\in(0,1)$ exist such that*
$$m(\mathcal{J}\xi,\mathcal{J}\ell)\neq 0\quad\Rightarrow\quad\gamma(m(\mathcal{J}\xi,\mathcal{J}\ell))\leq[\gamma(m(\xi,\ell))]^k$$
for all $\xi,\ell\in\mathcal{L}$. Thereupon, the set $Fix(\mathcal{J})$ has a unique element.

Next, Liu [30] realized that the condition (γ_3) could be relieved to $(\gamma_3{}')$ which is identified as follows:

$(\gamma_3{}')$ γ is continuous.

Let $\Theta = \{\gamma : (0, \infty) \to (1, \infty) :\ \gamma \text{ holds } (\gamma_1), (\gamma_2) \text{ and } (\gamma_3{}')\}$.

Example 1.15. For all $\chi > 0$; the following functions

- $\gamma_1(\chi) = e^{\chi}$,
- $\gamma_2(\chi) = e^{\sqrt{\chi}}$,
- $\gamma_3(\chi) = e^{\sqrt{\chi e^{\chi}}}$,
- $\gamma_4(\chi) = \cosh \chi$,
- $\gamma_5(\chi) = 1 + \ln(1 + \chi)$,
- $\gamma_6(\chi) = e^{\chi e^{\chi}}$,

belong to the class Θ.

2 Main Results

As can be understood from the definition, it is clear that the metric modular does not have to be finite. Considering this fact, it is needful to converse about the ensuing supplementary situations to ensure the existence and uniqueness of fixed points of contraction mappings on $\mathfrak{M}\mathfrak{M}\mathfrak{S}$s and $\mathfrak{M}_\flat\mathfrak{M}\mathfrak{S}$s.

$(\mathcal{S}_1)$ $\kappa_\hbar(\xi, \mathcal{J}\xi) < \infty$ for all $\hbar > 0$ and $\xi \in \mathcal{L}^*_\kappa$,

$(\mathcal{S}_2)$ $\kappa_\hbar(\xi, \ell) < \infty$ for all $\hbar > 0$ and $\xi, \ell \in \mathcal{L}^*_\kappa$.

We serve up some auxiliary functions to be employed in the following discussion.

The symbol $\Delta_{\mathbf{G}}$ is used to indicate the set of all functions $\mathbf{G} : [0, \infty)^4 \to [0, \infty)$ that own the continuity such that

$(\mathbf{G}_1)$ $\mathbf{G}$ is non-decreasing with respect to each variable;

($\mathbf{G}_2$) $\mathbf{G}(\mathfrak{a},\mathfrak{a},\mathfrak{a},\mathfrak{a}) \leq \mathfrak{a}$ for all $\mathfrak{a} \in [0,\infty)$.

Example 2.1. The followings are some instances of the function $\mathbf{G}$, which are included in the set $\Delta_{\mathbf{G}}$.

($\mathbf{G}_1$) $\mathbf{G}(\mathfrak{a}_1,\mathfrak{a}_2,\mathfrak{a}_3,\mathfrak{a}_4) = \max\{\mathfrak{a}_1,\mathfrak{a}_2,\mathfrak{a}_3,\mathfrak{a}_4\}$;

($\mathbf{G}_2$) $\mathbf{G}(\mathfrak{a}_1,\mathfrak{a}_2,\mathfrak{a}_3,\mathfrak{a}_4) = \max\{\mathfrak{a}_1+\mathfrak{a}_2,\mathfrak{a}_2+\mathfrak{a}_3,\mathfrak{a}_1+\mathfrak{a}_3,\mathfrak{a}_3+\mathfrak{a}_4\}$;

($\mathbf{G}_3$) $\mathbf{G}(\mathfrak{a}_1,\mathfrak{a}_2,\mathfrak{a}_3,\mathfrak{a}_4) = [\max\{\mathfrak{a}_1\mathfrak{a}_2,\mathfrak{a}_2\mathfrak{a}_3,\mathfrak{a}_1\mathfrak{a}_3,\mathfrak{a}_3\mathfrak{a}_4\}]^{\frac{1}{2}}$;

($\mathbf{G}_4$) $\mathbf{G}(\mathfrak{a}_1,\mathfrak{a}_2,\mathfrak{a}_3,\mathfrak{a}_4) = [\max\{\mathfrak{a}_1{}^q,\mathfrak{a}_2{}^q,\mathfrak{a}_3{}^q,\mathfrak{a}_4{}^q\}]^{\frac{1}{q}},\ q>0$;

($\mathbf{G}_5$) $\mathbf{G}(\mathfrak{a}_1,\mathfrak{a}_2,\mathfrak{a}_3,\mathfrak{a}_4) = \mathfrak{a}_1$;

($\mathbf{G}_6$) $\mathbf{G}(\mathfrak{a}_1,\mathfrak{a}_2,\mathfrak{a}_3,\mathfrak{a}_4) = \frac{\mathfrak{a}_2+\mathfrak{a}_3}{2}$;

($\mathbf{G}_7$) $\mathbf{G}(\mathfrak{a}_1,\mathfrak{a}_2,\mathfrak{a}_3,\mathfrak{a}_4) = \mathfrak{a}_1+\mathfrak{a}_2+\mathfrak{a}_3+\mathfrak{a}_4$;

($\mathbf{G}_8$) $\mathbf{G}(\mathfrak{a}_1,\mathfrak{a}_2,\mathfrak{a}_3,\mathfrak{a}_4) = p_1\mathfrak{a}_1+p_2\mathfrak{a}_2+p_3\mathfrak{a}_3+p_4\mathfrak{a}_4$, with $0<p_1+p_2+p_3+p_4<1$.

Let Ψ^* be denoted as the set of all φ self-mappings on $[1,+\infty)$ such that $\varphi(\mathfrak{a}) = 1 \Leftrightarrow \mathfrak{a} = 1$, which possesses strictly increasing and continuity properties.

We present a new class of functions that serve the family of $\mathcal{S}\mathfrak{f}$.

Definition 2.2. Let $\hat{Z}$ be the class of all mappings $\eta : [1,\infty)^2 \to \mathbb{R}$. Let a function $\varphi \in \Psi^*$ and a coefficient $\lambda \geq 1$ exist such that

(η_1) $\eta(1,1) = 1$;

(η_2) $\eta(\mathfrak{a},\mathfrak{t}) < \frac{\varphi(\mathfrak{t})}{\varphi(\mathfrak{a})}$, $\quad \forall\, \mathfrak{a},\mathfrak{t} > 1$;

($\eta_2{}'$) $\eta(\mathfrak{a},\mathfrak{t}) < \frac{\varphi(\mathfrak{t})}{\varphi(\mathfrak{c}^\lambda\mathfrak{a})}$, $\quad \forall\, \mathfrak{a},\mathfrak{t} > 1$;

(η_3) for all sequences $\{\mathfrak{a}_\mathfrak{z}\},\{\mathfrak{t}_\mathfrak{z}\} \subset (1,\infty)$ with $\mathfrak{a}_\mathfrak{z} \leq \mathfrak{t}_\mathfrak{z}$, $\quad \forall\, \mathfrak{z} = 1,2,3,...$

$$\lim_{\mathfrak{z}\to\infty}\mathfrak{a}_\mathfrak{z} = \lim_{\mathfrak{z}\to\infty}\mathfrak{t}_\mathfrak{z} > 1 \quad \Rightarrow \quad \limsup_{\mathfrak{z}\to\infty}\eta(\mathfrak{a}_\mathfrak{z},\mathfrak{t}_\mathfrak{z}) < 1.$$

If the function η satisfies (η_2)-(η_3), η is termed generalized $\Psi^* - \mathcal{S}\mathfrak{f}$.

Also, if η provides only the conditions (η_2')-(η_3), then η is called a generalized $\Psi^*\mathfrak{c} - \mathcal{S}\mathfrak{f}$.

If we choose $\varphi(\mathfrak{a}) = \mathfrak{a}$ for all a $t \geq 1$ and the features (η_1)-(η_2)-(η_3) are satisfied, then η is a generalized $\mathcal{Z}-$simulation function in the sense of Cho [16].

Example 2.3. Let $\eta_a, \eta_b, \eta_c, \eta_d : [1, \infty)^2 \to \mathbb{R}$ be functions defined as indicated below:

(e_1) $\eta_a(\mathfrak{a}, \mathfrak{t}) = \frac{\alpha\varphi(\mathfrak{t})}{\varphi(\mathfrak{c}\mathfrak{a})}$, $\quad \forall\ \mathfrak{a}, \mathfrak{t} \geq 1; \alpha \in (0, 1)$;

(e_2) $\eta_b(\mathfrak{a}, \mathfrak{t}) = \frac{\varphi(\mathfrak{t})}{\varphi\left(\mathfrak{c}^{\lambda}\mathfrak{a}\right)\phi(\mathfrak{t})}$, $\quad \forall\ \mathfrak{a}, \mathfrak{t} \geq 1$ and a coefficient $\lambda \geq 1$, where ϕ is a non-decreasing and lower semi-continuous self-mapping on $[1, \infty)$ such that $\phi^{-1}(\{1\}) = 1$;

(e_3) $\eta_c(\mathfrak{a}, \mathfrak{t}) = \frac{\phi(\mathfrak{t})}{\varphi\left(\mathfrak{c}^{\lambda}\mathfrak{a}\right)}$, $\quad \forall\ \mathfrak{a}, \mathfrak{t} \geq 1$ and a coefficient $\lambda \geq 1$, where $\phi : [1, \infty) \to [1, \infty)$ is a continuous function such that $\phi(\mathfrak{t}) < \varphi(\mathfrak{t})$ for all $\mathfrak{t} > 0$.

(e_4)

$$\eta_d(\mathfrak{a}, \mathfrak{t}) = \begin{cases} 1, & \text{if } (\mathfrak{a}, \mathfrak{t}) = (1, 1), \\ \frac{\varphi(\mathfrak{t})}{k\varphi(\mathfrak{a})}, & \text{if } \mathfrak{t} < \mathfrak{a}, \\ \frac{[\varphi(\mathfrak{t})]^p}{\varphi(\mathfrak{a})}, & \text{otherwise}, \end{cases}$$

$\forall\ \mathfrak{a}, \mathfrak{t} \geq 1$, where $k \geq 1$ and $p \in (0, 1)$.

Then, $\eta_a, \eta_b, \eta_c, \eta_d$ are generalized $\Psi^*\mathfrak{c} - \mathcal{S}\mathfrak{f}$. For $\mathfrak{c} = 1$, the above examples are generalized $\Psi^* - \mathcal{S}\mathfrak{f}$.

We now prepare to submit our principal theorem in this section. The definition required for this is as follows.

Definition 2.4. Let κ be a modular $b-$metric on a set $\mathcal{L}$ and $\mathcal{J}, \mathcal{K} : \mathcal{L}^*_\kappa \to \mathcal{L}^*_\kappa$ be two self-mappings. $\mathcal{J}$ and $\mathcal{K}$ are called generalized almost simulative $\hat{Z}^{\Theta}_{\Psi^*}$ −contraction mappings if a generalized $\Psi^*\mathfrak{c} - \mathcal{S}\mathfrak{f}$ and a constant $\rho \geq 0$ as well as, a $\gamma \in \Theta$ and $\mathbf{G} \in \Delta_{\mathbf{G}}$ exist such that

$$\frac{1}{2\mathfrak{c}} \min\left\{\kappa_\hbar(\xi, \mathcal{J}\xi), \kappa_\hbar(\ell, \mathcal{K}\ell)\right\} \leq \kappa_\hbar(\xi, \ell)$$

implies

$$\eta\left(\gamma\left(\mathfrak{c}^4\kappa_\hbar\left(\mathcal{J}\xi,\mathcal{K}\ell\right)\right),\left[\gamma\left(\mathbf{G}\left(\xi,\ell\right)+\rho\mathcal{N}\left(\xi,\ell\right)\right)\right]^k\right)\geq 1, \tag{2}$$

where

$$\mathbf{G}\left(\xi,\ell\right)=\left(\kappa_\hbar\left(\xi,\ell\right),\kappa_\hbar\left(\xi,\mathcal{J}\xi\right),\kappa_\hbar\left(\ell,\mathcal{K}\ell\right),\frac{\kappa_{2\hbar}\left(\xi,\mathcal{K}\ell\right)+\kappa_{2\hbar}\left(\ell,\mathcal{J}\xi\right)}{2\mathfrak{c}}\right)$$

and

$$\mathcal{N}\left(\xi,\ell\right)=\min\left\{\kappa_\hbar\left(\xi,\mathcal{J}\xi\right),\kappa_\hbar\left(\ell,\mathcal{K}\ell\right),\kappa_\hbar\left(\xi,\mathcal{K}\ell\right),\kappa_\hbar\left(\ell,\mathcal{J}\xi\right)\right\},$$

for all distinct $\xi,\ell\in\mathcal{L}_\kappa^*$, $k\in(0,1)$ and for all $\hbar>0$.

Theorem 2.5. *Let $\mathcal{L}_\kappa^*$ be a $\kappa-$complete $\mathfrak{M}_\flat\mathfrak{M}\mathfrak{S}$ with constant $\mathfrak{c}\geq 1$ and let $\mathcal{J}$ and $\mathcal{K}$ be generalized almost simulative $\hat{Z}_{\Psi^*}^{\Theta}-$contraction mappings. If the condition $(\mathcal{S}_1)$ is fulfilled, then $u\in\mathcal{L}_\kappa^*$ exists such that $u\in C_{Fix}\left(\mathcal{J},\mathcal{K}\right)$. If, in addition, the condition $(\mathcal{S}_2)$ is fulfilled, then $C_{Fix}\left(\mathcal{J},\mathcal{K}\right)=\{u\}$.*

Proof. Let $\xi_0\in\mathcal{L}_\kappa^*$ be an initial element, and we can construct a sequence $\{\xi_\mathfrak{z}\}$ by:

$$\xi_{2\mathfrak{z}+1}=\mathcal{J}\xi_{2\mathfrak{z}} \text{ and } \xi_{2\mathfrak{z}+2}=\mathcal{K}\xi_{2\mathfrak{z}+1},\quad \text{for all } \mathfrak{z}\in\mathbb{N}.$$

If there is some $\mathfrak{z}_0\in\mathbb{N}$ such that $\xi_{\mathfrak{z}_0}=\xi_{\mathfrak{z}_0+1}$, then $\mathfrak{z}_0$ becomes a common fixed point of $\mathcal{J}$ and $\mathcal{K}$. Herewith, we presume that $\xi_k\neq\xi_{k+1}$ for all $k\in\mathbb{N}$, and we have $\kappa_\hbar\left(\xi_k,\xi_{k+1}\right)>0$ for all $\hbar>0$.

Now, we will divide the proof into four steps to make sense more straightforward.

<u>**Step (1):**</u> We claim that $\lim_{k\to\infty}\kappa_\hbar\left(\xi_k,\xi_{k+1}\right)=0$ for all $\hbar>0$.

Thus, at first, we must show that

$$\kappa_\hbar\left(\xi_{k+1},\xi_{k+2}\right)<\kappa_\hbar\left(\xi_k,\xi_{k+1}\right),\quad \text{for all } k\in\mathbb{N}. \tag{3}$$

We presume that $k=2\mathfrak{z}$ for some $\mathfrak{z}\in\mathbb{N}$. So, we obtain

$$\begin{aligned}&\tfrac{1}{2\mathfrak{c}}\min\left\{\kappa_\hbar\left(\xi_{2\mathfrak{z}},\mathcal{J}\xi_{2\mathfrak{z}}\right),\kappa_\hbar\left(\xi_{2\mathfrak{z}+1},\mathcal{K}\xi_{2\mathfrak{z}+1}\right)\right\}\\ &\qquad=\tfrac{1}{2\mathfrak{c}}\min\left\{\kappa_\hbar\left(\xi_{2\mathfrak{z}},\xi_{2\mathfrak{z}+1}\right),\kappa_\hbar\left(\xi_{2\mathfrak{z}+1},\xi_{2\mathfrak{z}+2}\right)\right\}\\ &\qquad\leq\kappa_\hbar\left(\xi_{2\mathfrak{z}},\xi_{2\mathfrak{z}+1}\right).\end{aligned}$$

By using (2) and ($\eta_2{}'$), we get

$$1 \leq \eta\left(\gamma\left(\mathfrak{c}^4\kappa_\hbar\left(\mathcal{J}\xi_{2\mathfrak{z}}, \mathcal{K}\xi_{2\mathfrak{z}+1}\right)\right), \left(\gamma\left[\mathbf{G}\left(\xi_{2\mathfrak{z}}, \xi_{2\mathfrak{z}+1}\right) + \rho\mathcal{N}\left(\xi_{2\mathfrak{z}}, \xi_{2\mathfrak{z}+1}\right)\right]\right)^k\right)$$

$$< \frac{\varphi\left(\left(\gamma[\mathbf{G}(\xi_{2\mathfrak{z}}, \xi_{2\mathfrak{z}+1}) + \rho\mathcal{N}(\xi_{2\mathfrak{z}}, \xi_{2\mathfrak{z}+1})]\right)^k\right)}{\varphi\left(\mathfrak{c}^\lambda\gamma\left(\mathfrak{c}^4\kappa_\hbar(\xi_{2\mathfrak{z}+1}, \xi_{2\mathfrak{z}+2})\right)\right)},$$

that is,

$$\varphi\left(\mathfrak{c}^\lambda\gamma\left(\mathfrak{c}^4\kappa_\hbar\left(\xi_{2\mathfrak{z}+1}, \xi_{2\mathfrak{z}+2}\right)\right)\right) < \varphi\left(\left(\gamma\left[\mathbf{G}\left(\xi_{2\mathfrak{z}}, \xi_{2\mathfrak{z}+1}\right) + \rho\mathcal{N}\left(\xi_{2\mathfrak{z}}, \xi_{2\mathfrak{z}+1}\right)\right]\right)^k\right).$$

Due to features of the function φ and contemplating $k \in (0, 1)$, the above inequality yields

$$\begin{aligned}\mathfrak{c}^\lambda\gamma\left(\mathfrak{c}^4\kappa_\hbar\left(\xi_{2\mathfrak{z}+1}, \xi_{2\mathfrak{z}+2}\right)\right) &< \left(\gamma\left[\mathbf{G}\left(\xi_{2\mathfrak{z}}, \xi_{2\mathfrak{z}+1}\right) + \rho\mathcal{N}\left(\xi_{2\mathfrak{z}}, \xi_{2\mathfrak{z}+1}\right)\right]\right)^k \\ &< \gamma\left[\mathbf{G}\left(\xi_{2\mathfrak{z}}, \xi_{2\mathfrak{z}+1}\right) + \rho\mathcal{N}\left(\xi_{2\mathfrak{z}}, \xi_{2\mathfrak{z}+1}\right)\right].\end{aligned}$$

As $\gamma \in \Theta$, we derive that

$$\mathfrak{c}^{4+\lambda}\kappa_\hbar\left(\xi_{2\mathfrak{z}+1}, \xi_{2\mathfrak{z}+2}\right) < \mathbf{G}\left(\xi_{2\mathfrak{z}}, \xi_{2\mathfrak{z}+1}\right) + \rho\mathcal{N}\left(\xi_{2\mathfrak{z}}, \xi_{2\mathfrak{z}+1}\right), \tag{4}$$

where

$$\begin{aligned}\mathbf{G}\left(\xi_{2\mathfrak{z}}, \xi_{2\mathfrak{z}+1}\right) &= \begin{pmatrix} \kappa_\hbar\left(\xi_{2\mathfrak{z}}, \xi_{2\mathfrak{z}+1}\right), \kappa_\hbar\left(\xi_{2\mathfrak{z}}, \mathcal{J}\xi_{2\mathfrak{z}}\right), \kappa_\hbar\left(\xi_{2\mathfrak{z}+1}, \mathcal{K}\xi_{2\mathfrak{z}+1}\right), \\ \frac{\kappa_{2\hbar}(\xi_{2\mathfrak{z}}, \mathcal{K}\xi_{2\mathfrak{z}+1}) + \kappa_{2\hbar}(\xi_{2\mathfrak{z}+1}, \mathcal{J}\xi_{2\mathfrak{z}})}{2\mathfrak{c}} \end{pmatrix} \\ &= \begin{pmatrix} \kappa_\hbar\left(\xi_{2\mathfrak{z}}, \xi_{2\mathfrak{z}+1}\right), \kappa_\hbar\left(\xi_{2\mathfrak{z}}, \xi_{2\mathfrak{z}+1}\right), \kappa_\hbar\left(\xi_{2\mathfrak{z}+1}, \xi_{2\mathfrak{z}+2}\right), \\ \frac{\kappa_{2\hbar}(\xi_{2\mathfrak{z}}, \xi_{2\mathfrak{z}+2})}{2\mathfrak{c}} \end{pmatrix}.\end{aligned}$$

From (κ_3), note that $\kappa_{2\hbar}\left(\xi_{2\mathfrak{z}}, \xi_{2\mathfrak{z}+2}\right) \leq \mathfrak{c}\left[\kappa_\hbar\left(\xi_{2\mathfrak{z}}, \xi_{2\mathfrak{z}+1}\right) + \kappa_\hbar\left(\xi_{2\mathfrak{z}+1}, \xi_{2\mathfrak{z}+2}\right)\right]$. So, we gain

$$\mathbf{G}\left(\xi_{2\mathfrak{z}}, \xi_{2\mathfrak{z}+1}\right) = \begin{pmatrix} \kappa_\hbar\left(\xi_{2\mathfrak{z}}, \xi_{2\mathfrak{z}+1}\right), \kappa_\hbar\left(\xi_{2\mathfrak{z}}, \xi_{2\mathfrak{z}+1}\right), \kappa_\hbar\left(\xi_{2\mathfrak{z}+1}, \xi_{2\mathfrak{z}+2}\right), \\ \frac{\kappa_\hbar(\xi_{2\mathfrak{z}}, \xi_{2\mathfrak{z}+1}) + \kappa_\hbar(\xi_{2\mathfrak{z}+1}, \xi_{2\mathfrak{z}+2})}{2} \end{pmatrix}.$$

If we assume $\kappa_\hbar(\xi_{2\mathfrak{z}},\xi_{2\mathfrak{z}+1}) \leq \kappa_\hbar(\xi_{2\mathfrak{z}+1},\xi_{2\mathfrak{z}+2})$, we deduce that

$$\begin{aligned} \mathbf{G}(\xi_{2\mathfrak{z}},\xi_{2\mathfrak{z}+1}) &= \begin{pmatrix} \kappa_\hbar(\xi_{2\mathfrak{z}+1},\xi_{2\mathfrak{z}+2}), \kappa_\hbar(\xi_{2\mathfrak{z}+1},\xi_{2\mathfrak{z}+2}), \\ \kappa_\hbar(\xi_{2\mathfrak{z}+1},\xi_{2\mathfrak{z}+2}), \kappa_\hbar(\xi_{2\mathfrak{z}+1},\xi_{2\mathfrak{z}+2}) \end{pmatrix} \\ &\leq \kappa_\hbar(\xi_{2\mathfrak{z}+1},\xi_{2\mathfrak{z}+2}), \quad (\text{by } \mathbf{G} \in \Delta_{\mathbf{G}}). \end{aligned} \tag{5}$$

Nevertheless, we have

$$\begin{aligned} \mathcal{N}(\xi_{2\mathfrak{z}},\xi_{2\mathfrak{z}+1}) &= \min \left\{ \begin{array}{l} \kappa_\hbar(\xi_{2\mathfrak{z}},\mathcal{J}\xi_{2\mathfrak{z}}), \kappa_\hbar(\xi_{2\mathfrak{z}+1},\mathcal{K}\xi_{2\mathfrak{z}+1}), \\ \kappa_\hbar(\xi_{2\mathfrak{z}},\mathcal{K}\xi_{2\mathfrak{z}+1}), \kappa_\hbar(\xi_{2\mathfrak{z}+1},\mathcal{J}\xi_{2\mathfrak{z}}) \end{array} \right\} \\ &= \min \left\{ \begin{array}{l} \kappa_\hbar(\xi_{2\mathfrak{z}},\xi_{2\mathfrak{z}+1}), \kappa_\hbar(\xi_{2\mathfrak{z}+1},\xi_{2\mathfrak{z}+2}), \\ \kappa_\hbar(\xi_{2\mathfrak{z}},\xi_{2\mathfrak{z}+2}), 0 \end{array} \right\} = 0. \end{aligned} \tag{6}$$

Consequently, by using (5) and (6), inequality (4) becomes

$$\mathfrak{c}^{4+\lambda}\kappa_\hbar(\xi_{2\mathfrak{z}+1},\xi_{2\mathfrak{z}+2}) < \kappa_\hbar(\xi_{2\mathfrak{z}+1},\xi_{2\mathfrak{z}+2}).$$

This is a contradiction since $\mathfrak{c},\lambda \geq 1$. Then, our assumption is false, i.e., the expression (3) is verified while k is an even number. Similarly, it can be shown that (3) is held while k is an odd number. So, the sequence $\{\kappa_\hbar(\xi_{\mathfrak{z}},\xi_{\mathfrak{z}+1})\}_{\mathfrak{z}\geq 1}$ is non-decreasing and bounded below with 0. Hence there is a real number $\mathfrak{a} \geq 0$ such that $\lim_{\mathfrak{z}\to\infty} \kappa_\hbar(\xi_{\mathfrak{z}},\xi_{\mathfrak{z}+1}) = \mathfrak{a}$ for all $\hbar > 0$.

Now we confirm that $\mathfrak{a} = 0$. On the contrary, we claim $\mathfrak{a} > 0$. To see this, it is enough to mention the below two cases:

Case (1): Assume that $\mathfrak{c} > 1$.

As the expression (3) holds, we conclude that

$$\begin{aligned} &\mathbf{G}(\xi_{2\mathfrak{z}},\xi_{2\mathfrak{z}+1}) \\ &\quad = (\kappa_\hbar(\xi_{2\mathfrak{z}},\xi_{2\mathfrak{z}+1}), \kappa_\hbar(\xi_{2\mathfrak{z}},\xi_{2\mathfrak{z}+1}), \kappa_\hbar(\xi_{2\mathfrak{z}},\xi_{2\mathfrak{z}+1}), \kappa_\hbar(\xi_{2\mathfrak{z}},\xi_{2\mathfrak{z}+1})) \\ &\quad \leq \kappa_\hbar(\xi_{2\mathfrak{z}},\xi_{2\mathfrak{z}+1}), \quad (\text{by } \mathbf{G} \in \Delta_{\mathbf{G}}). \end{aligned}$$

So, by keeping in mind (4) and (6), we obtain

$$\mathfrak{c}^{4+\lambda}\kappa_\hbar\left(\xi_{2\mathfrak{z}+1},\xi_{2\mathfrak{z}+2}\right)<\kappa_\hbar\left(\xi_{2\mathfrak{z}},\xi_{2\mathfrak{z}+1}\right).$$

If the limit is taken for $\mathfrak{z}\to\infty$ in the above inequality, a contradiction in the form of $\mathfrak{c}^{4+\lambda}\mathfrak{a}<\mathfrak{a}$ is encountered.

Case (2): Presume that $\mathfrak{c}=1$. Using the expression (2), we achieve

$$1\leq\eta\left(\gamma\left(\kappa_\hbar\left(\mathcal{J}\xi_{2\mathfrak{z}},\mathcal{K}\xi_{2\mathfrak{z}+1}\right)\right),\left(\gamma\left[\mathbf{G}\left(\xi_{2\mathfrak{z}},\xi_{2\mathfrak{z}+1}\right)+\rho\mathcal{N}\left(\xi_{2\mathfrak{z}},\xi_{2\mathfrak{z}+1}\right)\right]\right)^k\right).$$

Similar to (5) and (6), we get

$$1\leq\eta\left(\gamma\left(\kappa_\hbar\left(\xi_{2\mathfrak{z}+1},\xi_{2\mathfrak{z}+2}\right)\right),\left(\gamma\left[\kappa_\hbar\left(\xi_{2\mathfrak{z}},\xi_{2\mathfrak{z}+1}\right)\right]\right)^k\right)\tag{7}$$

and so, from both $(\eta_2{}')$ and φ is strictly increasing, we have

$$\gamma\left(\kappa_\hbar\left(\xi_{\mathfrak{z}+1},\xi_{\mathfrak{z}+2}\right)\right)<\left[\gamma\left(\kappa_\hbar\left(\xi_{\mathfrak{z}},\xi_{\mathfrak{z}+1}\right)\right)\right]^k.\tag{8}$$

Let $\mathfrak{a}_\mathfrak{z}=\gamma\left(\kappa_\hbar\left(\xi_{\mathfrak{z}+1},\xi_{\mathfrak{z}+2}\right)\right)$ and $\mathfrak{t}_\mathfrak{z}=\left[\gamma\left(\kappa_\hbar\left(\xi_{\mathfrak{z}},\xi_{\mathfrak{z}+1}\right)\right)\right]^k$ for all $\mathfrak{z}\in\mathbb{N}$.

Now, since $\lim_{\mathfrak{z}\to\infty}\kappa_\hbar\left(\xi_{\mathfrak{z}},\xi_{\mathfrak{z}+1}\right)=\mathfrak{a}>0$, then it follows from (γ_2) that $\lim_{\mathfrak{z}\to\infty}\gamma\left(\kappa_\hbar\left(\xi_{\mathfrak{z}},\xi_{\mathfrak{z}+1}\right)\right)\neq1$ and so $\lim_{\mathfrak{z}\to\infty}\gamma\left(\kappa_\hbar\left(\xi_{\mathfrak{z}},\xi_{\mathfrak{z}+1}\right)\right)>1$.

Accordingly, from (8), we obtain

$$\begin{aligned}\mathfrak{a}_\mathfrak{z}=\gamma\left(\kappa_\hbar\left(\xi_{\mathfrak{z}+1},\xi_{\mathfrak{z}+2}\right)\right)&<\left[\gamma\left(\kappa_\hbar\left(\xi_{\mathfrak{z}},\xi_{\mathfrak{z}+1}\right)\right)\right]^k=\mathfrak{t}_\mathfrak{z}\\&<\left[\gamma\left(\kappa_\hbar\left(\xi_{\mathfrak{z}},\xi_{\mathfrak{z}+1}\right)\right)\right].\end{aligned}$$

Hence, considering as $\mathfrak{z}\to\infty$ in the above, we attain $\gamma(r)\leq\lim_{\mathfrak{z}\to\infty}\mathfrak{t}_\mathfrak{z}\leq\gamma(r)$. This implies that $\lim_{\mathfrak{z}\to\infty}\mathfrak{a}_\mathfrak{z}=\lim_{\mathfrak{z}\to\infty}\mathfrak{t}_\mathfrak{z}=\gamma(r)>1$. Hence, from (η_3), we deduce that $\limsup_{\mathfrak{z}\to\infty}\eta(\mathfrak{a}_\mathfrak{z},\mathfrak{t}_\mathfrak{z})<1$. Nevertheless, it is a contradiction due to (7).

Consequently, in both cases, we have a contradiction. For this reason we procure that $\mathfrak{a}=0$, that is, for all $\hbar>0$,

$$\lim_{\mathfrak{z}\to\infty}\kappa_\hbar\left(\xi_{\mathfrak{z}},\xi_{\mathfrak{z}+1}\right)=0.\tag{9}$$

Step (2): We assert that $\{\xi_{\mathfrak{z}}\}$ is a $\kappa-$Cauchy sequence. It is enough to verify $\{\xi_{2\mathfrak{z}}\}$ is a $\kappa-$Cauchy sequence. As opposed to our assertion, presume that $\{\xi_{2\mathfrak{z}}\}$ is not a $\kappa-$Cauchy sequence. An $\varepsilon > 0$ exists such that two sequences $\{\xi_{2\mathfrak{m}_q}\}$ and $\{\xi_{2\mathfrak{z}_q}\}$ of positive integers fulfilling $\mathfrak{z}_q > \mathfrak{m}_q > q$ can be formed. In this instance, $\mathfrak{z}_q$ exists as the smallest index for which

$$\kappa_\hbar\left(\xi_{2\mathfrak{m}_q}, \xi_{2\mathfrak{z}_q}\right) \geq \varepsilon \quad \text{and} \quad \kappa_\hbar\left(\xi_{2\mathfrak{m}_q}, \xi_{2\mathfrak{z}_q-2}\right) < \varepsilon, \quad \text{for all } \hbar > 0. \tag{10}$$

Now, without the loss of the generality from (10) and the modular inequality, we get

$$\varepsilon \leq \kappa_{2\hbar}\left(\xi_{2\mathfrak{m}_q}, \xi_{2\mathfrak{z}_q}\right) \leq \mathfrak{c}\kappa_\hbar\left(\xi_{2\mathfrak{m}_q}, \xi_{2\mathfrak{z}_q+1}\right) + \mathfrak{c}\kappa_\hbar\left(\xi_{2\mathfrak{z}_q+1}, \xi_{2\mathfrak{z}_q}\right).$$

Letting $q \to \infty$ and using (9) in the above, we obtain

$$\limsup_{q\to\infty} \kappa_\hbar\left(\xi_{2\mathfrak{z}_q+1}, \xi_{2\mathfrak{m}_q}\right) \geq \frac{\varepsilon}{\mathfrak{c}}. \tag{11}$$

Similarly, we have

$$\kappa_\hbar\left(\xi_{2\mathfrak{m}_q-1}, \xi_{2\mathfrak{z}_q}\right) \leq \mathfrak{c}\kappa_{\frac{\hbar}{2}}\left(\xi_{2\mathfrak{m}_q-1}, \xi_{2\mathfrak{m}_q}\right) + \mathfrak{c}^2\kappa_{\frac{\hbar}{4}}\left(\xi_{2\mathfrak{m}_q}, \xi_{2\mathfrak{z}_q-2}\right)$$
$$+\mathfrak{c}^3\kappa_{\frac{\hbar}{8}}\left(\xi_{2\mathfrak{z}_q-2}, \xi_{2\mathfrak{z}_q-1}\right) + \mathfrak{c}^3\kappa_{\frac{\hbar}{8}}\left(\xi_{2\mathfrak{z}_q-1}, \xi_{2\mathfrak{z}_q}\right).$$

If the limit superior for $q \to \infty$ in the above inequality is taken, then

$$\limsup_{q\to\infty} \kappa_\hbar\left(\xi_{2\mathfrak{m}_q-1}, \xi_{2\mathfrak{z}_q}\right) \leq \mathfrak{c}^2\varepsilon. \tag{12}$$

Moreover, like in the above, we achieve

$$\kappa_\hbar\left(\xi_{2_q}, \xi_{2_q}\right) \leq \mathfrak{c}\kappa_{\frac{\hbar}{2}}\left(\xi_{2_q}, \xi_{2_q-2}\right) + \mathfrak{c}^2\kappa_{\frac{\hbar}{4}}\left(\xi_{2_q-2}, \xi_{2_q-1}\right)$$
$$+\mathfrak{c}^2\kappa_{\frac{\hbar}{4}}\left(\xi_{2_q-1}, \xi_{2_q}\right).$$

So, by using (9), letting $q \to \infty$, we procure

$$\limsup_{q\to\infty} \kappa_\hbar\left(\xi_{2\mathfrak{m}_q-1}, \xi_{2\mathfrak{z}_q+1}\right) \leq \mathfrak{c}^2\varepsilon. \tag{13}$$

In conclusion, similar to the above calculations, it can be shown that

$$\limsup_{q\to\infty} \kappa_{\hbar}\left(\xi_{2\mathfrak{m}_q+1}, \xi_{2\mathfrak{z}_q+1}\right) \leq \mathfrak{c}^2\varepsilon. \tag{14}$$

Besides, we suggest that for sufficiently large $q \in \mathbb{N}$, if $\mathfrak{z}_q > \mathfrak{m}_q > q$, then

$$\frac{1}{2\mathfrak{c}} \min\left\{\kappa_{\hbar}\left(\xi_{2\mathfrak{z}_q}, \mathcal{J}\xi_{2\mathfrak{z}_q}\right), \kappa_{\hbar}\left(\xi_{2\mathfrak{m}_q-1}, \mathcal{K}\xi_{2\mathfrak{m}_q-1}\right)\right\} \leq \kappa_{\hbar}\left(\xi_{2\mathfrak{z}_q}, \xi_{2\mathfrak{m}_q-1}\right). \tag{15}$$

In fact, owing to $\mathfrak{z}_q > \mathfrak{m}_q$ and $\{\kappa_{\hbar}(\xi_{\mathfrak{z}}, \xi_{\mathfrak{z}+1})\}_{\mathfrak{z}\geq 1}$ is non-decreasing, we acquire

$$\begin{aligned}\kappa_{\hbar}\left(\xi_{2\mathfrak{z}_q}, \mathcal{J}\xi_{2\mathfrak{z}_q}\right) &= \kappa_{\hbar}\left(\xi_{2\mathfrak{z}_q}, \xi_{2\mathfrak{z}_q+1}\right) \leq \kappa_{\hbar}\left(\xi_{2\mathfrak{m}_q+1}, \xi_{2\mathfrak{m}_q}\right) \\ &\leq \kappa_{\hbar}\left(\xi_{2\mathfrak{m}_q}, \xi_{2\mathfrak{m}_q-1}\right) = \kappa_{\hbar}\left(\xi_{2\mathfrak{m}_q-1}, \mathcal{K}\xi_{2\mathfrak{m}_q-1}\right).\end{aligned}$$

Thereupon, the left-hand side of inequality (15) is

$$\frac{1}{2\mathfrak{c}}\kappa_{\hbar}\left(\xi_{2\mathfrak{z}_q}, \mathcal{J}\xi_{2\mathfrak{z}_q}\right) = \frac{1}{2\mathfrak{c}}\kappa_{\hbar}\left(\xi_{2\mathfrak{z}_q}, \xi_{2\mathfrak{z}_q+1}\right).$$

For sufficiently large $q \in \mathbb{N}$, it is necessary to indicate that if $\mathfrak{z}_q > \mathfrak{m}_q > q$, then

$$\kappa_{\hbar}\left(\xi_{2\mathfrak{z}_q}, \xi_{2\mathfrak{z}_q+1}\right) \leq \kappa_{\hbar}\left(\xi_{2\mathfrak{z}_q}, \xi_{2\mathfrak{m}_q-1}\right).$$

In accordance with (9), a natural number q_1 exists such that for any $q > q_1$,

$$\kappa_{\hbar}\left(\xi_{2\mathfrak{z}_q}, \xi_{2\mathfrak{z}_q+1}\right) < \frac{\varepsilon}{2\mathfrak{c}}.$$

There exists $q_2 \in \mathbb{N}$ such that for any $q > q_2$,

$$\kappa_{\hbar}\left(\xi_{2\mathfrak{m}_q-1}, \xi_{2\mathfrak{m}_q}\right) < \frac{\varepsilon}{2\mathfrak{c}}.$$

Therefore, for any $q > \max\{q_1, q_2\}$ and $\mathfrak{z}_q > \mathfrak{m}_q > q$, we have

$$\begin{aligned}\varepsilon \leq \kappa_{\hbar}\left(\xi_{2\mathfrak{z}_q}, \xi_{2\mathfrak{m}_q}\right) &\leq \mathfrak{c}\kappa_{\hbar}\left(\xi_{2\mathfrak{z}_q}, \xi_{2\mathfrak{m}_q-1}\right) + \mathfrak{c}\kappa_{\hbar}\left(\xi_{2\mathfrak{m}_q-1}, \xi_{2\mathfrak{m}_q}\right) \\ &\leq \mathfrak{c}\kappa_{\hbar}\left(\xi_{2\mathfrak{z}_q}, \xi_{2\mathfrak{m}_q-1}\right) + \mathfrak{c}\tfrac{\varepsilon}{2\mathfrak{c}}.\end{aligned}$$

So, one concludes that

$$\frac{\varepsilon}{2\mathfrak{c}} \leq \kappa_{\hbar}\left(\xi_{2\mathfrak{z}_q}, \xi_{2\mathfrak{m}_q-1}\right).$$

Hence, we deduce that for any $q > \max\{q_1, q_2\}$ and $\mathfrak{z}_q > \mathfrak{m}_q > q$,

$$\kappa_\hbar\left(\xi_{2\mathfrak{z}_q}, \xi_{2\mathfrak{z}_q+1}\right) \le \frac{\varepsilon}{2\mathfrak{c}} \le \kappa_\hbar\left(\xi_{2\mathfrak{z}_q}, \xi_{2\mathfrak{m}_q-1}\right)$$

that is, expression (15) is proved. So, from (2), it implies that

$$\eta\left(\begin{array}{l}\gamma\left(\mathfrak{c}^4\kappa_\hbar\left(\mathcal{J}\xi_{2_q}, \mathcal{K}\xi_{2_q-1}\right)\right),\\ \left(\gamma\left[\mathbf{G}\left(\xi_{2_q}, \xi_{2_q-1}\right) + \rho\mathcal{N}\left(\xi_{2_q}, \xi_{2_q-1}\right)\right]\right)^k\end{array}\right) \ge 1.$$

By using $(\eta_2{}')$ and taking into account the properties of φ and γ with $k \in (0,1)$, we obtain

$$\mathfrak{c}^{4+\lambda}\kappa_\hbar\left(\mathcal{J}\xi_{2\mathfrak{z}_q}, \mathcal{K}\xi_{2\mathfrak{m}_q-1}\right) < \mathbf{G}\left(\xi_{2\mathfrak{z}_q}, \xi_{2\mathfrak{m}_q-1}\right) + \rho\mathcal{N}\left(\xi_{2\mathfrak{z}_q}, \xi_{2\mathfrak{m}_q-1}\right), \quad (16)$$

where

$$\begin{aligned}
&\mathbf{G}\left(\xi_{2\mathfrak{z}_q}, \xi_{2\mathfrak{m}_q-1}\right)\\
&= \left(\begin{array}{l}\kappa_\hbar\left(\xi_{2\mathfrak{z}_q}, \xi_{2\mathfrak{m}_q-1}\right), \kappa_\hbar\left(\xi_{2\mathfrak{z}_q}, \mathcal{J}\xi_{2\mathfrak{z}_q}\right), \kappa_\hbar\left(\xi_{2\mathfrak{m}_q-1}, \mathcal{K}\xi_{2\mathfrak{m}_q-1}\right),\\ \frac{\kappa_{2\hbar}\left(\xi_{2\mathfrak{z}_q}, \mathcal{K}\xi_{2\mathfrak{m}_q-1}\right) + \kappa_{2\hbar}\left(\xi_{2\mathfrak{m}_q-1}, \mathcal{J}\xi_{2\mathfrak{z}_q}\right)}{2\mathfrak{c}}\end{array}\right)\\
&= \left(\begin{array}{l}\kappa_\hbar\left(\xi_{2\mathfrak{z}_q}, \xi_{2\mathfrak{m}_q-1}\right), \kappa_\hbar\left(\xi_{2\mathfrak{z}_q}, \xi_{2\mathfrak{z}_q+1}\right), \kappa_\hbar\left(\xi_{2\mathfrak{m}_q-1}, \xi_{2\mathfrak{m}_q}\right),\\ \frac{\kappa_{2\hbar}\left(\xi_{2\mathfrak{z}_q}, \xi_{2\mathfrak{m}_q}\right) + \kappa_{2\hbar}\left(\xi_{2\mathfrak{m}_q-1}, \xi_{2\mathfrak{z}_q+1}\right)}{2\mathfrak{c}}\end{array}\right).
\end{aligned}$$

By taking the limit superior as $q \to \infty$ in above and using (9), (12),(13), and (14), we derive the following:

$$\limsup_{q\to\infty} \mathbf{G}\left(\xi_{2\mathfrak{z}_q}, \xi_{2\mathfrak{m}_q-1}\right) \le \left(\mathfrak{c}^2\varepsilon, 0, 0, \frac{\mathfrak{c}\varepsilon + \mathfrak{c}^2\varepsilon}{2\mathfrak{c}}\right) \le \mathfrak{c}^2\varepsilon. \quad (17)$$

Also,

$$\begin{aligned}
\mathcal{N}\left(\xi_{2\mathfrak{z}_q}, \xi_{2\mathfrak{m}_q-1}\right) &= \min\left\{\begin{array}{l}\kappa_\hbar\left(\xi_{2\mathfrak{z}_q}, \mathcal{J}\xi_{2\mathfrak{z}_q}\right), \kappa_\hbar\left(\xi_{2\mathfrak{m}_q-1}, \mathcal{K}\xi_{2\mathfrak{m}_q-1}\right),\\ \kappa_\hbar\left(\xi_{2\mathfrak{z}_q}, \mathcal{K}\xi_{2\mathfrak{m}_q-1}\right) \kappa_\hbar\left(\xi_{2\mathfrak{m}_q-1}, \mathcal{J}\xi_{2\mathfrak{z}_q}\right)\end{array}\right\}\\
&= \min\left\{\begin{array}{l}\kappa_\hbar\left(\xi_{2\mathfrak{z}_q}, \xi_{2\mathfrak{z}_q+1}\right), \kappa_\hbar\left(\xi_{2\mathfrak{m}_q-1}, \xi_{2\mathfrak{m}_q}\right),\\ \kappa_\hbar\left(\xi_{2\mathfrak{z}_q}, \xi_{2\mathfrak{m}_q}\right), \kappa_\hbar\left(\xi_{2\mathfrak{m}_q-1}, \xi_{2\mathfrak{z}_q+1}\right)\end{array}\right\}
\end{aligned}$$

thus we have

$$\limsup_{q\to\infty} \mathcal{N}\left(\xi_{2\mathfrak{z}_q}, \xi_{2\mathfrak{m}_q-1}\right) = 0. \tag{18}$$

Consequently, if we take the limit as $q \to \infty$, and considering (11), (17), and (18), inequality (16) becomes $\mathfrak{c}^{4+\lambda}\frac{\varepsilon}{\mathfrak{c}} < \mathfrak{c}^2\varepsilon + \rho 0$. This is a contradiction; $\{\xi_{2\mathfrak{z}}\}$ is a $\kappa-$Cauchy sequence. Thereby, $\{\xi_{\mathfrak{z}}\}$ is a $\kappa-$Cauchy sequence in $\mathcal{L}^*_\kappa$. Because $\mathcal{L}^*_\kappa$ is a $\kappa-$complete $\mathfrak{M}_\flat\mathfrak{M}\mathfrak{G}$, $u \in \mathcal{L}^*_\kappa$ exists such that

$$\lim_{\mathfrak{z}\to\infty} \xi_{\mathfrak{z}} = u. \tag{19}$$

Step (3): In this step, we will prove that $u \in C_{Fix}(\mathcal{J}, \mathcal{K})$.

First, we shall demonstrate that $u \in Fix(\mathcal{K})$. Conversely, this statement is not true. We assert that for all $\mathfrak{z} \geq 0$, at least one of the following inequalities is true:

$$\frac{1}{2\mathfrak{c}}\kappa_\hbar\left(\xi_{2\mathfrak{z}}, \xi_{2\mathfrak{z}+1}\right) \leq \kappa_\hbar\left(\xi_{2\mathfrak{z}}, u\right), \tag{20}$$

or

$$\frac{1}{2\mathfrak{c}}\kappa_\hbar\left(\xi_{2\mathfrak{z}+1}, \xi_{2\mathfrak{z}+2}\right) \leq \kappa_\hbar\left(\xi_{2\mathfrak{z}}, u\right). \tag{21}$$

Unlike, let for some $\mathfrak{z}_0 \geq 0$, both of them are not provided. Hence, we say that

$$\begin{aligned} \kappa_\hbar\left(\xi_{2\mathfrak{z}_0}, \xi_{2\mathfrak{z}_0+1}\right) &\leq \mathfrak{c}\kappa_\hbar\left(\xi_{2\mathfrak{z}_0}, u\right) + \mathfrak{c}\kappa_\hbar\left(u, \xi_{2\mathfrak{z}_0+1}\right) \\ &< \tfrac{1}{2}\kappa_\hbar\left(\xi_{2\mathfrak{z}_0}, \xi_{2\mathfrak{z}_0+1}\right) + \tfrac{1}{2}\kappa_\hbar\left(\xi_{2\mathfrak{z}_0+1}, \xi_{2\mathfrak{z}_0+2}\right) \\ &< \tfrac{1}{2}\kappa_\hbar\left(\xi_{2\mathfrak{z}_0}, \xi_{2\mathfrak{z}_0+1}\right) + \tfrac{1}{2}\kappa_\hbar\left(\xi_{2\mathfrak{z}_0}, \xi_{2\mathfrak{z}_0+1}\right) \\ &= \kappa_\hbar\left(\xi_{2\mathfrak{z}_0}, \xi_{2\mathfrak{z}_0+1}\right), \end{aligned}$$

such that it is a contradiction. Whence the assertion is true. From this point, the following two subcases can be considered.

Subcase (3.1): The inequality (20) is valid for infinitely many $\mathfrak{z} \geq 0$. In this instance, for infinitely many $\mathfrak{z} \geq 0$, we attain

$$\begin{aligned} \tfrac{1}{2\mathfrak{c}}\min\left\{\kappa_\hbar\left(\xi_{2\mathfrak{z}}, \mathcal{J}\xi_{2\mathfrak{z}}\right), \kappa_\hbar\left(u, \mathcal{K}u\right)\right\} &= \tfrac{1}{2\mathfrak{c}}\min\left\{\kappa_\hbar\left(\xi_{2\mathfrak{z}}, \xi_{2\mathfrak{z}+1}\right), \kappa_\hbar\left(u, \mathcal{K}u\right)\right\} \\ &\leq \kappa_\hbar\left(\xi_{2\mathfrak{z}}, u\right) \end{aligned}$$

which yields that

$$\eta\left(\gamma\left(\mathfrak{c}^{4}\kappa_{\hbar}\left(\mathcal{J}\xi_{2\mathfrak{z}},\mathcal{K}u\right)\right),\left(\gamma\left[\mathbf{G}\left(\xi_{2\mathfrak{z}},u\right)+\rho\mathcal{N}\left(\xi_{2\mathfrak{z}},u\right)\right]\right)^{k}\right)\geq 1.$$

By using $(\eta_2{}')$ and keep in mind that $\varphi\in\Psi^{*}$ and $\gamma\in\Theta$ with $k\in(0,1)$, then we conclude that

$$\mathfrak{c}^{4+\lambda}\kappa_{\hbar}\left(\xi_{2\mathfrak{z}+1},\mathcal{K}u,\right)<\mathbf{G}\left(\xi_{2\mathfrak{z}},u\right)+\rho\mathcal{N}\left(\xi_{2\mathfrak{z}},u\right), \tag{22}$$

where

$$\begin{aligned}\mathbf{G}\left(\xi_{2\mathfrak{z}},u\right)&=\mathbf{G}\begin{pmatrix}\kappa_{\hbar}\left(\xi_{2\mathfrak{z}},u\right),\kappa_{\hbar}\left(\xi_{2\mathfrak{z}},\mathcal{J}\xi_{2\mathfrak{z}}\right),\kappa_{\hbar}\left(u,\mathcal{K}u\right),\\ \frac{\kappa_{2\hbar}(\xi_{2\mathfrak{z}},\mathcal{K}u)+\kappa_{2\hbar}(u,\mathcal{J}\xi_{2\mathfrak{z}})}{2\mathfrak{c}}\end{pmatrix}\\ &\leq\mathbf{G}\begin{pmatrix}\kappa_{\hbar}\left(\xi_{2\mathfrak{z}},u\right),\kappa_{\hbar}\left(\xi_{2\mathfrak{z}},\xi_{2\mathfrak{z}+1}\right),\kappa_{\hbar}\left(u,\mathcal{K}u\right),\\ \frac{\mathfrak{c}[\kappa_{\hbar}(\xi_{2\mathfrak{z}},\xi_{2\mathfrak{z}+})+\kappa_{\hbar}(\xi_{2\mathfrak{z}+1},\mathcal{K}u)]+\kappa_{2\hbar}(u,\xi_{2\mathfrak{z}+1})}{2\mathfrak{c}}\end{pmatrix},\end{aligned}$$

and

$$\begin{aligned}\limsup_{\mathfrak{z}\to\infty}\mathbf{G}\left(u,\xi_{2\mathfrak{z}+1}\right)&=\mathbf{G}\left(0,0,\kappa_{\hbar}\left(u,\mathcal{J}u\right),\tfrac{\kappa_{\hbar}(u,\mathcal{K}u)}{2}\right)\\ &\leq\kappa_{\hbar}\left(u,\mathcal{K}u\right).\end{aligned} \tag{23}$$

Also,

$$\begin{aligned}&\mathcal{N}\left(\xi_{2\mathfrak{z}},u\right)\\ &\quad=\min\left\{\kappa_{\hbar}\left(\xi_{2\mathfrak{z}},\mathcal{J}\xi_{2\mathfrak{z}}\right),\kappa_{\hbar}\left(u,\mathcal{K}u\right),\kappa_{\hbar}\left(\xi_{2\mathfrak{z}},\mathcal{K}u\right),\kappa_{\hbar}\left(u,\mathcal{J}\xi_{2\mathfrak{z}}\right)\right\}\\ &\quad=\min\left\{\kappa_{\hbar}\left(x_{2\mathfrak{z}},x_{2\mathfrak{z}+1}\right),\kappa_{\hbar}\left(u,\mathcal{K}u\right),\kappa_{\hbar}\left(\xi_{2\mathfrak{z}},\mathcal{K}u\right),\kappa_{\hbar}\left(u,\xi_{2\mathfrak{z}+1}\right)\right\}\end{aligned}$$

and so

$$\limsup_{\mathfrak{z}\to\infty}\mathcal{N}\left(\xi_{2\mathfrak{z}},u\right)=\min\left\{0,\kappa_{\hbar}\left(u,\mathcal{J}u\right),\kappa_{\hbar}\left(u,\mathcal{J}u\right),0,0\right\}=0. \tag{24}$$

Next, by (23) and (24), if we take the limit superior as $\mathfrak{z}\to\infty$ in (22), then it gives a contradiction since $\mathfrak{c}^{4+\lambda}\kappa_{\hbar}\left(u,\mathcal{J}u\right)<\kappa_{\hbar}\left(u,\mathcal{J}u\right)$.

Eventually, we have $u \in Fix(\mathcal{K})$. Pursuing the similar method above, $u \in Fix(\mathcal{J})$ is obvious.

Subcase (3.2): The expression (20) exclusively fulfills for finitely many $\mathfrak{z} \geq 0$.

Thereupon, one can find $\mathfrak{z}_0 \geq 0$ such that (21) holds for any $n \geq \mathfrak{z}_0$. In the same way, as in Subcase (3.1), it follows that (21) also causes a contradiction unless $u \in Fix(\mathcal{J})$ or $u \in Fix(\mathcal{K})$.

So, in both subcases, we achieve that $u \in C_{Fix}(\mathcal{J}, \mathcal{K})$.

Step (4): We claim that the set of $C_{Fix}(\mathcal{J}, \mathcal{K})$ has a unique element. Suppose, on the contrary. Then, there is a point r belongs to $\mathcal{L}^*_\kappa$ by $r \neq u$ such that $r \in C_{Fix}(\mathcal{J}, \mathcal{K})$. Since

$$0 = \frac{1}{2\mathfrak{c}} \min\{\kappa_\hbar(u, \mathcal{J}u), \kappa_\hbar(r, \mathcal{K}r)\} \leq \kappa_\hbar(u, r),$$

it implies that

$$\eta\left(\gamma\left(\mathfrak{c}^4 \kappa_\hbar(\mathcal{J}u, \mathcal{K}r)\right), (\gamma[\mathbf{G}(u, r) + \rho\mathcal{N}(u, r)])^k\right) \geq 1, \tag{25}$$

where

$$\begin{aligned}\mathbf{G}(u, r) &= \mathbf{G}\left(\kappa_\hbar(u, r), \kappa_\hbar(u, \mathcal{J}u), \kappa_\hbar(r, \mathcal{K}r), \tfrac{\kappa_{2\hbar}(u, \mathcal{K}r) + \kappa_{2\hbar}(r, \mathcal{J}u)}{2\mathfrak{c}}\right) \\ &\leq \kappa_\hbar(u, r)\end{aligned} \tag{26}$$

and also,

$$\begin{aligned}\mathcal{N}(u, r) &= \min\{\kappa_\hbar(u, \mathcal{J}u), \kappa_\hbar(r, \mathcal{K}r), \kappa_\hbar(u, \mathcal{K}r), \kappa_\hbar(r, \mathcal{J}u)\} \\ &= \min\{0, 0, \kappa_\hbar(u, r), \kappa_\hbar(u, r)\} = 0.\end{aligned} \tag{27}$$

Now, by using $(\eta_2{}')$ and the properties of φ and γ with $k \in (0, 1)$, inequality (25) turns into

$$\mathfrak{c}^{4+\lambda} \kappa_\hbar(u, r) < \mathbf{G}(u, r) + \rho\mathcal{N}(u, r).$$

Finally, from (26) and (27), we deduce that $\mathfrak{c}^{4+\lambda} \kappa_\hbar(u, r) < \kappa_\hbar(u, r)$, which causes a contradiction, that is, our assertion is false. Hence, $u = r$ and $C_{Fix}(\mathcal{J}, \mathcal{K}) = \{u\}$. $\square$

3 Consequences

This section illustrates the applicability and validity of our main theorem and supports it with several conclusions, which permits us to cover some obtained findings in the literature.

Initially, if we remove the restriction

$$\frac{1}{2\mathfrak{c}}\min\left\{\kappa_\hbar\left(\xi,\mathcal{J}\xi\right),\kappa_\hbar\left(\ell,\mathcal{K}\ell\right)\right\}\le\kappa_\hbar\left(\xi,\ell\right),$$

then the subsequent consequence can be acquired immediately from Theorem 2.5.

Corollary 3.1. *Let $\mathcal{L}_\kappa^*$ be a $\kappa-$complete $\mathfrak{M}_\flat\mathfrak{M}\mathfrak{S}$ with constant $\mathfrak{c}\ge 1$ and $\mathcal{J},\mathcal{K}:\mathcal{L}_\kappa^*\to\mathcal{L}_\kappa^*$ be two self-mappings. Let a generalized $\Psi^*\mathfrak{c}-\mathcal{S}\mathfrak{f}$ and a constant $\rho\ge 0$ as well as, a $\gamma\in\Theta$, a $\mathbf{G}\in\Delta_{\mathbf{G}}$ and $k\in(0,1)$ exist such that for all distinct $\xi,\ell\in\mathcal{L}_\kappa^*$, and for all $\hbar>0$*

$$\eta\left(\gamma\left(\mathfrak{c}^4\kappa_\hbar\left(\mathcal{J}\xi,\mathcal{K}\ell\right)\right),\left[\gamma\left(\mathbf{G}\left(\xi,\ell\right)+\rho\mathcal{N}\left(\xi,\ell\right)\right)\right]^k\right)\ge 1, \tag{28}$$

where $\mathbf{G}(\xi,\ell)$ and $\mathcal{N}(\xi,\ell)$ are defined as in Theorem 2.5. If condition $(\mathcal{S}_1)$ is satisfied, then the set $C_{Fix}(\mathcal{J},\mathcal{K})$ has at least one element. Also, together with the condition $(\mathcal{S}_2)$, the set $C_{Fix}(\mathcal{J},\mathcal{K})$ has a unique element.

If $\mathcal{J}=\mathcal{K}$ in Theorem 2.5, the following consequence and its proof are evident.

Corollary 3.2. *Let $\mathcal{L}_\kappa^*$ be a $\kappa-$complete $\mathfrak{M}_\flat\mathfrak{M}\mathfrak{S}$ with constant $\mathfrak{c}\ge 1$ and $\mathcal{J}:\mathcal{L}_\kappa^*\to\mathcal{L}_\kappa^*$ be a self-mapping. Let a generalized $\Psi^*\mathfrak{c}-\mathcal{S}\mathfrak{f}$ and a constant $\rho\ge 0$ as well as, a $\gamma\in\Theta$, a $\mathbf{G}\in\Delta_{\mathbf{G}}$ and $k\in(0,1)$ exist such that for all distinct $\xi,\ell\in\mathcal{L}_\kappa^*$, and for all $\hbar>0$*

$$\frac{1}{2\mathfrak{c}}\kappa_\hbar\left(\xi,\mathcal{J}\xi\right)\le\kappa_\hbar\left(\xi,\ell\right)$$

implies

$$\eta\left(\gamma\left(\mathfrak{c}^4\kappa_\hbar\left(\mathcal{J}\xi,\mathcal{J}\ell\right)\right),\left[\gamma\left(\mathbf{G}\left(\xi,\ell\right)+\rho\mathcal{N}\left(\xi,\ell\right)\right)\right]^k\right)\ge 1, \tag{29}$$

where

$$\mathbf{G}(\xi,\ell)=\left(\begin{array}{c}\kappa_{\hbar}(\xi,\ell),\kappa_{\hbar}(\xi,\mathcal{J}\xi),\kappa_{\hbar}(\ell,\mathcal{J}\ell),\\ \frac{\kappa_{2\hbar}(\xi,\mathcal{J}\ell)+\kappa_{2\hbar}(\ell,\mathcal{J}\xi)}{2\mathfrak{c}}\end{array}\right)$$

and

$$\mathcal{N}(\xi,\ell)=\min\{\kappa_{\hbar}(\xi,\mathcal{J}\xi),\kappa_{\hbar}(\ell,\mathcal{J}\ell),\kappa_{\hbar}(\xi,\mathcal{J}\ell),\kappa_{\hbar}(\ell,\mathcal{J}\xi)\}.$$

Thereby, under condition $(\mathcal{S}_1)$, the set $Fix(\mathcal{J})$ has at least one element. If the condition $(\mathcal{S}_2)$ is met in addition to $(\mathcal{S}_1)$, the set $Fix(\mathcal{J})$ consists of a unique element.

Corollary 3.3. *Let $\mathcal{L}^*_\kappa$ be a $\kappa-$complete $\mathfrak{M}_\flat\mathfrak{M}\mathfrak{S}$ with constant $\mathfrak{c}\geq 1$ and $\mathcal{J},\mathcal{K}:\mathcal{L}^*_\kappa\to\mathcal{L}^*_\kappa$ be two self- mappings. Let a generalized $\Psi^*\mathfrak{c}-\mathcal{S}\mathfrak{f}$ and a constant $\rho\geq 0$ as well as, a $\gamma\in\Theta$, a $\mathbf{G}\in\Delta_{\mathbf{G}}$ and $k\in(0,1)$ exist such that for all distinct $\xi,\ell\in\mathcal{L}^*_\kappa$, and for all $\hbar>0$,*

$$\frac{1}{2\mathfrak{c}}\min\{\kappa_{\hbar}(\xi,\mathcal{J}\xi),\kappa_{\hbar}(\ell,\mathcal{K}\ell)\}\leq\kappa_{\hbar}(\xi,\ell)$$

implies

$$\eta\left(\gamma\left(\mathfrak{c}^4\kappa_{\hbar}(\mathcal{J}\xi,\mathcal{K}\ell)\right),[\gamma(\mathbf{G}(\xi,\ell)+\rho\mathcal{N}(\xi,\ell))]^k\right)\geq 1, \tag{30}$$

where

$$\mathbf{G}(\xi,\ell)=\max\left\{\begin{array}{c}\kappa_{\hbar}(\xi,\ell),\kappa_{\hbar}(\xi,\mathcal{J}\xi),\kappa_{\hbar}(\ell,\mathcal{K}\ell),\\ \frac{\kappa_{2\hbar}(\xi,\mathcal{K}\ell)+\kappa_{2\hbar}(\ell,\mathcal{K}\xi)}{2\mathfrak{c}}\end{array}\right\}$$

and $\mathcal{N}(\xi,\ell)$ is defined as in Theorem 2.5. Under condition $(\mathcal{S}_1)$, the set $C_{Fix}(\mathcal{J},\mathcal{K})$ admits at least one element, and together with $(\mathcal{S}_2)$, the element of $C_{Fix}(\mathcal{J},\mathcal{K})$ is unique.

Proof. If we prefer $\mathbf{G}\in\Delta_{\mathbf{G}}$ as $\mathbf{G}(\mathfrak{a}_1,\mathfrak{a}_2,\mathfrak{a}_3,\mathfrak{a}_4)=\max\{\mathfrak{a}_1,\mathfrak{a}_2,\mathfrak{a}_3,\mathfrak{a}_4\}$, then it follows from Theorem 2.5. □

As in Corollary 3.3, if we choose $\mathbf{G}\in\Delta_{\mathbf{G}}$ as $\mathbf{G}(\mathfrak{a}_1,\mathfrak{a}_2,\mathfrak{a}_3,\mathfrak{a}_4)=\mathfrak{a}_1$, then the subsequent consequence can be attained as an immediate outcome of Theorem 2.5.

Corollary 3.4. *Let $\mathcal{L}_{\kappa}^{*}$ be a $\kappa-$complete $\mathfrak{M}_{\flat}\mathfrak{M}\mathfrak{S}$ with constant $\mathfrak{c} \geq 1$ and $\mathcal{J}, \mathcal{K} : \mathcal{L}_{\kappa}^{*} \to \mathcal{L}_{\kappa}^{*}$ be two self-mappings. Let a generalized $\Psi^{*}\mathfrak{c} - \mathcal{S}\mathfrak{f}$, a constant $\rho \geq 0$, a $\gamma \in \Theta$ and $k \in (0,1)$ exist such that for all distinct $\xi, \ell \in \mathcal{L}_{\kappa}^{*}$, and for all $\hbar > 0$,*

$$\frac{1}{2\mathfrak{c}} \min \left\{ \kappa_{\hbar}\left(\xi, \mathcal{J}\xi\right), \kappa_{\hbar}\left(\ell, \mathcal{K}\ell\right) \right\} \leq \kappa_{\hbar}\left(\xi, \ell\right)$$

implies

$$\eta\left(\gamma\left(\mathfrak{c}^{4}\kappa_{\hbar}\left(\mathcal{J}\xi, \mathcal{K}\ell\right)\right), \left[\gamma\left(\kappa_{\hbar}\left(\xi, \ell\right) + \rho\mathcal{N}\left(\xi, \ell\right)\right)\right]^{k}\right) \geq 1, \tag{31}$$

where $\mathcal{N}(\xi, \ell)$ is defined as in Theorem 2.5. With conditions $(\mathcal{S}_1)$ and $(\mathcal{S}_2)$, the set $C_{Fix}(\mathcal{J}, \mathcal{K})$ admits a unique element.

Next, we give some new corollaries dependent on the choice of generalized $\Psi^{*}\mathfrak{c} - \mathcal{S}\mathfrak{f}$.

Before continuing with the conclusions in this section, first, we acquaint a novel notion that we called Suzuki type (φ, γ) $-$contraction in the setting of an $\mathfrak{M}_{\flat}\mathfrak{M}\mathfrak{S}$, as indicated below.

Definition 3.5. Let $\mathcal{J}$ and $\mathcal{K}$ be two self-mappings on $\mathcal{L}_{\kappa}^{*}$. The mappings $\mathcal{J}$ and $\mathcal{K}$ are called Suzuki type (φ, γ) $-$contraction mappings if $\varphi \in \Psi^{*}$, $\gamma \in \Theta$ and $k \in (0,1)$ exist such that for all $\lambda \in (0,1)$

$$\frac{1}{2\mathfrak{c}} \min \left\{ \kappa_{\hbar}\left(\xi, \mathcal{J}\xi\right), \kappa_{\hbar}\left(\ell, \mathcal{K}\ell\right) \right\} \leq \kappa_{\hbar}\left(\xi, \ell\right)$$

implies

$$\varphi\left(\mathfrak{c}\gamma\left(\mathfrak{c}^{4}\kappa_{\hbar}\left(\mathcal{J}\xi, \mathcal{K}\ell\right)\right)\right) \leq \lambda\varphi\left(\left[\gamma\left(\kappa_{\hbar}\left(\xi, \ell\right)\right)\right]^{k}\right), \tag{32}$$

for all distinct $\xi, \ell \in \mathcal{L}_{\kappa}^{*}$ and for all $\hbar > 0$.

Corollary 3.6. *Let $\mathcal{J}$ and $\mathcal{K}$ be Suzuki type (φ, γ) $-$contraction mappings on $\mathcal{L}_{\kappa}^{*}$, which is a $\kappa-$complete $\mathfrak{M}_{\flat}\mathfrak{M}\mathfrak{S}$ with the constant $\mathfrak{c} \geq 1$.*

If condition $(\mathcal{S}_1)$ is satisfied, there is at least one element in the set $C_{Fix}(\mathcal{J}, \mathcal{K})$. Besides, together with $(\mathcal{S}_2)$, $C_{Fix}(\mathcal{J}, \mathcal{K})$ consists of only one element.

Proof. If we choose $\eta(t,v) = \frac{\lambda\varphi(v)}{\varphi(\mathfrak{c}t)}$, for all $t,v > 1$; $\lambda \in (0,1)$ as well as $\mathbf{G} \in \Delta_{\mathbf{G}}$ with $\mathbf{G}(\mathfrak{a}_1,\mathfrak{a}_2,\mathfrak{a}_3,\mathfrak{a}_4) = \mathfrak{a}_1$ and $\rho = 0$, then it follows from Theorem 2.5. □

The subsequent corollary is a generalization of the Suzuki type (φ,γ)-contraction mapping, which is also a consequence of Theorem 2.5.

Corollary 3.7. *Let $\mathcal{J}$ and $\mathcal{K}$ be two self-mappings on $\mathcal{L}^*_\kappa$, which is a $\kappa-$complete $\mathfrak{M}_\flat\mathfrak{M}\mathfrak{S}$ with constant $\mathfrak{c} \geq 1$. Presume that there exist $\varphi \in \Psi^*$, $\mathbf{G} \in \Delta_{\mathbf{G}}$, $\gamma \in \Theta$ and $k \in (0,1)$ such that*

$$\frac{1}{2\mathfrak{c}}\min\{\kappa_\hbar(\xi,\mathcal{J}\xi),\kappa_\hbar(\ell,\mathcal{K}\ell)\} \leq \kappa_\hbar(\xi,\ell)$$

implies

$$\varphi\left(\mathfrak{c}\gamma\left(\mathfrak{c}^4\kappa_\hbar(\mathcal{J}\xi,\mathcal{K}\ell)\right)\right) \leq \lambda\varphi\left([\gamma(\mathbf{G}(\xi,\ell))]^k\right), \tag{33}$$

where

$$\mathbf{G}(\xi,\ell) = \max\left\{\begin{array}{l}\kappa_\hbar(\xi,\ell),\kappa_\hbar(\xi,\mathcal{J}\xi),\kappa_\hbar(\ell,\mathcal{K}\ell), \\ \\ \frac{\kappa_{2\hbar}(\xi,\mathcal{K}\ell)+\kappa_{2\hbar}(\ell,\mathcal{K}\xi)}{2\mathfrak{c}}\end{array}\right\}$$

and $\lambda \in (0,1)$, for all distinct $\xi,\ell \in \mathcal{L}^*_\kappa$ and for all $\hbar > 0$.

If conditions $(\mathcal{S}_1)$ and $(\mathcal{S}_2)$ are satisfied, the set $C_{Fix}(\mathcal{J},\mathcal{K})$ exactly has a unique element.

Proof. As with the proof of Corollary 3.6, the proof is comprehensible if $\mathbf{G}(\mathfrak{a}_1,\mathfrak{a}_2,\mathfrak{a}_3,\mathfrak{a}_4) = \max\{\mathfrak{a}_1,\mathfrak{a}_2,\mathfrak{a}_3,\mathfrak{a}_4\}$ is settled on specifically. □

We also signify the Suzuki type $(\varphi,\phi,\gamma)-$contraction mapping, which is a finding of Theorem 2.5, as noted below.

Definition 3.8. Let κ be a modular $b-$metric on $(\mathcal{L},\kappa)$, and $\mathcal{J},\mathcal{K} : \mathcal{L}^*_\kappa \to \mathcal{L}^*_\kappa$ be two self-mappings. $\mathcal{J}$ and $\mathcal{K}$ are called Suzuki type $(\varphi,\phi,\gamma)-$contraction mappings if $\varphi \in \Psi^*$, $\gamma \in \Theta$ and $k \in (0,1)$ exist such that for all distinct $\xi,\ell \in \mathcal{L}^*_\kappa$ and for all $\hbar > 0$,

$$\frac{1}{2\mathfrak{c}}\min\{\kappa_\hbar(\xi,\mathcal{J}\xi),\kappa_\hbar(\ell,\mathcal{K}\ell)\} \leq \kappa_\hbar(\xi,\ell)$$

implies

$$\varphi\left(\mathfrak{c}^\lambda\gamma\left(\mathfrak{c}^4\kappa_\hbar(\mathcal{J}\xi,\mathcal{K}\ell)\right)\right) \leq \phi\left([\gamma(\kappa_\hbar(\xi,\ell))]^k\right), \tag{34}$$

where $\phi : [1, \infty) \to [1, \infty)$ is a continuous mapping satisfying $\phi(\mathfrak{t}) < \varphi(\mathfrak{t})$ for all $\mathfrak{t} > 0$.

Corollary 3.9. *Let $\mathcal{J}$ and $\mathcal{K}$ be Suzuki type $(\varphi, \phi, \gamma)-$contraction mappings on $\mathcal{L}_{\kappa}^{*}$, which is a $\kappa-$complete $\mathfrak{M}_{\flat}\mathfrak{M}\mathfrak{S}$ with constant $\mathfrak{c} \geq 1$. If $(\mathcal{S}_1)$ and $(\mathcal{S}_2)$ are met, $C_{Fix}(\mathcal{J}, \mathcal{K})$ definitely admits a unique element.*

Proof. If we choose $\eta(t, v) = \frac{\phi(v)}{\varphi(\mathfrak{c}^{\lambda} t)}$, for all $t, v > 1$ as well as $\mathbf{G} \in \Delta_{\mathbf{G}}$ with

$$\mathbf{G}(\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3, \mathfrak{a}_4) = \mathfrak{a}_1$$

and $\rho = 0$, then it follows from Theorem 2.5. □

Corollary 3.10. *Let $\mathcal{J}$ and $\mathcal{K}$ be two self-mappings on $\mathcal{L}_{\kappa}^{*}$, which is a $\kappa-$complete $\mathfrak{M}_{\flat}\mathfrak{M}\mathfrak{S}$ with coefficient $\mathfrak{c} \geq 1$. Presume that $\varphi \in \Psi^{*}$, $\mathbf{G} \in \Delta_{\mathbf{G}}$, $\gamma \in \Theta$ and $k \in (0, 1)$ exist such that for all distinct $\xi, \ell \in \mathcal{L}_{\kappa}^{*}$ and for all $\hbar > 0$,*

$$\frac{1}{2\mathfrak{c}} \min\{\kappa_{\hbar}(\xi, \mathcal{J}\xi), \kappa_{\hbar}(\ell, \mathcal{K}\ell)\} \leq \kappa_{\hbar}(\xi, \ell)$$

implies

$$\varphi\left(\mathfrak{c}^{\lambda}\gamma\left(\mathfrak{c}^{4}\kappa_{\hbar}(\mathcal{J}\xi, \mathcal{K}\ell)\right)\right) \leq \phi\left([\gamma(\mathbf{G}(\xi, \ell))]^{k}\right), \tag{35}$$

where

$$\mathbf{G}(\xi, \ell) = \max\left\{\begin{array}{c} \kappa_{\hbar}(\xi, \ell), \kappa_{\hbar}(\xi, \mathcal{J}\xi), \kappa_{\hbar}(\ell, \mathcal{K}\ell), \\ \frac{\kappa_{2\hbar}(\xi, \mathcal{K}\ell) + \kappa_{2\hbar}(\ell, \mathcal{K}\xi)}{2\mathfrak{c}} \end{array}\right\}$$

and $\phi : [1, \infty) \to [1, \infty)$ is a continuous mapping, which has the property $\phi(\mathfrak{t}) < \varphi(\mathfrak{t})$ for all $\mathfrak{t} > 0$. Thereupon, by considering conditions $(\mathcal{S}_1)$ and $(\mathcal{S}_2)$, the set $C_{Fix}(\mathcal{J}, \mathcal{K})$ holds exactly a unique element.

Proof. Using the analog method of the proof of Corollary 3.9, it is adequate to choose $\mathbf{G} \in \Delta_{\mathbf{G}}$ as $\mathbf{G}(\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3, \mathfrak{a}_4) = \max\{\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3, \mathfrak{a}_4\}$. Thus the proof is evident. □

The following theorem indicates that Corollary 3.4 can be satisfied as an outcome of Theorem 2.5 without the constant k.

Theorem 3.11. *Let $\mathcal{L}^*_\kappa$ be a $\kappa-$complete $\mathfrak{M}_\flat\mathfrak{M}\mathfrak{S}$ with coefficient $\mathfrak{c} \geq 1$ and $\mathcal{J}, \mathcal{K} : \mathcal{L}^*_\kappa \to \mathcal{L}^*_\kappa$ be two self-mappings. Presume that $\eta \in \Psi^*$, $\gamma \in \Theta$ and a constant $\rho \geq 0$ exist such that for all distinct $\xi, \ell \in \mathcal{L}^*_\kappa$ and for all $\hbar > 0$*

$$\frac{1}{2\mathfrak{c}} \min \left\{ \kappa_\hbar \left(\xi, \mathcal{J}\xi\right), \kappa_\hbar \left(\ell, \mathcal{K}\ell\right) \right\} \leq \kappa_\hbar \left(\xi, \ell\right)$$

implies

$$\eta \left(\gamma \left(\mathfrak{c}^4 \kappa_\hbar \left(\mathcal{J}\xi, \mathcal{K}\ell \right) \right), \gamma \left[\kappa_\hbar \left(\xi, \ell\right) + \rho \mathcal{N} \left(\xi, \ell\right) \right] \right) \geq 1, \tag{36}$$

where $\mathcal{N}(\xi, \ell)$ is defined as in Theorem 2.5.

Thereby, the set $C_{Fix}(\mathcal{J}, \mathcal{K})$ precisely admits only one element together with conditions $(\mathcal{S}_1)$ and $(\mathcal{S}_2)$.

Proof. We shall show that Theorem 3.11 can be achieved from Corollary 3.4. Owing to the fact that $\varphi \in \Psi^*$ in expression (31), we get

$$\varphi \left(\gamma \left(\mathfrak{c}^4 \kappa_\hbar \left(\mathcal{J}\xi, \mathcal{K}\ell \right) \right) \right) \leq \varphi \left(\left[\gamma \left(\kappa_\hbar \left(\xi, \ell\right) + \rho \mathcal{N} \left(\xi, \ell\right) \right) \right]^k \right),$$

which yields

$$\varphi \left(\gamma \left(\mathfrak{c}^4 \kappa_\hbar \left(\mathcal{J}\xi, \mathcal{K}\ell \right) \right) \right) \leq \varphi \left(\gamma \left(\kappa_\hbar \left(\xi, \ell\right) + \rho \mathcal{N} \left(\xi, \ell\right) \right) \right)$$

because $k \in (0, 1)$.

Under the same conditions, the above inequality can also be acquired from the expression (36). Therefore, Theorem 3.11 is demonstrated by applying a method similar to the proof of Corollary 3.4. □

Now, we put forward an example that fulfills the statement of Theorem 3.11 under specific states.

Example 3.12. Let $\mathcal{L} = [0, 1]$ and regard the modular $b-$metric by

$$\kappa_\hbar(\xi, \ell) = \frac{|\xi - \ell|^2}{\hbar},$$

for all distinct $\xi, \ell \in \mathcal{L}^*_\kappa$, and for all $\hbar > 0$. Observe that $(\mathcal{L}^*_\kappa, \kappa)$ is a $\kappa-$complete $\mathfrak{M}_\flat\mathfrak{M}\mathfrak{S}$ with the constant $\mathfrak{c} = 2$. Also, let the mappings $\mathcal{J}, \mathcal{K} : \mathcal{L}^*_\kappa \to \mathcal{L}^*_\kappa$ be defined by

$$\mathcal{J}\xi = \frac{\xi}{4} \quad \text{and} \quad \mathcal{K}\ell = 2\ell.$$

Now, we propose to prove the contractivity conditions

$$\frac{1}{2\mathfrak{c}}\min\left\{\kappa_{\hbar}\left(\xi,\mathcal{J}\xi\right),\kappa_{\hbar}\left(\ell,\mathcal{K}\ell\right)\right\}\leq\kappa_{\hbar}\left(\xi,\ell\right) \tag{37}$$

implies

$$\eta\left(\gamma\left(\mathfrak{c}^{4}\kappa_{\hbar}\left(\mathcal{J}\xi,\mathcal{K}\ell\right)\right),\gamma\left[\kappa_{\hbar}\left(\xi,\ell\right)+\rho\mathcal{N}\left(\xi,\ell\right)\right]\right)\geq 1, \tag{38}$$

where

$$\mathcal{N}\left(\xi,\ell\right)=\min\left\{\kappa_{\hbar}\left(\xi,\mathcal{J}\xi\right),\kappa_{\hbar}\left(\ell,\mathcal{J}\ell\right),\kappa_{\hbar}\left(\xi,\mathcal{J}\ell\right),\kappa_{\hbar}\left(\ell,\mathcal{J}\xi\right)\right\},$$

via $\eta\left(\mathfrak{a},\mathfrak{t}\right)=\frac{\psi(\mathfrak{t})^{\lambda}}{\psi(\mathfrak{a})}$, $\forall\ \mathfrak{a},\mathfrak{t}>1,\lambda\in(0,1)$ and $\psi\in\Psi^{*}$ with $\psi\left(\mathfrak{a}\right)=\frac{\mathfrak{a}}{2}$, holds for all distinct $\xi,\ell\in\mathcal{L}_{\kappa}^{*}$ and for all $\hbar>0$.

Notice that η belongs to the class of generalized $\Psi^{*}-\mathcal{S}\mathfrak{f}$ (also $\Psi^{*}\mathfrak{c}-\mathcal{S}\mathfrak{f}$).

Also, we define the function $\gamma:(0,\infty)\to(1,\infty)$ by $\gamma\left(\mathfrak{a}\right)=e^{\mathfrak{a}}$.

Hence, we will first attest that expression (37) is satisfied. Then, we have

$$\kappa_{\hbar}\left(\xi,\mathcal{J}\xi\right)=\frac{\left|\xi-\mathcal{J}\xi\right|^{2}}{\hbar}=\frac{\left|\xi-\frac{\xi}{4}\right|^{2}}{\hbar}=\frac{9\xi^{2}}{16\hbar}$$

and

$$\kappa_{\hbar}\left(\ell,\mathcal{K}\ell\right)=\frac{\left|\ell-\mathcal{K}\ell\right|^{2}}{\hbar}=\frac{\left|\ell-2\ell\right|^{2}}{\hbar}=\frac{\ell^{2}}{\hbar}.$$

Subsequently, expression (37) yields

$$\frac{1}{4}\min\left\{\frac{9\xi^{2}}{16\hbar},\frac{\ell^{2}}{\hbar}\right\}=\min\left\{\frac{9\xi^{2}}{64\hbar},\frac{\ell^{2}}{4\hbar}\right\}\leq\frac{\left|\xi-\ell\right|^{2}}{\hbar}. \tag{39}$$

Without ignoring the general case, we esteem that $\xi>\ell\geq 0$.

Case (1): Let $\min\left\{\frac{9\xi^{2}}{64\hbar},\frac{\ell^{2}}{4\hbar}\right\}=\frac{\ell^{2}}{4\hbar}$. So, from (39), we obtain that

$$\frac{\ell^{2}}{4\hbar}\leq\frac{\left(\xi-\ell\right)^{2}}{\hbar}\quad\Leftrightarrow\quad\frac{\ell}{2}\leq\left|\xi-\ell\right|=\xi-\ell\quad\Leftrightarrow\quad\ell\leq\frac{2}{3}\xi<\xi,$$

which is true as $\ell<\xi$.

Case (2): Let $\min\left\{\frac{9\xi^2}{64\hbar}, \frac{\ell^2}{4\hbar}\right\} = \frac{9\xi^2}{64\hbar}$. Then, by (39), we conclude that

$$\frac{9\xi^2}{64\hbar} \leq \frac{(\xi-\ell)^2}{\hbar} \quad \Leftrightarrow \quad \frac{3\xi}{8} \leq |\xi-\ell| = \xi-\ell \quad \Leftrightarrow \quad \ell \leq \frac{5}{8}\xi < \xi,$$

which holds because $\ell < \xi$.

Consequently, in any case, expression (37) is valid for all distinct $\xi, \ell \in \mathcal{L}^*_\kappa$ and for all $\hbar > 0$.

Next, we will prove that expression (38) is satisfied. Using the above choices for the mappings $\mathcal{J}$, $\mathcal{K}$ and modular $b-$metric, we attain

$$\kappa_\hbar(\mathcal{J}\xi, \mathcal{K}\ell) = \frac{|\mathcal{J}\xi - \mathcal{K}\ell|^2}{\hbar} = \frac{\left|\frac{\xi}{4} - 2\ell\right|^2}{\hbar} = \frac{(\xi-8\ell)^2}{16\hbar},$$

and

$$\begin{aligned} \mathcal{N}(\xi,\ell) &= \min\{\kappa_\hbar(\xi,\mathcal{J}\xi), \kappa_\hbar(\ell,\mathcal{J}\ell), \kappa_\hbar(\xi,\mathcal{J}\ell), \kappa_\hbar(\ell,\mathcal{J}\xi)\} \\ &= \min\left\{\frac{|\xi-\frac{\xi}{4}|^2}{\hbar}, \frac{|\ell-2\ell|^2}{\hbar}, \frac{|\xi-2\ell|^2}{\hbar}, \frac{|\ell-\frac{\xi}{4}|^2}{\hbar}\right\} \\ &= \min\left\{\frac{9\xi^2}{16\hbar}, \frac{\ell^2}{\hbar}, \frac{|\xi-2\ell|^2}{\hbar}, \frac{|4\ell-\xi|^2}{16\hbar}\right\} \\ &= \min\left\{\frac{\ell^2}{\hbar}, \frac{|4\ell-\xi|^2}{16\hbar}\right\}. \end{aligned} \tag{40}$$

Case (3): We assume that $\xi < 4\ell$, and so, we get $|4\ell-\xi| = 4\ell-\xi$. Using (40), we conclude that

$$\min\left\{\frac{\ell^2}{\hbar}, \frac{|4\ell-\xi|^2}{16\hbar}\right\} = \frac{(4\ell-\xi)^2}{16\hbar}.$$

Thereby, the expression (38) provides

$$\eta\left(\gamma\left(\mathfrak{c}^4\kappa_\hbar(\mathcal{J}\xi,\mathcal{K}\ell)\right), \gamma\left(\kappa_\hbar(\xi,\ell) + \rho\mathcal{N}(\xi,\ell)\right)\right)$$

$$= \frac{\psi(\gamma(\kappa_\hbar(\xi,\ell)+\rho\mathcal{N}(\xi,\ell)))^\lambda}{\psi(\gamma(\mathfrak{c}^4\kappa_\hbar(\mathcal{J}\xi,\mathcal{K}\ell)))} = 2^{1-\lambda}\frac{e^{[\kappa_\hbar(\xi,\ell)+\rho\mathcal{N}(\xi,\ell)]\lambda}}{e^{\mathfrak{c}^4\kappa_\hbar(\mathcal{J}\xi,\mathcal{K}\ell)}} = 2^{1-\lambda}\frac{e^{\left[\frac{(\xi-\ell)^2}{\hbar}+\rho\frac{(4\ell-\xi)^2}{16\hbar}\right]\lambda}}{e^{2^4\frac{(\xi-8\ell)^2}{16\hbar}}}$$

$$= 2^{1-\lambda}e^{\left[\frac{(\xi-\ell)^2}{\hbar}+\rho\frac{(4\ell-\xi)^2}{16\hbar}\right]\lambda - \frac{(\xi-8\ell)^2}{\hbar}} \geq 1,$$

where $\rho \geq 2^4$ and $\lambda = \frac{1}{2} \in (0,1)$. Thus, the desired state is achieved.

Case (4): We presume that $4\ell < \xi$, i.e., $|4\ell - \xi| = \xi - 4\ell$. In this case, if we choose $\xi = 8\ell$, then we get

$$\frac{\ell^2}{\hbar} = \frac{|4\ell - \xi|^2}{16\hbar}.$$

Therefore, we discuss the following subcases.

Subcase (4.1): If $\xi > 8\ell$, by (40), we have

$$\min\left\{\frac{\ell^2}{\hbar}, \frac{|4\ell - \xi|^2}{16\hbar}\right\} = \frac{\ell^2}{\hbar}.$$

Then, similar to the above, we get

$$\eta\left(\gamma\left(\mathfrak{c}^4\kappa_\hbar\left(\mathcal{J}\xi, \mathcal{K}\ell\right)\right), \gamma\left(\kappa_\hbar\left(\xi,\ell\right) + \rho\mathcal{N}\left(\xi,\ell\right)\right)\right)$$

$$= \frac{\psi(\gamma(\kappa_\hbar(\xi,\ell)+\rho\mathcal{N}(\xi,\ell)))^\lambda}{\psi(\gamma(\mathfrak{c}^4\kappa_\hbar(\mathcal{J}\xi,\mathcal{K}\ell)))} = 2^{1-\lambda}\frac{e^{[\kappa_\hbar(\xi,\ell)+\rho\mathcal{N}(\xi,\ell)]\lambda}}{e^{\mathfrak{c}^4\kappa_\hbar(\mathcal{J}\xi,\mathcal{K}\ell)}} = 2^{1-\lambda}\frac{e^{\left[\frac{(\xi-\ell)^2}{\hbar}+\rho\frac{\ell^2}{\hbar}\right]\lambda}}{e^{2^4\frac{(\xi-8\ell)^2}{16\hbar}}}$$

$$= 2^{1-\lambda}e^{\left[\frac{(\xi-\ell)^2}{\hbar}+\rho\frac{\ell^2}{\hbar}\right]\lambda - \frac{(\xi-8\ell)^2}{\hbar}} \geq 1.$$

Subcase (4.2): If $\xi < 8\ell$, by (40), we have

$$\min\left\{\frac{\ell^2}{\hbar}, \frac{|4\ell - \xi|^2}{16\hbar}\right\} = \frac{(4\ell - \xi)^2}{16\hbar}.$$

This case has been proved to be satisfied in Case (3).

Consequently, the mappings $\mathcal{J}$ and $\mathcal{K}$ satisfy the hypotheses of Theorem 3.11 associated with the function $\eta(\mathfrak{a}, \mathfrak{t}) = \frac{\psi(\mathfrak{t})^\lambda}{\psi(\mathfrak{a})}, \quad \forall\ \mathfrak{a}, \mathfrak{t} > 1, \lambda \in (0,1)$ and $\psi \in \Psi^*$ with $\psi(\mathfrak{a}) = \frac{\mathfrak{a}}{2}$.

4 An Application to Integral Equations

Initially, a novel consequence has been put forward that can be enforced to the nonlinear integral equations.

Corollary 4.1. *Let $\mathcal{L}^*_\kappa$ be a $\kappa-$complete $\mathfrak{M}_\flat\mathfrak{M}\mathfrak{S}$ with constant $\mathfrak{c} \geq 1$ and $\mathcal{J} : \mathcal{L}^*_\kappa \to \mathcal{L}^*_\kappa$ be a self-mapping. Suppose that the ensuing circumstances be provided:*

i. A generalized $\Psi^*-\mathcal{S}\mathfrak{f}$, a constant $\rho \geq 0$, and $\gamma \in \Theta$ exist such that for all distinct $\xi, \ell \in \mathcal{L}^*_\kappa$, $k \in (0,1)$ and for all $\hbar > 0$

$$\eta \left(\gamma \left(\mathfrak{c}^4 \kappa_\hbar \left(\mathcal{J}\xi, \mathcal{J}\ell \right) \right), \left[\gamma \left(\kappa_\hbar \left(\xi, \ell \right) \right) \right]^k \right) \geq 1; \tag{41}$$

ii. $(\mathcal{S}_1)$ and $(\mathcal{S}_2)$ are provided.

In this case, the set $Fix(\mathcal{J})$ enjoys a unique element in $\mathcal{L}^*_\kappa$.

Proof. Without Suzuki restriction, if we choose $\mathbf{G} \in \Delta_{\mathbf{G}}$ as

$$\mathbf{G}(\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3, \mathfrak{a}_4) = \mathfrak{a}_1,$$

$\rho = 0$, and $\mathcal{J} = \mathcal{K}$ in Theorem 2.5, the requested result is acquired. □

We shall verify an existence theorem for the solution of the ensuing nonlinear integral equation via Corollary 4.1.

$$\xi(\mathfrak{a}) = \int_{\hat{m}}^{\hat{n}} \mathcal{E}(\mathfrak{a}, \mathfrak{t}, \xi(\mathfrak{t}))\, d\mathfrak{t}, \tag{42}$$

where $\hat{m}, \hat{n} \in \mathbb{R}$ by $\hat{m} < \hat{n}$, $\xi \in C[\hat{m}, \hat{n}]$ (the set of all continuous functions from $[\hat{m}, \hat{n}]$ into $\mathbb{R}$) and $\mathcal{E} : [\hat{m}, \hat{n}] \times [\hat{m}, \hat{n}] \times \mathbb{R} \to \mathbb{R}$ are given mappings. We endow $\mathcal{L}^*_\kappa = C[\hat{m}, \hat{n}]$ with

$$\kappa_\hbar(x, y) = \frac{|x(\mathfrak{a}) - y(\mathfrak{a})|^p}{\hbar}, \quad (p > 1),$$

for all $x, y \in \mathcal{L}^*_\kappa$ and for all $\hbar > 0$. Precisely, $(\mathcal{L}^*_\kappa, \kappa)$ is a $\kappa-$complete modular $b-$metric space with the constant $\mathfrak{c} = 2^{p-1}$.

Additionally, let $f : \mathcal{L}^*_\kappa \to \mathcal{L}^*_\kappa$ be defined by

$$f(\xi(\mathfrak{a})) = \int_{\hat{m}}^{\hat{n}} \mathcal{E}(\mathfrak{a}, \mathfrak{t}, \xi(\mathfrak{t}))\, d\mathfrak{t}$$

for all $\xi \in \mathcal{L}^*_\kappa$ and $\mathfrak{a} \in [\hat{m}, \hat{n}]$. Accordingly, the existence of a solution to (42) is equivalent to the existence of a fixed point of the function f.

Theorem 4.2. *Contemplate the nonlinear integral equation (42). Let the ensuing circumstances be met:*

i. $\mathcal{E} : [\hat{m}, \hat{n}] \times [\hat{m}, \hat{n}] \times \mathbb{R} \to \mathbb{R}$ *is continuous and non-decreasing in the third order,*

ii. *there exists* $p > 1$ *satisfying the following condition: for each* $\mathfrak{a}, \mathfrak{t} \in [\hat{m}, \hat{n}]$ *and* $\xi, \ell \in \mathcal{L}_\kappa^*$ *with* $\xi(w) \leq \ell(w)$ *for all* $w \in [\hat{m}, \hat{n}]$*, we have*

$$|\mathcal{E}(\mathfrak{a}, \mathfrak{t}, \xi(\mathfrak{t})) - \mathcal{E}(\mathfrak{a}, \mathfrak{t}, \ell(\mathfrak{t}))| \leq \Lambda(\mathfrak{a}, \mathfrak{t}) |\xi(\mathfrak{t}) - \ell(\mathfrak{t})|, \tag{43}$$

where $\Lambda : [\hat{m}, \hat{n}] \times [\hat{m}, \hat{n}] \to [0, \infty)$ *is a continuous function defined by*

$$\sup_{\mathfrak{a} \in [\hat{m}, \hat{n}]} \left(\int_{\hat{m}}^{\hat{n}} \Lambda(\mathfrak{a}, \mathfrak{t})^p d\mathfrak{t} \right) \leq \frac{k}{2^{4p-4}}, \quad (k \in (0,1)). \tag{44}$$

Thereby, the nonlinear integral equation (42) enjoys a solution.

Proof. For all $\mathfrak{a} \in [\hat{m}, \hat{n}]$, by (43) and (44), we derive

$$\begin{aligned}
e^{2^{4p-4}\kappa_\hbar(f(\xi(\mathfrak{a})), f(\ell(\mathfrak{a})))^p} &= e^{2^{4p-4} p \frac{|f(\xi(\mathfrak{a})) - f(\ell(\mathfrak{a}))|}{\hbar}^p} \\
&= e^{2^{4p-4} . p . \frac{1}{\hbar} \left| \int_{\hat{m}}^{\hat{n}} \mathcal{E}(\mathfrak{a}, \mathfrak{t}, \xi(\mathfrak{t})) d\mathfrak{t} - \int_{\hat{m}}^{\hat{n}} \mathcal{E}(\mathfrak{a}, \mathfrak{t}, \ell(\mathfrak{t})) d\mathfrak{t} \right|^p} \\
&\leq e^{2^{4p-4} . p . \frac{1}{\hbar} \left| \int_{\hat{m}}^{\hat{n}} [\mathcal{E}(\mathfrak{a}, \mathfrak{t}, \xi(\mathfrak{t})) - \mathcal{E}(\mathfrak{a}, \mathfrak{t}, \ell(\mathfrak{t}))] d\mathfrak{t} \right|^p} \\
&\leq e^{2^{4p-4} . p . \frac{1}{\hbar} \left(\int_{\hat{m}}^{\hat{n}} |\mathcal{E}(\mathfrak{a}, \mathfrak{t}, \xi(\mathfrak{t})) - \mathcal{E}(\mathfrak{a}, \mathfrak{t}, \ell(\mathfrak{t}))| d\mathfrak{t} \right)^p} \\
&\leq e^{2^{4p-4} . p . \frac{1}{\hbar} \left(\int_{\hat{m}}^{\hat{n}} \Lambda(\mathfrak{a}, \mathfrak{t}) |\xi(\mathfrak{t}) - \ell(\mathfrak{t})| d\mathfrak{t} \right)^p} \\
&\leq e^{2^{4p-4} . p . \left(\int_{\hat{m}}^{\hat{n}} \Lambda(\mathfrak{a}, \mathfrak{t})^p d\mathfrak{t} \right) \frac{|\xi(\mathfrak{t}) - \ell(\mathfrak{t})|^p}{\hbar}} \\
&\leq e^{2^{4p-4} . p \frac{k}{2^{4p-4}} \kappa_\hbar(\xi(\mathfrak{a}), \ell(\mathfrak{a}))} = \left[\left(e^{\kappa_\hbar(\xi(\mathfrak{a}), \ell(\mathfrak{a}))} \right)^k \right]^p.
\end{aligned}$$

Now, let η be a $\Psi^*-\mathcal{S}\mathfrak{f}$ with $\eta\left(x,y\right)=\frac{\varphi(y)}{\varphi(x)},\quad\left(\forall\ x,y>1\right)$, where $\varphi\in\Psi^*$ such that $\varphi\left(q\right)=q^p,\ \left(q>1\right)$. We define $\gamma\in\Theta$ by $\gamma\left(\alpha\right)=e^{\alpha},\ \left(\alpha>0\right)$. This suggests that

$$\begin{aligned}\varphi\left(\gamma\left(2^{4p-4}\kappa_{\hbar}\left(f\left(\xi\left(\mathfrak{a}\right)\right),f\left(\ell\left(\mathfrak{a}\right)\right)\right)\right)\right) &= \left(\gamma\left(2^{4p-4}\kappa_{\hbar}\left(f\left(\xi\left(\mathfrak{a}\right)\right),f\left(\ell\left(\mathfrak{a}\right)\right)\right)\right)\right)^p \\ &= e^{2^{4p-4}\kappa_{\hbar}\left(f\left(\xi\left(\mathfrak{a}\right)\right)-f\left(\ell\left(\mathfrak{a}\right)\right)\right)^p} \\ &\leq \left[\left(e^{\kappa_{\hbar}\left(\xi\left(\mathfrak{a}\right),\ell\left(\mathfrak{a}\right)\right)}\right)^k\right]^p \\ &= \left(\left[\gamma\left(\kappa_{\hbar}\left(\xi,\ell\right)\right)\right]^k\right)^p \\ &= \varphi\left(\left[\gamma\left(\kappa_{\hbar}\left(\xi,\ell\right)\right)\right]^k\right).\end{aligned}$$

With this last equation, we deduce that all circumstances of Corollary 4.1 are met. Herewith, a unique $\xi\in\mathcal{L}^*_{\kappa}$ exists such that $\xi\in Fix\left(f\right)$, which conveys that ξ is the unique solution for the integral equation (42). □

Acknowledgements
The authors appreciate anonymous reviewers for their remarks to cultivate the study.

References

[1] A. A. N. Abdou, Fixed points of Kannan maps in modular metric spaces, *AIMS Mathematics*, 5(6) (2020), 6395-6403.

[2] H. Afshari, H. Aydi and E. Karapınar, On generalized $\alpha,\psi-$Geraghty contractions on $b-$metric spaces, *Georgian Math. Journal*, 27(1) (2020), 9-21.

[3] R. P. Agarwal and E. Karapınar, Interpolative Rus-Reich-Ciric type contractions via simulation functions, *An. St. Univ. Ovidius Constanta*, 27(3) (2019), 137-152.

[4] A. Aghajani, M. Abbas and J. R. Roshan, Common fixed point of generalized weak contractive mappings in partially ordered b-metric spaces, *Math. Slovaca*, 64(4) (2014), 941–960.

[5] J. Ahmad, A. E. Al-Mazrooei, Y. J. Cho and Y. O. Yang, Fixed point results for generalized $\Theta-$contractions, *J. Nonlinear Sci. Appl.*, 10 (2017), 2350–2358.

[6] U. Aksoy, I. M. Erhan and V. Rakocevic, Meir-Keeler type contractions on modular metric spaces, *Filomat*, 32(10) (2018), 3697–3707.

[7] H. Argoubi, B. Samet and C. Vetro, Nonlinear contractions involving simulation functions in a metric space with a partial order, *J. Nonlinear Sci. Appl.,* 8 (2015), 1082–1094.

[8] H. Aydi, M. F. Bota, E. Karapınar and S. Moradi, A common fixed point for weak $\phi-$contractions on $b-$metric spaces, *Fixed Point Theory*, 13(2) (2012), 337-346.

[9] H. Aydi, E. Karapınar and V. Rakocevic, Nonunique fixed point theorems on $b-$metric spaces via simulation functions, *Jordan J. Math. Stat.*, 12(3) (2019), 265-288.

[10] S. Banach, Sur les operations dans les emsembles abstraits et leurs applications aux equations integrales, *Fund. Math.,* 1 (1922), 133-181.

[11] I. A. Bakhtin, The contraction mapping principle in quasimetric spaces, *Funct. Anal. Unianowsk Gos. Ped. Inst.,* 30 (1989), 26–37.

[12] V. V. Chistyakov, Modular metric spaces generated by F-modulars, *Folia Math.,* 15 (2008), 3-24.

[13] V. V. Chistyakov, Modular metric spaces, I: Basic concepts, *Nonlinear Anal.,* 72 (2010), 1-14.

[14] V. V. Chistyakov, Modular metric spaces, II: Application to superposition operators, *Nonlinear Anal.,* 72 (2010), 15-30.

[15] V. V. Chistyakov, Fixed points of modular contractive maps, *Dokl. Math.,* 86 (2012), 515-518.

[16] S. H. Cho, Fixed point theorem for $Z-$contraction generalized metric spaces, *Abstr. Appl. Anal.*, 2018 (2018), DOI: 10.1155/2018/1327691.

[17] S. Czerwik, Contraction mappings in b-metric spaces, *Acta. Math. Inform. Univ. Ostrav.,* 1(1) (1993), 5-11.

[18] S. Czerwik, Nonlinear set-valued contraction mappings in b-metric spaces, *Atti Semin. Mat. Fis. Univ. Modena*, 46 (1998), 263-276.

[19] M. E. Ege and C. Alaca, Fixed point results and an application to homotopy in modular metric spaces, *J. Nonlinear Sci. Appl.*, 8(6) (2015), 900-908.

[20] M. E. Ege and C. Alaca, Some results for modular $b-$metric spaces and an application to system of linear equations, *Azerbaijan J. Math.*, 8(1) (2018), 3-14.

[21] G. H. Joonaghany, A. Farajzadeh, M. Azhini, and F. Khojasteh, A new common fixed point theorem for Suzuki type contractions via generalized $\Psi-$simulation functions, *Sahand Commun. Math. Anal.*, 16(1) (2019), 129-148.

[22] A. Gholidahneh, S. Sedghi, O. Ege, Z. D. Mitrovic and M. de la Sen, The Meir-Keeler type contractions in extended modular $b-$metric spaces with an application, *AIMS Mathematics*, 6(2) (2021), 1781-1799.

[23] H. Hosseinzadeh and V. Parvaneh, Meir-Keeler type contractive mappings in modular and partial modular metric spaces, *Asian-Eur. J. Math.*, 13(5) (2020), 2050087.

[24] M. Iqbal, A. Batool, O. Ege and M. de la Sen, Fixed point of generalized weak contraction in $b-$metric spaces, *J. Funct. Spaces*, 2021 (2021), DOI: 10.1155/2021/2042162.

[25] M. Iqbal, A. Batool, O. Ege and M. de la Sen, Fixed point of almost contraction in $b-$metric spaces, *Journal of Mathematics*, 2020 (2020), DOI: 10.1155/2020/3218134.

[26] M. Jleli and B. Samet, A new generalization of the Banach contraction principle, *J. Inequal. Appl.*, 38 (2014), 1–8.

[27] E. Karapınar, A. Fulga and A. Petrusel, On Istratescu type contractions in $b-$metric spaces., *Mathematics*, 8(388) (2020), 1-11.

[28] E. Karapınar and C. Chifu, Results in $wt-$distance over $b-$metric Spaces, *Mathematics*, 8(2) (2020), 220.

[29] F. Khojasteh, S. Shukla and S. Radenovic, A new approach to the study of fixed point theorems for simulation functions, *Filomat*, 29 (2015), 1189-1194.

[30] X. D. Liu, S. S. Chang, Y. Xiao and L. C. Zhao, Some fixed point theorems concerning $(\psi, \phi)-$type contraction in complete metric spaces, *J. Nonlinear Sci. Appl.*, 9 (2016), 4127–4136.

[31] C. Mongkolkeha, W. Sintunavarat and P. Kumam, Fixed point theorems for contraction mappings in modular metric spaces, *Fixed Point Theory Appl.,* 93 (2011), 1-9.

[32] V. Parvaneh, N. Hussain, M. Khorshidi, N. Mlaiki and H. Aydi, Fixed point results for generalized $F-$contractions in modular $b-$metric spaces with applications, *Mathematics*, 7(10) (2019), 1-16.

[33] A. F. Roldan-Lopez-de-Hierro, E. Karapınar, C. Roldan-Lopez-de-Hierro and J. Martínez-Moreno, Coincidence point theorems on metric spaces via simulation functions, *J. Comput. Appl. Math.,* 275 (2015), 345–355.

[34] H. N. Saleh, M. Imdad, T. Abdeljawad and M. Arif, New contractive mappings and their fixed points in Branciari metric spaces, *J. Funct. Spaces*, 2020 (2020), DOI: 10.1155/2020/9491786.

[35] K. Zoto, N. Mlaiki, and H. Aydi, Related fixed point theorems via general approach of simulation functions, *Journal of Mathematics*, 2020 (2020), DOI: 10.1155/2020/4820191.

Mahpeyker Öztürk
Department of Mathematics
Associate Professor of Mathematics
Sakarya University
Sakarya, Turkey
E-mail: mahpeykero@sakarya.edu.tr

Farhan Golkarmanesh
Department of Mathematics
Assistant Professor of Mathematics
Sanandaj Branch, Islamic Azad University
Sanandaj, Iran
E-mail: Fgolkarmanesh@yahoo.com

Abdurrahman Büyükkaya
Department of Mathematics
Karadeniz Technical University
Trabzon, Turkey
E-mail: abdurrahman.giresun@hotmail.com

Vahid Parvaneh
Department of Mathematics
Assistant Professor of Mathematics
Gilan-E-Gharb Branch, Islamic Azad University
Gilan-E-Gharb, Iran
E-mail: zam.dalahoo@gmail.com