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On a Uniform Fuzzy Direct Method for Stability of Functional Equations

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Abstract. In this paper, we consider the Hyers-Ulam-Rassias stability of the following equation

$$f(x) = af(h(x)) + bf(-h(x))$$

in the fuzzy normed spaces with some conditions imposed on the constants a, b and the function h on a nonempty set X.

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1. Introduction

In 1940, the problem of the stability of functional equations has been first posed by Ulam ([12]). In 1941, Hyers ([3]) showed that if $\delta > 0$ and $f : E_1 \to E_2$ is a mapping between Banach spaces E_1 and E_2 with $||f(x+y) - f(x) - f(y)|| \leqslant \delta$ for all $x, y \in E_1$, then there exists a unique additive mapping $T : E_1 \to E_2$ such that $||f(x) - T(x)|| \leqslant \delta$ for all $x, y \in E_1$. Aoki ([1]), in 1950 investigated about the stability problem for linear maps in Banach spaces. In 1978, a generalized solution to Ulam's

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problem for approximately linear mappings was given by Th. M. Rassias [10].

Fuzzy notion introduced firstly by Zadeh in [14]. Later, in 1984, Katsaras [4] defined a fuzzy norm on a linear space to construct a fuzzy vector topological structure on the space. For more details of stability problem in fuzzy normed spaces see [5,6,7,9].

In 2012, J. Sikorska [11] provided the stability of the functional equation $f(x) = af(h(x)) + bf(-h(x))$ for the constants a, b and the function h on a nonempty set X. Here, by defining the class of approximate solutions of a given functional equation, one can ask if every mapping from this class can be somehow approximated by an exact solution of the considered equation in the fuzzy Banach space. To answer this question, we use the definition of fuzzy normed spaces given in [4] to exhibit some reasonable notions of fuzzy direct method and we will prove that under some suitable conditions an approximate function f from space X into a fuzzy Banach space Y can be approximated in a fuzzy sense by an exact solution T from X to Y.

2. Preliminaries

For the sake of simplicity we provide some definitions and theorems.

Definition 2.1. ([2])*Let X be a real linear space. A function $N : X \times \mathbb{R} \to [0, 1]$ is said to be a fuzzy norm on X, if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,*
(N_1) $N(x, 0) = 0$ *for* $c \leqslant 0$*;*
(N_2) $x = 0$ *if and only if* $N(x, c) = 1$ *for all* $c > 0$*;*
(N_3) $N(cx, t) = N(x, \frac{t}{|c|})$ *if* $c \neq 0$*;*
(N_4) $N(x + y, s + t) \geqslant \min\{N(x, s), N(y, t)\}$*;*
(N_5) $N(x, .)$ *is a non-decreasing function on* $\mathbb{R}$ *and* $\lim_{t\to\infty} N(x, t) = 1$*;*
(N_6) *for* $x \neq 0$, $N(x, .)$ *is (upper semi) continuous on* $\mathbb{R}$.

The pair (X, N) is called a fuzzy normed linear space.

Example 2.2. Let X be a linear space and $x \to ||x||$ from X t $[0, \infty)$

be a mapping. Define

$$N(x,t)=\begin{cases} 0, & t\leqslant 0;\\ \frac{t}{t+||x||}, & t>0. \end{cases}$$

It is easy too see that N is a fuzzy norm on X.

Example 2.3. Let X be the space of complex-valued continuous functions on the real line. Then X is not normable [13]. Define

$$N(f,t)=\begin{cases} 0, & t\geqslant 0;\\ sup\{\frac{n}{n+1}:||f||_n\leqslant t\}, & t<0, \end{cases}$$

where $||.||_n$ denotes the sup-norm on $[-n,n], n\in\mathbb{N}$. By [7], an easy computation shows that (X,N) is a fuzzy normed algebra.

Definition 2.4. ([2]) *Let (X,N) be fuzzy normed linear space, and $\{x_n\}$ a sequence in X. Then $\{x_n\}$ is said to be convergent if there exists $x\in X$ such that $\lim_{n\to\infty}N(x_n-x,t)=1$, for all $t>0$. In that case, x is called the limit of the sequence $\{x_n\}$ and we denote it by $N-\lim_{n\to\infty}x_n=x$.*

Definition 2.5. *A sequence $\{x_n\}$ in X is called Cauchy, if for each $\varepsilon>0$ and each $t>0$, there exists a positive integer n_0 such that for all $n\geqslant n_0$ and all $p>0$, we have $N(x_{n+p}-x_n,t)>1-\varepsilon$.*

It is known that every convergent sequence in a fuzzy normed space is Cauchy and if each Cauchy sequence is convergent, then, the fuzzy norm is said to be complete and furthermore the fuzzy normed space is called a fuzzy Banach space.

Theorem 2.6. ([8]) *Let X be a linear space and (Y,N) a fuzzy Banach space. Let $\varphi:X\times X\to[0,\infty)$ be a control function such that*

$$\widetilde{\varphi}(x,y)=\sum_{n=0}^{\infty}2^{-n}\varphi(2^nx,2^ny) \qquad (x,y\in X).$$

Assume that $f:X\to Y$ is a uniformly approximately additive mapping with respect to φ in the sense that:

$$\lim_{n\to\infty}N(f(x+y)-f(x)-f(y),t\varphi(x,y))=1,$$

uniformly on $X \times X$*. Then, for each* $x \in X$, $T(x) := N - \lim_{n \longrightarrow \infty} \frac{f(2^n x)}{2^n}$ *exists and defines an additive mapping* $T : X \to Y$*. Also, if for some* $\delta > 0, \alpha > 0$

$$N(f(x+y) - f(x) - f(y), \delta\varphi(x, y)) > \alpha, \text{ where } x \in X \text{ and } t > 0$$

then, we have

$$N(f(x) - T(x), \tfrac{\delta}{2}\widetilde{\varphi}(x, y)) > \alpha, \ x \in X.$$

3. Main Results

In this section, we deal with uniform fuzzy version of the stability of the equation

$$f(x) = af(h(x)) + bf(-h(x)),$$

in which a, b are constants and h is a function on a nonempty set X.

Theorem 3.1. *Let* X *be a nonempty set with an involution denoted by a map from* X *into* X *such that* $-(-(x)) = x$ *for all* $x \in X$*, and* (Y, N) *a fuzzy Banach space. Let* φ*:* $X \to [0, \infty)$ *be a control function such that*

$$\widetilde{\varphi}(x) = \sum_{k=0}^{\infty}[\mid \alpha_k \mid \varphi(h^k(x)) + \mid \beta_k \mid \varphi(-h^k(x))] < \infty, \qquad x \in X. \quad (1)$$

Consider the mapping f: $X \to Y$ *such that the following condition holds:*

$$\lim_{t \to \infty} N(f(x) - \alpha f(h(x)) - \beta f(-h(x)), t\varphi(x)) = 1, \qquad x \in X \qquad (2)$$

where α *and* β *are real constants, and h:* $X \to X$ *is an odd function. Set*

$$\alpha_0 := 1, \alpha_n = 1/2[(\alpha + \beta)^n + (\alpha - \beta)^n],$$

$$\beta_0 := 0, \beta_n = 1/2[(\alpha + \beta)^n - (\alpha - \beta)^n].$$

Then there exists a uniquely determined function T: $X \to Y$ *such that, for all* $x \in X$

$$T(x) = N - \lim_{k\to\infty}(\alpha_k f(h^k(x)) + \beta_k f(-h^k(x))).$$

Also, if for some $\delta > 0, \gamma > 0$

$$N(f(x)) - \alpha f((h(x)) - \beta f(-h(x), \delta\varphi(x)) > \gamma, \qquad x \in X \qquad (3)$$

then

$$N(T(x) - f(x), \tfrac{\delta}{2}\widetilde{\varphi}(x)) > \gamma, \qquad x \in X.$$

Proof. Given $\varepsilon > 0$, by Definition 2.1, we can find some $t_0 > 0$ such that

$$N(f(x)) - \alpha f((h(x)) - \beta f(-h(x), t\varphi(x)) \geqslant 1 - \varepsilon, \qquad (4)$$

for all $x \in X$ and $t \geqslant t_0$. By induction on n, we shall show that

$$N(f(x) - \alpha_n f(h^n(x)) - \beta_n f(-h^n(x))), \qquad (5)$$

$$t\sum_{k=0}^{n-1}[\mid \alpha_k \mid \varphi(h^k(x)) + \mid \beta_k \mid \varphi(-h^k(x))]) \geqslant 1 - \varepsilon,$$

for all $t > t_0$, $x \in X$ and all positive integers n. Putting $n = 1$ in (5), we get (4). Let (5) holds for some positive integer n. It is clear that

$$\alpha_{n+1} = \alpha\alpha_n + \beta\beta_n, \qquad \beta_{n+1} = \alpha\beta_n + \beta\alpha_n,$$

and

$$\alpha_{n+m} = \alpha_m\alpha_n + \beta_m\beta_n, \qquad \beta_{n+m} = \alpha_m\beta_n + \beta_m\alpha_n.$$

Now, we have
$N(f(x) - \alpha_{n+1}f(h^{n+1}(x)) - \beta_{n+1}f(-h^{n+1}(x)), t\sum_{k=0}^{n}[\mid \alpha_k \mid \varphi(h^k(x)) + \mid \beta_k \mid \varphi(-h^k(x))]) = N(f(x) - \alpha\alpha_n - f(h^{n+1}(x)) - \beta\beta_n f(h^{n+1}(x)) - \alpha\beta_n f(-h^{n+1}(x)) - \beta\alpha_n f(-h^{n+1}(x)), t\sum_{k=0}^{n-1}[\mid \alpha_k \mid \varphi(h^k(x)) + \mid \beta_k \mid \varphi(-h^k(x))] + [\mid \alpha_n \mid \varphi(h^n(x)) + \mid \beta_n \mid \varphi(-h^n(x))]) \geqslant \min\{N(f(x) - \alpha_n f(h^n(x)) - \beta_n f(-h^n(x)), t\sum_{k=0}^{n-1}[\mid \alpha_k \mid \varphi(h^k(x)) + \mid \beta_k \mid \varphi(-h^k(x))]), N(f(h^n(x)) - \alpha f(h^{n+1}(x)) - \beta f(-h^{n+1}(x)), t\varphi(h^n(x))),$
and

$$N(f(-h^n(x)) - \alpha f(-h^{n+1}(x)) - \beta f(h^{n+1}(x)), t\varphi(-h^n(x)))\} \geqslant 1-\varepsilon.$$

This completes the induction argument. Putting $n = n+p$ in (5), we obtain

$$N(f(x) - \alpha_{n+p} f(h^{n+p}(x)) - \beta_{n+p} f(-h^{n+p}(x)), \tag{6}$$

$$t\textstyle\sum_{k=0}^{n+p}[\mid \alpha_k \mid \varphi(h^k(x)) + \mid \beta_k \mid \varphi(-h^k(x))]) \geqslant 1-\varepsilon.$$

By the item (N_4) of Definition 2.1, we have

$N(\alpha_{n+p} f(h^{n+p}(x)) + \beta_{n+p} f(-h^{n+p}(x)) - \alpha_n f(h^n(x)) -$

$\beta_n f(-h^n(x)), t\sum_{k=0}^{n+p-1}[\mid \alpha_k \mid \varphi(h^k(x)) + \mid \beta_k \mid \varphi(-h^k(x))] + t\sum_{k=0}^{n-1}[\mid$

$\alpha_k \mid \varphi(h^k(x)) + \mid \beta_k \mid \varphi(-h^k(x))]) \geqslant \min\{N(f(x) - \alpha_{n+p} f(h^{n+p}(x)) -$

$\beta_{n+p} f(-h^{n+p}(x)), t\sum_{k=0}^{n+p-1}[\mid \alpha_k \mid \varphi(h^k(x)) + \mid \beta_k \mid$

$\varphi(-h^k(x))]), N(f(x) - \alpha_n f(h^n(x)) - \beta_n f(-h^n(x)), t\sum_{k=0}^{n-1}[\mid \alpha_k \mid$

$\varphi(h^k(x)) + \mid \beta_k \mid \varphi(-h^k(x))])\} \geqslant 1-\varepsilon,$

for all integers $n \geqslant 0$ and $p > 0$. For given $\delta > 0$, there is $n_0 \in \mathbb{N}$ such that

$$2t\textstyle\sum_{k=0}^{n-1}[\mid \alpha_k \mid \varphi(h^k(x)) + \mid \beta_k \mid \varphi(-h^k(x))] + t\sum_{k=n}^{n+p-1}[\mid \alpha_k \mid \\ \varphi(h^k(x)) + \mid \beta_k \mid \varphi(-h^k(x))] < \delta,$$

for all $n \geqslant n_0$ and $p > 0$. Now we deduce from (6) that

$$N(\alpha_{n+p} f(h^{n+p}(x)) + \beta_{n+p} f(-h^{n+p}(x)) - \alpha_n f(h^n(x)) - \beta_n f(-h^n(x)), \delta)$$

$$\geqslant N(\alpha_{n+p} f(h^{n+p}(x)) + \beta_{n+p} f(-h^{n+p}(x)) - \alpha_n f(h^n(x)) - \beta_n f(-h^n(x)),$$

$$t\sum_{k=0}^{n+p-1}[\mid \alpha_k \mid \varphi(h^k(x)) + \mid \beta_k \mid \varphi(-h^k(x))] + t\sum_{k=0}^{n-1}[\mid \alpha_k \mid \varphi(h^k(x)) +$$

$$\mid \beta_k \mid \varphi(-h^k(x))]) \geqslant 1-\varepsilon.$$

It is clear that $\{\alpha_n f(h^n(x)) + \beta_n f(-h^n(x))\}$ is a Cauchy sequence in Y. Since Y is a fuzzy Banach space, this sequence converges to some $T(x) \in Y$, such that T is defined by

$$T(x) := N - \lim_{n\to\infty}(\alpha_n f(h^n(x)) + \beta_n f(-h^n(x))).$$

Next, let $x \in X$. Temporality fix $t > 0$ and $0 < \varepsilon < 1$. Since

$$\lim_{n\to\infty}[| \alpha_n | \varphi(h^n(x)) + | \beta_n | \varphi(-h^n(x))] = 0,$$

there are some $n_1 > n_0$ and $n_2 > n_0$ such that $t_0\varphi(-h^n(x) < \frac{t}{5|\beta_n|}$ and $t_0\varphi(h^n(x) < \frac{t}{5|\alpha_n|}$ for all $n \geqslant \max\{n_1, n_2\}$. We have

$$\begin{aligned}
&N(T(x) - \alpha T(h(x)) - \beta T(-h(x)), t)\\
&\geqslant \min\{N(T(x) - \alpha_n f(h^n(x)) - \beta_n f(-h^n(x)), \frac{t}{5}),\\
&N(T(h(x)) - \alpha_n f(h^{n+1}(x)) - \beta_{n+1} f(-h^{n+1}(x)), \frac{t}{5\alpha}),\\
&N(T(-h(x)) - \alpha_n f(-h^{n+1}(x)) - \beta_n f(h^{n+1}(x)), \frac{t}{5\beta}),\\
&N(f(h^n(x)) - \alpha f(h^{n+1}(x)) - \beta f(-h^{n+1}(x)), \frac{t}{5\alpha_n}),\\
&N(f(-h^n(x)) - \alpha f(-h^{n+1}(x)) - \beta f(h^{n+1}(x)), \frac{t}{5\beta_n})\}.
\end{aligned}$$

The first three terms on the right-hand side of the above inequality tend to 1 as $n \to \infty$, and the forth term is greater than

$$N(f(h^n(x)) - \alpha f(h^{n+1}(x)) - \beta f(-h^{n+1}(x)), \frac{t}{5\alpha_n}),$$

$$N(f(-h^n(x)) - \alpha f(-h^{n+1}(x)) - \beta f(h^{n+1}(x)), \frac{t}{5\beta_n})\},$$

which is, by (2), greater than or equal to $1 - \varepsilon$. Thus for all $t > 0$ and $0 < \varepsilon < 1$, $N(T(x) - \alpha T(h(x)) - \beta T(-h(x)), t) \geqslant 1 - \varepsilon$. It follows that $N(T(x) - \alpha T(h(x)) - \beta T(-h(x)), t) = 1$ for all $t > 0$. So, by item (N_2) of Definition 2.1, we have $T(x) = \alpha T(h(x)) + \beta T(-h(x))$.

To end the proof, let (3) holds for some positive δ and γ. Let

$$\varphi_(x) = \sum_{k=0}^{n-1}[| \alpha_k | \varphi(h^k(x)) + | \beta_k | \varphi(-h^k(x))].$$

By a similar argument as in the beginning of the proof, one can conclude from (3) that:

$$N(f(x) - \alpha_n f(h^n(x)) - \beta_n f(-h^n(x)), \delta \sum_{k=0}^{n-1} [\mid \alpha_k \mid \varphi(h^k(x)), \qquad (7)$$

$$+ \mid \beta_k \mid \varphi(-h^k(x)]) \geqslant \gamma,$$

for all positive integers n. Let $s > 0$, we have

$$N(f(x)-T(x), \delta\varphi_n(x)+s) \geqslant min\{N(f(x)-\alpha_n f(h^n(x))-\beta_n f(-h^n(x)),$$
$$\delta\varphi_n(x)), N(T(x) - \alpha_n f(h^n(x)) - \beta_n f(-h^n(x)), s)\}. \qquad (8)$$

Combining (7), (8) and the fact that

$$\lim_{n\to\infty} N(\alpha_n f(h^n(x)) + \beta_n f(-h^n(x)) - T(x), s) = 1,$$

we observe that

$$N(f(x) - T(x), \delta\varphi_n(x) + s) \geqslant \gamma,$$

for large enough n. Regarding to the (upper semi) continuity of the real function $N(f(x) - T(x), .)$, we see that

$$N(f(x) - T(x), \tfrac{\delta}{2}\widetilde{\varphi}(x) + s) \geqslant \gamma.$$

Letting $s \to 0$, we conclude that

$$N(f(x) - T(x), \tfrac{\delta}{2}\widetilde{\varphi}(x)) \geqslant \gamma. \quad \square$$

Theorem 3.2. ([6]) *Let (X, N) be a fuzzy normed linear space such that 1) $N(x, t) > 0$ for each $t > 0$ implies that $x = 0$ and*
2) $N(x, t)$ is a continuous function of $\mathbb{R}$ and strictly increasing on $\{t : 0 < N(x, t) < 1\}$ for each nonzero $x \in X$.
Let $||x||_\nu = inf\{t : N(x, t) \geqslant \nu\}(\nu \in (0, 1), x \in X)$ and $N_1 : X \times \mathbb{R} \to [0, 1]$ be defined by

$$N_1(x, t) = \begin{cases} 0, & (x, t) = (0, 0); \\ sup\{\nu \in (0, 1) : ||x||_\nu \leqslant t\}, & (x, t) \neq (0, 0). \end{cases}$$

Then

(i) $\{||.||_\nu\}_{\nu\in(0,1)}$ *is an increasing family of norms on the linear space* X*.*

(ii) $N = N_1$*.*

Moreover, (X, N) *is complete if and only if* $(X, ||.||_N)$ *is complete for each* $\nu \in (0, 1)$*.*

The following result gives a relation between fuzzy and non fuzzy stability.

Theorem 3.3. *Let conditions of Theorem 3.2 hold and let* N *and* N' *be the norms satisfying the conditions of Theorem 3.2. If* $\{||.||_\nu\}_{\nu\in(0,1)}$ *and* $\{||.||'_\nu\}_{\nu\in(0,1)}$ *are the increasing norms corresponding to* N *and* N'*, respectively, then there is a unique mapping* T *such that for each* $\nu \in (0, 1)$*,*

$$||f(x) - T(x)||_\nu \leqslant \tfrac{\delta||\widetilde{\varphi}(x)||'_\nu}{2}.$$

Proof. By Theorem 3.1, there is a unique mapping T such that

$$N(T(x) - f(x), \tfrac{\delta}{2}\widetilde{\varphi}(x)) > \gamma, \qquad x \in X.$$

By definition of $||.||_\nu$ and $||.||'_\nu$,

$$||f(x) - T(x)||_\nu \leqslant \tfrac{\delta||\widetilde{\varphi}(x)||'_\nu}{2}, \qquad x \in X.$$

If $T' : X \to Y$ satisfies

$$||f(x) - T'(x)||_\nu \leqslant \tfrac{\delta||\widetilde{\varphi}(x)||'_\nu}{2}, \qquad x \in X,$$

for each $\nu \in (0, 1)$, then by definition and property (ii) of Theorem 3.2,

$$N(f(x) - T'(x), t) \geqslant N'(\tfrac{\delta\widetilde{\varphi}(x)}{2}, t), \qquad x \in X, t > 0.$$

Hence $T = T'$. □

Corollary 3.4. *Let* $(X, +)$ *be a group and* (Y, N) *be a fuzzy Banach space. Assume that f:* $X \to Y$ *is a function such that the following condition holds:*

$$\lim_{t\to\infty} N(f(x) - \frac{a+1}{2a^2}f(ax) + \frac{a-1}{2a^2}f(-ax), t\varphi(x)) = 1, \qquad x \in X, \tag{9}$$

where a is a positive integer different from 1, and φ: $X \to [0, \infty)$ is a control function such that the series $\sum_{i=0}^{\infty} \frac{1}{a^i}\varphi(a^i x)$ is convergent for any $x \in X$. Then there exists a uniquely determined function T: $X \to Y$ such that

$$T(x) = N - \lim_{k\to\infty}(\tfrac{a^n+1}{2a^{2n}} f(a^n x) - \tfrac{a^n-1}{2a^{2n}} f(-a^n x)) \text{ for all } x \in X.$$

Also, if for some $\delta > 0, \gamma > 0$

$$N(f(x) - \frac{a+1}{2a^2} f(ax) + \frac{a-1}{2a^2} f(-ax), \delta\varphi(x)) > \gamma, \qquad x \in X \quad (10)$$

then

$$N(T(x) - f(x), \tfrac{\delta}{2}\widetilde{\varphi}(x)) > \gamma, \qquad x \in X.$$

Proof. It is enough to apply Theorem 2.1 with $\alpha := \frac{a+1}{2a^2}$, $\beta := \frac{a-1}{2a^2}$ and $h(x) := ax$. It is easy to see that

$$\alpha_n := \frac{a^n+1}{2a^{2n}}, \qquad \beta_n := \frac{a^n-1}{2a^{2n}}, \qquad n \in \mathbb{N}.$$

Since the series $\sum_{i=0}^{\infty} \frac{1}{a^i}\varphi(a^i x)$ is convergent, so $\sum_{i=0}^{\infty} \frac{1}{a^{2i}}\varphi(a^i x)$ is convergent too, and by Theorem 2.1, there exists a uniquely $T : X \to Y$ such that

$$T(x) = N - \lim_{k\to\infty}(\tfrac{a^n+1}{2a^{2n}} f(a^n x) - \tfrac{a^n-1}{2a^{2n}} f(-a^n x)),$$

for all $x \in X$, and

$$T(x) = \frac{a+1}{2a^2} T(ax) + \frac{a-1}{2a^2} T(-ax),$$

If for some $\delta > 0$ and $\gamma > 0$,

$$N(f(x) - \frac{a+1}{2a^2} f(ax) + \frac{a-1}{2a^2} f(-ax), \delta\varphi(x)) > \gamma, \qquad x \in X,$$

then

$$N(T(x) - f(x), \tfrac{\delta}{2}\widetilde{\varphi}(x)) > \gamma, \qquad x \in X. \quad \square$$

Corollary 3.5. *Let (X,+) be a group and (Y,N) a fuzzy Banach space. Assume that $f : X \to Y$ is a function such that the following condition holds:*

$$\lim_{t\to\infty} N(f(x) - \frac{a+1}{2a^2} f(ax) + \frac{a-1}{2a^2} f(-ax), t\varphi) = 1, \qquad x \in X \qquad (11)$$

where a is an integer different from 1,-1,0 and φ is a nonnegative constant. Then there exists a uniquely determined function T: $X \to Y$ such that

$$T(x) = N - \lim_{k\to\infty} (\tfrac{|a|^n+1}{2|a|^{2n}} f(| a |^n x) - \tfrac{|a|^n-1}{2|a|^{2n}} f(- | a |^n x))$$

for all $x \in X$. Also, if for some $\delta > 0$ and $\gamma > 0$,

$$N(f(x) - \tfrac{|a|+1}{2|a|^2} f(| a | x) + \tfrac{|a|-1}{2|a|^2} f(- | a | x), \delta\varphi) > \gamma, \qquad x \in X$$

then

$$N(T(x) - f(x), \tfrac{\delta}{2} \textstyle\sum_{i=0}^{\infty} \tfrac{1}{a^{2i}} \varphi) > \gamma, \qquad x \in X.$$

Corollary 3.6. *Let $(X, +)$ be a uniquely $2-$divisible group and (Y,N) a fuzzy Banach space. Assume that f: $X \to Y$ is a function such that the following condition holds:*

$$\lim_{t\to\infty} N(f(x) - \tfrac{a^2+a}{2} f(\tfrac{1}{a}x) - \tfrac{a^2-a}{2} f(-\tfrac{1}{a}x), t\varphi(x)) = 1, \qquad x \in X$$

where a is a positive integer different from 1, and φ: $X \to [0, \infty)$ is a control function such that the series $\sum_{i=0}^{\infty} a^{2i} \varphi(\frac{1}{a^i}x)$ is convergent for any $x \in X$. Then there exists a uniquely determined function T: $X \to Y$ such that

$$T(x) = N - \lim_{k\to\infty} \tfrac{a^2+a}{2} T(\tfrac{1}{a}x) - \tfrac{a^2-a}{2} T(-\tfrac{1}{a}x)$$

for all $x \in X$. Also, if for some $\delta > 0$ and $\gamma > 0$,

$$N(f(x) - \tfrac{a^2+a}{2} f(\tfrac{1}{a}x) - \tfrac{a^2-a}{2} f(-\tfrac{1}{a}x), \delta\varphi(x)) > \gamma, \qquad x \in X$$

then

$$N(T(x) - f(x), \tfrac{\delta}{2}\widetilde{\varphi}(x)) > \gamma, \qquad x \in X.$$

Proof. It is enough to apply Theorem 2.1 with $\alpha\colon=\frac{a^2+a}{2}$, $\beta\colon=\frac{a^2-a}{2}$ and $h(x)\colon=\frac{1}{a}x$. It is easy to see that $\alpha\colon=\frac{a^{2n}+a^n}{2}$, $\beta\colon=\frac{a^{2n}-a^n}{2}$. Since the series $\sum_{i=0}^{\infty}a^{2i}\varphi(\frac{1}{a^i}x)$ is convergent, so, by Theorem 2.1, there exists a uniquely mapping $T\colon X\to Y$ such that

$$T(x)=N-\lim_{k\to\infty}\tfrac{a^{2n}+a^n}{2}T(\tfrac{1}{a^n}x)-\tfrac{a^{2n}-a^n}{2}T(-\tfrac{1}{a^n}x),$$

for all $x\in X$, and

$$T(x)=\frac{a+1}{2a^2}T(ax)+\frac{a-1}{2a^2}T(-ax).$$

If for some $\delta>0$ and $\gamma>0$

$$N(f(x)-\frac{a^2+a}{2}f(\frac{1}{a}x)-\frac{a^2-a}{2}f(-\frac{1}{a}x),\delta\varphi(x))>\gamma,\qquad x\in X$$

then

$$N(T(x)-f(x),\tfrac{\delta}{2}\widetilde{\varphi}(x))>\gamma,\qquad x\in X.\quad\square$$

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