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Original Research Paper

Du Fort-Frankel Scheme for the Variable Order Time Fractional Diffusion Equation

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Abstract. In this paper, the variable order time-fractional diffusion equations in a finite domain are considered. A Du Fort-Frankel finite difference method is introduced to solve these equations. The stability condition for this scheme is discussed and proved via the approach of Fourier analysis. Numerical examples are prepared to illsutrate that the numerical method is computationally efficient.

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Keywords and Phrases: Variable order time fractional diffusion equation, Du Fort-Frankel method, Stability, Fourier analysis

1 Introduction

Scientific problems related to the field of fractional calculus have become very important in recent years. Description of the memory and hereditary properties of various materials and processes are the main advantages of fractional derivatives [1, 2, 3, 10, 15, 17, 18, 19, 20, 21, 23, 24, 27, 29, 30, 37].
Many dynamic processes have the behavior of time and space fractional derivatives therefore, it is crucial for researchers to develop the concept

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of variable-order calculus. Recently, variable-order calculus has been utilized in many various fields such as geographic data [6], viscoelastic mechanics [7], signal and confirmation [36].
Samko and Ross [31] firstly suggested the perception of variable order operators that allow the order to be a function of place, time, or some other variable rather than an arbitrary order constant [26, 28].
Compared with the fractional variable order operators and the constant order operators which based on their non-stationary power-law kernel is more careful to describe these complex physical processes and systems [34, 35]. Afterward, Lorenzo and Hartley [26] investigated another kinds of variable order fractional operator definitions and made some theoretical studies by the iterative Laplace transform. These probes show that the variable-order operator is a new advancement of learning in science [4, 9, 11, 14].
Many various numerical methods have been suggested to solve variable order time fractional diffusion equations [13, 16, 22, 32, 38, 39, 41, 42, 43, 44].
The aim of this article is to utilize a finite difference scheme based on Du Fort-Frankel for solving variable order time fractional diffusion equation

$$ {}_0D_t^{\alpha(x,t)}\, u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} + p(x,t), \;\; x \in (0, L], \;\; t \in (0, T], \qquad (1) $$

with initial and boundary conditions

$$ u(x, 0) = f(x),\, x \in [0, L], \qquad (2) $$

$$ u(0, t) = g(x), \;\; u(L, t) = h(t), \;\; t \in [0, T], \qquad (3) $$

where $0 < \alpha(x,t) \leq 1$, $p(x,t)$ is a source term and ${}_0D_t^{\alpha(x,t)}$ denotes the Caputo variable order time fractional derivative defined by [32]

$$ {}_0D_t^{\alpha(x,t)}\, u(x,t) = \frac{1}{\Gamma(1-\alpha(x,t))} \int_0^t (t-s)^{-\alpha(x,t)} \frac{\partial u(x,s)}{\partial s}\, ds. \qquad (4) $$

It should be noted that the Du Fort and Frankel scheme first published in 1953 [8] and used for solving a lot of equations [12, 25, 40].

2 Description of the Method

In this section, the process of solving the variable order time fractional diffusion equations is considered. For describing the Du Fort-Frankel method we suppose that

$$x_i = ih, i = 0, 1, ..., N,$$

$$t_n = nk, n = 0, 1, ..., M.$$

The second-order spatial derivative can be approximated by the Du Fort-Frankel finite difference:

$$\frac{\partial^2 u(x_i,t_n)}{\partial x^2} = \frac{u(x_{i+1},t_{n-1}) - u(x_i,t_n) - u(x_i,t_{n-2}) + u(x_{i-1},t_{n-1})}{h^2} + O(h^2) + O(k^2) + O(\frac{k^2}{h^2}). \tag{5}$$

Now we discretize the variable order time fractional derivative as

$$D_t^{\alpha(x_i,t_n)} u(x_i,t_n) = \frac{k^{-\alpha(x_i,t_n)}}{\Gamma(2-\alpha(x_i,t_n))} \sum_{j=1}^{n} b_j(x_i,t_n) \left[u(x_i,t_{n-j+1}) - u(x_i,t_{n-j})\right], \tag{6}$$

where $b_j(x_i,t_n) = j^{1-\alpha(x_i,t_n)} - (j-1)^{1-\alpha(x_i,t_n)}$.
We supposed that $u(x_i,t_n) = u_i^n$, $\alpha(x_i,t_n) = \alpha_i^n$, $b_j(x_i,t_n) = b_j^{i,n}$ and $p(x_i,t_n) = p_i^n$ therefore, from (5) and (6) we obtain the following approximate method for Eq. (1):

$$\frac{k^{-\alpha_i^n}}{\Gamma(2-\alpha_i^n)}\{(u_i^n - u_i^{n-1}) + \sum_{j=2}^{n} b_j^{i,n}\left[u_i^{n-j+1} - u_i^{n-j}\right]\} = \frac{u_{i+1}^{n-1} - u_i^n - u_i^{n-2} + u_{i-1}^{n-1}}{h^2} + p_i^n, \tag{7}$$

if put $\gamma_i^n = 1 + \frac{h^2}{k^{\alpha_i^n}\Gamma(2-\alpha_i^n)}$ hence we have

$$\gamma_i^n u_i^n = (\gamma_i^n - 1)u_i^{n-1} + (1-\gamma_i^n)\sum_{j=2}^{n} b_j^{i,n}\left[u_i^{n-j+1} - u_i^{n-j}\right] + u_{i+1}^{n-1} - u_i^{n-2} + u_{i-1}^{n-1} + h^2\, p_i^n, \tag{8}$$

for $i = 1, 2, ..., N-1$ and $n = 1, 2, ..., M$.
Expression $\sum_{j=2}^{n} b_j^{i,n}\left[u_i^{n-j+1} - u_i^{n-j}\right]$ in (8) can be expand and rework by the following approch:

$$\begin{aligned}
&\sum_{j=2}^{n} b_j^{i,n}\left[u_i^{n-j+1} - u_i^{n-j}\right] \\
&= b_2^{i,n}(u_i^{n-1} - u_i^{n-2}) + b_3^{i,n}(u_i^{n-2} - u_i^{n-3}) + \ldots + b_{n-1}^{i,n}(u_i^2 - u_i^1) \\
&+ b_n^{i,n}(u_i^1 - u_i^0) \\
&= b_2^{i,n}\, u_i^{n-1} + (b_3^{i,n} - b_2^{i,n})u_i^{n-2} + (b_4^{i,n} - b_3^{i,n})u_i^{n-3} \\
&+ ... + (b_{n-1}^{i,n} - b_{n-2}^{i,n})u_i^2 + (b_n^{i,n} - b_n^{i,n})u_i^1 - b_n^{i,n}\, u_i^0 \\
&= b_2^{i,n}\, u_i^{n-1} - b_n^{i,n}\, u_i^0 + \sum_{j=2}^{n-1} d_{j+1}^{i,n}\, u_i^{n-j},
\end{aligned} \tag{9}$$

where $d_i^{i,n} = b_{nj}^{i,n} - b_{j-1}^{i,n}$. Now from (8) and (9) we obtain

$$\begin{aligned}
\gamma_i^n u_i^n = {} & (b_2^{i,n} - 1)(1-\gamma_i^n)\, u_i^{n-1} + u_{i+1}^{n-1} + u_{i-1}^{n-1} - u_i^{n-2} \\
& + (1-\gamma_i^n)\textstyle\sum_{j=2}^{n-1} d_{j+1}^{i,n}\, u_i^{n-j} + (\gamma_i^n - 1)b_n^{i,n}\, u_i^0 + h^2\, p_i^n,
\end{aligned} \tag{10}$$

for $i = 1, 2, ..., N-1$ and $n = 1, 2, ..., M$. Finally (10) can be rewrite by the following expression:

$$\left\{\begin{array}{lr}
\gamma_i^1 u_i^1 = (b_2^{i,1} - 1)(1-\gamma_i^1)\, u_i^0 + u_{i+1}^0 + u_{i-1}^0 - u_i^{-1} + h^2\, p_i^1, & n = 1, \\
\gamma_i^n u_i^n = (b_2^{i,n} - 1)(1-\gamma_i^n)\, u_i^{n-1} + u_{i+1}^{n-1} + u_{i-1}^{n-1} - u_i^{n-2} & \\
\quad + (1-\gamma_i^n)\sum_{j=2}^{n-1} d_{j+1}^{i,n}\, u_i^{n-j} + (\gamma_i^n - 1)b_n^{i,n}\, u_i^0 + h^2\, p_i^n, & 2 \le n \le M.
\end{array}\right. \tag{11}$$

3 Stability Analysis of the Scheme

In this section, the technique of Fourier analysis utilizes for discussing the stability of the approximate method (11). Let

$$w_l^n = u_l^n - U_l^n,$$

where U_l^n is approximate solution in (x_l, t_n) for $n = 1, 2, ..., M,\ l = 1, 2, ..., N-1$ and

$$w^n = \left[w_1^n, w_2^n, ..., w_{N-1}^n\right]^T.$$

Consider the following equation

$$\left\{ \begin{array}{ll} \gamma_l^1 w_l^1 = (b_2^{l,1} - 1)(1-\gamma_l^1)\, w_l^0 + w_{l+1}^0 + w_{l-1}^0 - w_l^{-1}, & n = 1, \\ \begin{array}{c} \gamma_l^n w_l^n = (b_2^{l,n} - 1)(1-\gamma_l^n)\, w_l^{n-1} + w_{l+1}^{n-1} + w_{l-1}^{n-1} - w_l^{n-2} \\ +(1-\gamma_l^n)\sum_{j=2}^{n-1} d_{j+1}^{l,n}\, w_l^{n-j} + (\gamma_l^n - 1) b_n^{l,n}\, w_l^0 \end{array}, & 2 \le n \le M \end{array} \right. \tag{12}$$

For $n = 1, 2, ..., M$ we define the following grid function:

$$w^n(x) = \left\{ \begin{array}{ll} w_l^n, & x_l - \frac{h}{2} < x \le x_l + \frac{h}{2}, \\ \\ 0, & 0 \le x \le \frac{h}{2} \ or\ L - \frac{h}{2} < x \le L. \end{array} \right.$$

Therefore, $w^n(x)$ can be expanded in a Fourier series:

$$w^n(x) = \sum_{m=-\infty}^{+\infty} \delta_n(m)\, e^{\frac{i2\pi m x}{L}}\ ,\ n = 1, 2, ..., M,$$

where

$$\delta_n(m) = \frac{1}{L}\int_0^L w^n(x)\ e^{\frac{i2\pi m x}{L}} dx.$$

It can be proved that [5]

$$\|w^n\|_2^2 = \sum_{m=-\infty}^{+\infty} |\delta_n(m)|^2. \tag{13}$$

Suppose that the solution of the (12) has the form
$w_l^n = \delta_n e^{i\sigma l h}$, where $\sigma = \frac{2\pi m}{L}$.

Substituting the above expression into (12) gives
for $n = 1$:

$$(\gamma_l^1 + 1)\delta_1 e^{i\sigma l h} = (b_2^{l,1} - 1)(3 - \gamma_l^1)\delta_0 e^{i\sigma l h} + \delta_0 e^{i\sigma(l+1)h} + \delta_0 e^{i\sigma(l-1)h}, \quad (14)$$

and for $n = 2, 3, .., M$

$$\begin{aligned} \gamma_l^n \, \delta_n \, e^{i\sigma l h} &= (b_2^{l,n} - 1)(1 - \gamma_l^n)\delta_{n-1} \, e^{i\sigma l h} + \delta_{n-1} \, e^{i\sigma(l+1)h} \\ &+ \delta_{n-1} e^{i\sigma(l-1)h} - \delta_{n-2} \, e^{i\sigma l h} \\ &+ (1 - \gamma_l^n) \sum_{j=2}^{n-1} d_{j+1}^{l,n} \, \delta_{n-j} \, e^{i\sigma l h} + (\gamma_l^n - 1) b_n^{l,n} \, \delta_0 \, e^{i\sigma l h}. \end{aligned} \quad (15)$$

Lemma 3.1. *Let δ_n be the solution of (14, 15), then*

$$|\delta_n| \leq |\delta_0|, \quad n = 1, 2, .., M.$$

Proof. For $n = 1$, in view of (14) , we have

$$\delta_1 = \frac{(b_2^{l,1} - 1)(3 - \gamma_l^1) + 2\cos(\sigma h)}{1 + \gamma_l^1} \, \delta_0.$$

Therefore, we obtain

$$|\delta_1| = |\frac{(b_2^{l,1} - 1)(3 - \gamma_l^1) + 2\cos(\sigma h)}{1 + \gamma_l^1} \, \delta_0| \leq \delta_0.$$

Now suppose that

$$|\delta_n| \leq |\delta_0|, \qquad m = 2, 3, ..., n-1,$$

by mathematical induction and view of (15) show that $|\delta_n| \leq |\delta_0|$, for $m = n$.

$$|\gamma_l^n \, \delta_n| = \left| \begin{array}{c} (b_2^{l,n} - 1)(1 - \gamma_l^n)\delta_{n-1} + \delta_{n-1} \, e^{i\sigma h} + \delta_{n-1} e^{-i\sigma h} \\ -\delta_{n-2} + (1 - \gamma_l^n) \sum_{j=2}^{n-1} d_{j+1}^{l,n} \, \delta_{n-j} + (\gamma_l^n - 1) b_n^{l,n} \delta_0 \end{array} \right|$$

$$\leq \left| (b_2^{l,n} - 1)(1 - \gamma_l^n) + 2\cos(\sigma h) - 1 + (1 - \gamma_l^n) \sum_{j=2}^{n-1} d_{j+1}^{l,n} + (\gamma_l^n - 1) b_n^{l,n} \right| |\delta_0|,$$

then we obtain

$$|\delta_n| \leq \left| \frac{(1-\gamma_l^n)\overbrace{\left[(b_2^{l,n}-1)+\sum_{j=2}^{n-1} d_{j+1}^{l,n} - b_n^{l,n}\right]}^{=1} - \sin^2(\frac{\sigma h}{2})}{\gamma_l^n} \right| \leq |\delta_0|.$$

The proof of Lemma (3.1) is completed. □

Pursuant to Lemma (3.1) and (13), it can be found that the solution of Eq. (12) satisfies

$$\|w^n\|_2 \leq \|w^0\|_2, \quad n = 1, 2, ..., M.$$

Hence, we obtain the following result:

Theorem 3.2. *The approximate method (11) is unconditionally stable.*

4 Numerical Examples

In this section of the paper, in order to show the ability and efficiency of the proposed scheme, we present two examples. Also, the results of the suggested method and other schemes are compared.

Example 4.1. Consider the following variable order time fractional diffusion equation: [33]

$$\begin{cases} {}_0D_t^{\alpha(x,t)}u(x,t) = K\frac{\partial^2 u(x,t)}{\partial x^2} + p(x,t), \;\; x \in (0, L], \;\; t \in (0, T], \\ \quad u(x,0) = 0\,, \quad x \in [0, L], \\ \quad u(0,t) = u(L,t) = 0, \quad t \in [0, T], \end{cases} \tag{16}$$

where $0 < \alpha(x,t) \leq 1$ for $\forall(x,t)$ and

$$p(x,t) = \frac{2}{\Gamma(3-\alpha(x,t))} t^{2-\alpha(x,t)} sin(\frac{x\pi}{L}) + \frac{K\pi^2 t^2}{L^2} sin(\frac{x\pi}{L}). \tag{17}$$

Table 1: Absolute errors of the proposed method and method of [33] for constant-order fractional diffusion equation at $x = 5$. The time step length $k = 0.01$, the space step size $h = 0.1$ and the time fractional $\alpha(x,t) = 0.8$.

Time	**Explicit method**	**Implicit method**	**Crank-Nicholson method**	**Proposed method**
t=0.1	0.007e-2	0.526e-3	0.465e-3	3.224e-5
t=0.2	0.314e-2	1.080e-3	0.919e-3	5.640e-5
t=0.3	1.114e-2	1.836e-3	1.167e-3	7.811e-5
t=0.4	2.556e-2	2.932e-3	1.109e-3	9.848e-5
t=0.5	4.766e-2	4.434e-3	0.620e-3	1.783e-5

The exact solution of this equation can be stated as

$$u(x,t) = t^2 sin(\frac{x\pi}{L}). \tag{18}$$

In this example we supposed that $k = 0.01$, $L = 10$ and $T = 0.5$. Comparison of the absolute errors of the proposed method and the method of [33] for $\alpha(x,t) = 0.8$, $\alpha(x,t) = 0.8 + \frac{0.2t}{T}$ and $\alpha(x,t) = 0.8 + \frac{0.2tx}{LT}$ is provided in Table 1-3 respectively.

Example 4.2. Consider the following variable order time fractional diffusion equation: [32]

$$\begin{cases} {}_0D_t^{\alpha(x,t)}\, u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} + p(x,t), \quad x \in (0,1], \ t \in (0,1], \\ u(x,0) = 10x^2(1-x), \quad x \in [0,1], \\ u(0,t) = u(1,t) = 0, \quad t \in [0,1], \end{cases} \tag{19}$$

where $\alpha(x,t) = \frac{2+sin(xt)}{4}$ and

$$p(x,t) = 20x^2(1-x)\left[\frac{t^{2-\alpha(x,t)}}{\Gamma(3-\alpha(x,t))} + \frac{t^{1-\alpha(x,t)}}{\Gamma(2-\alpha(x,t))}\right] - 20(t+1)^2(1-3x). \tag{20}$$

The exact solution is

$$u(x,t) = 10x^2(1-x)(t^2+1)^2. \tag{21}$$

Table 2: Absolute errors of the proposed method and method of [33] for variable order time fractional diffusion equation at $x = 5$. The time step length $k = 0.01$, the space step size $h = 0.1$ and $\alpha(x, t) = 0.8 + \frac{0.2t}{T}$.

Time	**Explicit method**	**Implicit method**	**Crank-Nicholson method**	**Proposed method**
t=0.1	0.012e-2	0.574e-3	0.418e-3	3.775-5
t=0.2	0.356e-2	1.304e-3	0.698e-3	7.623e-5
t=0.3	1.257e-2	2.453e-3	0.568e-3	1.279e-5
t=0.4	2.898e-2	4.282e-3	0.233e-3	1.937e-4
t=0.5	5.425e-2	7.118e-3	2.027e-3	2.821e-4

Table 3: Absolute errors of the proposed method and method of [33] for variable order time fractional diffusion equation at $x = 5$. The time step length $k = 0.01$, the space step size $h = 0.1$ and $\alpha(x, t) = 0.8 + \frac{0.2tx}{LT}$.

Time	**Explicit method**	**Implicit method**	**Crank-Nicholson method**	**Proposed method**
t=0.1	0.009e-2	0.550e-3	0.442e-3	3.469e-5
t=0.2	0.335e-2	1.188e-3	0.813e-3	6.571e-5
t=0.3	1.184e-2	2.132e-3	0.885e-3	9.984e-5
t=0.4	2.722e-2	3.556e-3	0.484e-3	1.346e-4
t=0.5	5.085e-2	5.650e-3	0.578e-3	1.812e-4

Table 4: The error, numerical solution and exact solution, when $t = 1, h = 0.1, k = 0.001$.

Space ($\mathbf{x}_i$)	**method of ref [12]**	**Present method**
0.1000	0.00002996	0.00004895
0.2000	0.00005972	0.00005532
0.3000	0.00008803	0.00007239
0.4000	0.00011251	0.00006591
0.5000	0.00012981	0.00001013
0.6000	0.00013595	0.00001771
0.7000	0.00012705	0.00001083
0.8000	0.00010048	0.00002839
0.9000	0.00005643	0.00008319

In table 4, comparison of the absolute errors of the proposed method and the method of [32] is provided.

5 Conclusion

In this paper, Du Fort-Frankel finite difference method for the variable order time fractional diffusion equation has been proposed. The stability of the numerical scheme has been considered by the technique of Fourier analysis. Two numerical examples have been given. The results of the proposed method and some other methods have been compared and the results have indicated the effectiveness of the theoretical analysis. All things considered, the proposed method can be used for other equations in the fields of physics and engineering.

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