Journal of Mathematical Extension
Vol. 17, No. 2, (2023) (6)1-30
URL: https://doi.org/10.30495/JME.2023.2417
ISSN: 1735-8299
Original Research Paper

A Novel Fully Fuzzy DEA Approach for Measuring Cost and Revenue Efficiency with Target Setting

J. Gerami*
Shiraz Branch, Islamic Azad University

Abstract. In this paper, we present cost and revenue efficiency evaluation models and target setting in data envelopment analysis (DEA) in the presence of fuzzy inputs and outputs that the corresponding prices of inputs and outputs are also fuzzy numbers. We proposed a fuzzy value-based technology based on fuzzy input and output data at corresponding prices. We provide an approach for calculating fuzzy cost (revenue) efficiency based on value fuzzy based technology. The proposed fully fuzzy model is transformed into a four-objective model of non-fuzzy linear programming and solved by the weighted sum method. We show that the proposed approach is suitable than the previous approaches based on traditional models from a computational point of view. The innovation of this research is to present and solve a fully fuzzy model to obtain fuzzy cost (revenue) efficiency score as a fuzzy number and we do not need additional comparisons to detect the efficient unit in previous approaches. We also obtain benchmark corresponding to the all DMUs. In the following, with two numerical examples, we obtain the results of the presented approach and compare it with the results of the previous approaches, and at the end, we present the results of the research.

AMS Subject Classification: 90C05; 90C08.
Keywords and Phrases: Data envelopment analysis; Fully fuzzy DEA; Fuzzy cost efficiency; Fuzzy revenue efficiency; Target setting.

Received: June 2021; Accepted: December 2022
*Corresponding Author

1 Introduction

DEA is a non-parametric method for evaluating the performance of a set of homogeneous DMUs. This method was first proposed by Charnes et al. [5]. This technique is based on linear programming. DEA, by accepting the underlying assumptions for estimating the production possibility set (PPS), including: inclusion of observations axiom, possibility (free-disposal) axiom, convexity axiom, the DEA enables the production of a set called the PPS, and considers the frontier of this set as the efficient frontier. DMUs that are on this frontier are efficient units, these units are the units that produce the most output with the least amount of input. Other units are inefficient units. Traditional DEA models in the envelopment form depicts inefficient units on the efficiency frontier based on decreasing inputs (input-oriented models) or increasing outputs (output-oriented models) and obtaining the corresponding efficiency score for each DMU. These models propose a benchmark corresponding to each inefficient DMUs (Zhu [35], Cooper et al. [6]). There are several ways to project inefficient DMUs on the efficiency frontier, which can be referred to as the following approaches Radial efficiency approach (Korhonen et al. [22]), the multidirectional efficiency approach (Nasseri et al. [24]) or the potential efficiency approach (Hatami-Marbini, et al.[15]), etc. In the basic models of DEA, the input and output data are exact numbers. But in the real world, some input or output data may be often imprecise. One way to deal with this uncertainty is to use fuzzy sets. Therefore, imprecise and vague data in DEA can be represented as logical expressions with fuzzy numbers (Hatami-Marbini, et al. [15]). The different approaches in DEA are presented to deal with fuzzy input and output data (Emrouznejad et al. [8]). In the following, we will introduce some of the available approaches for solving fuzzy DEA models. -level set approach is one of the common methods in solving fuzzy DEA (FDEA) models. This method presents a pair of parametric programming problems corresponding to each -level. Kao and Liu [20] used the -level set approach to solve FDEA models. Their method became a widely used method in solving FDEA models. Saati and Memariani [30] used the α-level set approach to solve the fuzzy SBM model based on the constant returns to scale technology. Ghasemi et al. [14] used fuzzy ranking methods to solve FDEA models. Lozano [23] proposed fuzzy network DEA

(FNDEA) evaluation models. In all the studies presented above, only part of the model is fuzzy and not all parts are simultaneously fuzzy. Hosseinzadeh et al. [16] presented a fully FDEA model in which all variables and parameters were fuzzy and the numbers are the triangular fuzzy numbers. They used multi-objective linear programming (MOLP) to solve their model. Also, each fuzzy number was approximated to the nearest symmetric triangular fuzzy number. The proposed approach by them also placed into the category of the fuzzy ranking approach. Kumar et al. [21] tried to solve the problems with the model presented by Hosseinzadeh et al.[16]. The problem with their method was the lack of an accurate solution. Hatami-Marbini et al. [15] reviewed two decades of research and advances in FDEA. They examined FDEA methods in four different classes. Ezzati et al. [9] examined the problem of fully fuzzy linear programming and defined a new order on fuzzy numbers. They tried to turn a fully fuzzy linear programming problem into a MOLP problem and calculate the optimal solutions using lexicography. Bhardwaj and Kumar [4] showed that in the presence of unequal constraints in the model of Ezzati et al. [9], optimal solutions may not be obtained. They consider all the parameters and variables as fuzzy, also all the constraints are equal. They apply the method proposed by Ezzati et al. [9] in the paper. Hosseinzadeh and Edalatpanah [17] proposed a method to solve the problem of fully fuzzy linear programming with fuzzy L-R numbers, which used MOLP and lexicography to solve it. Ruiz and Sirvent [29] examined the concept of fuzzy cross efficiency and proposed the possibility approach based on benevolent and aggressive possibility-level cross efficiency scores. They presented their model in the multiplier form and in the input orientation and in constant returns to scale technology and in the form of radial. Barak and Heidary Dahooei [3] evaluated the security of 7 different airlines in Iran. In this evaluation, each airline is considered as a DMU and FDEA is used to calculate the weight of the criteria. Finally, the airlines were ranked and the safest were introduced. Zhu et al. [36] have used a method with triangular fuzzy data to evaluate the performance of teaching in colleges, which has led to a significant practical improvement in the university. Izadikhah and Khoshroo [18] proposed a novel FDEA model in the form of non-radial, non-oriented modified ERGM envelopment formulation; in CRS technology in the presence of undesirable outputs Non-radial, non-oriented modified ERGM envelopment formulation; in constant returns to scale technology; undesirable outputs; possibility level super-efficiency scores. Wang et al. [34] proposed a two-stage granular consensus model for minimum adjustment and minimum cost under Pythagorean fuzzy linguistic information. They concentrate on designing a two-stage consensus optimization model combining minimum adjustment and minimum cost under Pythagorean fuzzy linguistic preference information in order to realize the consensus of the group decision-making problems in a complex and uncertain environment. The designed model by them satisfies the need for minimum costs for the mediator and considers experts' adjustment amount to shorten the time consumption, retain initial preference, and maximize the balance between the minimum amount of adjustment and the minimum cost. Zou et al. [37] developed a life-cycle cost model for evaluation system for power grid assets based on fuzzy membership degree. They propose a life-cycle cost assessment model for the management of electric power plant equipment during its service life. They used a membership function method based on fuzzy logic to improve the allocation of modernization and overhaul projects to multiple equipment assets. Arana-Jim´enez et al. [1] proposed a fully FDEA model to obtain efficiency score and targets in the presence of fuzzy data. They presented a two-step method for solving their model, the models presented by them in each stage was a multi-objective model, and they used the lexicographic gravimetric method to solve their models. They obtained fuzzy efficiency scores for each DMUs in the first stage and fuzzy goals in the second stage. In the first step, obtain the fuzzy radial efficiency scores and in the second step, the maximum values of fuzzy slack corresponding to the inputs and outputs to present the fuzzy efficiency goals. The proposed approach by them is in the fuzzy ranking category. Arana-Jim´enez et al. [2] presented a fully FDEA model based on the concept of inefficiency measure in the additive model. They obtained efficiency scores and fuzzy targets corresponding to each of the DMUs. They defined the concept of fuzzy Pareto solutions for FDEA model. If some price information about inputs and outputs is available, important results will be obtained in evaluating the efficiency of DMUs, and DMUs can be evaluated in terms of price and value corresponding to inputs and outputs. The cost efficiency model evaluates the ability of a DMU to produce the current outputs at minimal cost, given its input prices. The concept of cost efficiency was first introduced by Farrell [12] and later developed by Fare et al. [10]. In the following, Tone [33] introduced a cost efficiency model based on the concept of base cost. In contrast to previous approaches, which were considered to correspond to fixed input components, in the proposed approach by Tone [33], input components could have different prices. The revenue efficiency evaluation model is also determined based on the price of outputs. The revenue efficiency corresponding to each DMU are defined as the ratio of actual observed revenue to the maximum revenue from a DMU based on input and output data values and output prices. Some studies on cost efficiency evaluation are listed in the following articles. Fukuyama and Weber [13], Fare and Grosskopf [11], Paradi and Zhu [25]. Fukuyama and Weber [13] proposed a cost efficiency model based on the value-based technology and concept of directional distance function. They obtained a new vector by multiplying the cost vector by the input vector, which was then considered as a new input vector and the new PPS was introduced based on new input vector. Sahoo et al. [31] considered a situation in which all input and output data and their corresponding prices were known for each of the DMUs. They propose cost, revenue and profit efficiency measurement in DEA based on the directional distance function approach. In many cases in the real world, the values corresponding to the input and output data are often inaccurate, and fuzzy set theory is a good strategy for dealing with inaccurate data. For example, profitability and activity in hospitals, banks, schools, etc. can be fuzzy data. In this case, fuzzy sets are a realistic strategy for incorporating data uncertainty. In recent years, some research has been done to evaluate the cost efficiency of the presence of inaccurate data in fuzzy situations. Jahanshahloo et al. [19] presented a cost efficiency evaluation model in a situation where the input and output data are exact numbers but the prices corresponding to the inputs are triangular fuzzy numbers. Paryab et al.[26] presented a fuzzy cost efficiency evaluation model in the presence of input and output data and fuzzy prices. They proposed two methods based on convex and non-convex approaches, and the variables in their model were also fuzzy. Puri and Yadav [28] developed cost and revenue efficiency evaluation models in a fully fuzzy environment where input and output data and their corresponding prices were triangular fuzzy numbers. Pourmahmoud and Bafekr Sharak [27] presented a model for evaluating cost efficiency in the presence of inputs and outputs and fuzzy prices. They considered the data as triangular fuzzy numbers and offered a new definition of fuzzy cost efficiency. They used a -level-based approach to convert their model to an into an interval model, and the interval model was a parametric model that, by properly selecting the values, could provide cost efficiency scores corresponding to the values. Their model provided cost efficiency scores in the form of fuzzy numbers, but components of fuzzy numbers corresponding to cost efficiency scores could be numbers greater than one then we could not be used and looked at as cost efficiency scores, because, given that cost efficiency scores resulting in the cost efficiency evaluation traditional model were less than or equal to one. They called a DMU under evaluation cost efficient if the cost efficiency score minimum obtained of the cost efficiency evaluation model be located in the interval where the lower and upper bounds are the lower and upper bounds of the fuzzy number corresponding to actual observed fuzzy cost vector. Similarly, the fuzzy cost efficiency scores obtained in the approach presented by Puri and Yadav [28] may be greater than or equal to one, while the cost efficiency scores are less than or equal to one in the traditional model for evaluation cost efficiency. These approaches also do not represent the fuzzy targets corresponding to each of the DMUs. It can be said that the contribution of this paper compared to previous studies is as follows. In this paper, we estimate the cost efficiency of firms in a non-competitive market with fuzzy heterogeneous inputs and outputs along with their variable prices, which prices are fuzzy numbers. We proposed the factor-based technology set in FDEA, and then we proposed a new approach for calculating fuzzy revenue and cost efficiency. The proposed approach is suitable than the previous approaches from a computational point of view. The innovation of this research is that by solving only one model, we obtain the fuzzy cost (revenue) efficiency score corresponding to under evaluation unit, if all the components of that fuzzy number are equal to one then this unit is fully fuzzy cost (revenue) efficient, and if the center of this fuzzy number is equal to one, then the unit is eval-

uated as fuzzy cost (revenue) efficient, otherwise the unit will be fuzzy cost (revenue) inefficient. The proposed models are always feasible and easily become a linear programming model. Also, with a simple comparison of the observed cost (revenue) vectors and the minimum cost (maximum revenue), we can determine whether the unit under evaluation is efficient or not. We also obtain benchmark corresponding to the all inefficient DMUs.
The remainder of the paper organized as follows. In the preliminary section, we first introduce the basic notations and definitions of the fuzzy set theory, and then briefly introduce traditional DEA models to calculate cost and revenue efficiency. In the third section, which is the main section of the paper, we first present fuzzy cost and revenue efficiency evaluation models based on traditional DEA models, then we examine the properties of the proposed models. In the fourth section, we present two numerical examples, we use the proposed approach to calculate fuzzy cost-effectiveness and revenue, and finally bring the results of the research.

2 Preliminary

In this section, we first introduce the basic concepts required for fuzzy sets and then describe the concepts of cost and revenue efficiency based on traditional DEA models.

2.1 Fuzzy Set Theory

A fuzzy set can be defined as a mapping $\mu : R^n \to [0,1]$. For each fuzzy set and for each $\alpha \in [0,1]$, we define the α-Level set as follows.
$\mu_\alpha = \{x \in R^n | \mu(x) \geq \alpha\}$. Suppose we denote the support of μ by $supp(\mu)$ where $supp(\mu) = \{x \in R^n | \mu(x) \geq 0\}$. The closure of $supp(\mu)$ defines the 0-Level set of μ that is $[\mu]^0 = cl(supp(\mu))$, that $cl(M)$ represents the closure of the subset $M \subseteq R^n$. A fuzzy number is a type of fuzzy set and is defined as follows (Dubois and Prade [7]).

Definition 2.1. *A fuzzy set μ on R is called a fuzzy number if*
1) μ is normal, meaning that there exists $x_o \in R$ such that $\mu(x_o) = 1$.
2) μ is an upper semi-continuous function.

3) $\min\{\mu(x), \mu(y)\} \le \mu(\lambda x + (1-\lambda)y)$, $x, y \in R$, $\lambda \in [0, 1]$.
4) $[\mu]^0$ is compact.

Definition 2.2. *A fuzzy number $\mu = (\mu_1, \mu_2, \mu_3, \mu_4)$ is called a trapezoidal fuzzy number, if its membership function is as follows.*

$$\tilde{\mu} = \begin{cases} (x-\mu^1)/(\mu^2-\mu^1) & if \quad \mu^1 \le x \le \mu^2 \\ 1 & if \quad \mu^2 \le x \le \mu^3 \\ (\mu^4 - x)/(\mu^4-\mu^3) & if \quad \mu^3 \le x \le \mu^4 \\ 0 & else \end{cases}$$

Corresponding to a fuzzy number $\tilde{\mu}$, α-Level set with it is defined as follows.
$[\mu]^\alpha = [\mu^1 + \alpha(\mu^2 - \mu^1), \mu^4 - \alpha(\mu^4 - \mu^3)]$.
We show the set of all trapezoidal fuzzy numbers as $TF(R)$. We represent the subset of non-negative fuzzy numbers $TF(R)$ as $TF^+(R)$. A trapezoidal fuzzy number μ is called a triangular fuzzy number if and only if $\mu^2 = \mu^3$.

Definition 2.3. *Let two trapezoidal fuzzy numbers $(a^1, a^2, a^3, a^4) \in TF(R)$ and $(b^1, b^2, b^3, b^4) \in TF(R)$, we define the arithmetical operations as follows.*

i) Addition $\tilde{a} + \tilde{b} = (a^1 + b^1, a^2 + b^2, a^3 + b^3, a^4 + b^4)$.
ii) Multiplication by a scalar

$$\Gamma \in R,\ \Gamma\tilde{a} = \begin{cases} (\Gamma a^1, \Gamma a^2, \Gamma a^3, \Gamma a^4) & if \quad \Gamma > 0 \\ (\Gamma a^4, \Gamma a^3, \Gamma a^2, \Gamma a^1) & if \quad \Gamma < 0 \end{cases}$$

iii) Multiplication of two $\tilde{a} \in TF(R)$, $\tilde{b} \in TF(R)$.
$\tilde{a}\tilde{b} = \tilde{c} = (c^1, c^2, c^3, c^4)$, where
$c^1 = \min\{a^1b^1, a^1b^4, a^4b^1, a^4b^4\}$, $\quad c^2 = \min\{a^2b^2, a^2b^3, a^3b^2, a^3b^3\}$
$c^3 = \min\{a^2b^2, a^2b^3, a^3b^2, a^3b^3\}$, $\quad c^4 = \min\{a^1b^1, a^1b^4, a^4b^1, a^4b^4\}$
In the particular that $\tilde{a} \in TF^+(R)$, $\tilde{b} \in TF^+(R)$, we have
$\tilde{a}\tilde{b} = (a^1b^1, a^2b^2, a^3b^3, a^4b^4)$.
iv) Division of two $\tilde{a} \in TF^+(R)$, $\tilde{b} \in TF^+(R)$, that all components of these numbers are opposite to zero. We have
$\tilde{a}/\tilde{b} = (a^1/b^4, a^2/b^3, a^3/b^2, a^4/b^1)$.
In the present paper, we consider the input and output variables and some model variables as non-negative fuzzy trapezoidal numbers, which

belong to $TF^+(R)$. The arithmetic operations between them are those established in Definition 2.3. Besides, we provide a partial order relationship between two trapezoidal fuzzy numbers. In this way, we will use LU-fuzzy partial orders, which are well known in the literature (see, e.g., Stefanini and Arana-Jime´nez [32]).
$u \leqq (\geqq) v$ if $\underline{u}^\alpha \leq (\geq) \underline{v}^\alpha$, $\bar{u}^\alpha \leq (\geq) \bar{v}^\alpha$, for $\alpha \in [0, 1]$.
In the special case for two trapezoidal number (u^1, u^2, u^3, u^4) and (v^1, v^2, v^3, v^4),we have $\tilde{u} \leqq (\geqq) \tilde{v}$ if $u^i \leq (\geq) v^i$, $i = 1, 2, 3, 4$.

2.2 Cost and revenue efficiency measurement

In this section, we propose cost and revenue evaluation models. We assume that the prices corresponding to the input and output components are exact numbers. Also, we suppose that input and output prices are available. Suppose we have n $DMUs$ as $DMU_j = (x_j, y_j)$, $j = 1, \cdots, n$. The input and output vectors corresponding to DMU_j, $j = 1, \cdots, n$, as $x_j = (x_{1j}, \cdots, x_{mj})$ and $y_j = (y_{1j}, \cdots, y_{sj})$ respectively. We consider the non-negative price vectors of input and output of DMU_j, $j = 1, \cdots, n$ are equal to $c_j = (c_{1j}, \cdots, c_{mj})^T \in R^m_+$, $p_j = (p_{1j}, \cdots, p_{sj})^T \in R^s_+$, respectively. The superscript T stands for a vector transpose. Suppose, input-cost (input-spending) and output-revenue (output-earnings) of DMU_j, $j = 1, \cdots, n$, are as $\bar{x}_j = c_j * x_j$, $\bar{y}_j = p_j * y_j$, $j = 1, \cdots, n$. Where $*$ shows the component-wise multiplication of vectors. We consider o as the index of $DMUs$ under evaluation. Now, we define four production technologies depending upon data availability (see, Sahoo et al. [31]).
If physical outputs are observed and are homogeneous but not physical inputs, we show the technology by considering all feasible input-spending and physical output vectors as follows.

$$T_{\bar{x},y} = \{(\bar{x}, y) \mid (\bar{x}, y) \in R^{m+s}_+, \sum_{j=1}^{n} \lambda_j \bar{x}_j \leq \bar{x}, \sum_{j=1}^{n} \lambda_j y_j \geq y, \sum_{j=1}^{n} \lambda_j = 1\}. \quad (1)$$

If physical inputs are observed and are homogeneous but not physical outputs, then we propose the technology by considering all feasible physical input and output-earnings vectors as follows.

$$T_{x,\bar{y}} = \{(x,\bar{y}) \mid (x,\bar{y}) \in R_{+}^{m+s}, \sum_{j=1}^{n} \lambda_j x_j \le x, \sum_{j=1}^{n} \lambda_j \bar{y}_j \ge \bar{y}, \sum_{j=1}^{n} \lambda_j = 1\}. \tag{2}$$

We consider the assumption of VRS in above DEA technology constructs, because the assumption of CRS is not consistent with some directional DEA models based on specific direction vectors, which all directly deal with both positive and negative data (Sahoo et al. [31]), and the real situations do not always display CRS. In order to face a situation that input prices change between firms or to reflect the qualitative differences in the resources, the alternative value-based cost efficiency model of Tone [33] that is based on $T_{\bar{x},y}$ should be used. This alternative cost efficiency model can be represented as

$$\begin{aligned} \delta_o^{CE} = \min \quad & \frac{\sum_{i=1}^{m} \bar{x}_i}{\sum_{i=1}^{m} \bar{x}_{io}}, \\ s.t. \quad & \sum_{j=1}^{n} \lambda_j \bar{x}_{ij} \le \bar{x}_i, \quad i = 1, \cdots, m, \\ & \sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{ro}, \quad r = 1, \cdots, s, \\ & \sum_{j=1}^{n} \lambda_j = 1, \quad \lambda_j \ge 0, \quad j = 1, \cdots, n. \end{aligned} \tag{3}$$

Where $\sum_{i=1}^{m} \bar{x}_{io}$ is the amount of cost observed from the unit under considering. Suppose that $\bar{x}_i^*, i = 1, \cdots, m$, is an optimal solution of model (3), in this situation, we define the cost efficiency corresponding to DMU_o, i.e. the under evaluation unit, as follows.

$$\delta_o^{CE} = \frac{\sum_{i=1}^{m} \bar{x}_i^*}{\sum_{i=1}^{m} \bar{x}_{io}}. \tag{4}$$

It should be noted that $\sum_{i=1}^{m} \bar{x}_i^*$ is the minimum cost of model (3). It is clear that DMU_o means that the under-evaluation unit will be cost-efficient in evaluation with model (3) if δ_o^{CE} otherwise DMU_o is called cost inefficient. If (physical) outputs are heterogeneous, in order to face a situation that output prices change between firms to reflect the qualitative differences in their products, the alternative value-based revenue efficiency model of Tone [33] that is based on $T_{x,\bar{y}}$ should be used. This alternative value-based revenue efficiency measure can be represented as

$$\begin{aligned} \frac{1}{\tau_o^{RE}} = \max \quad & \frac{\sum_{r=1}^{s} \bar{y}_r}{\sum_{r=1}^{s} \bar{y}_{ro}}, \\ s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} \leq x_{io}, \quad i = 1, \cdots, m, \\ & \sum_{j=1}^{n} \lambda_j \bar{y}_{rj} \geq \bar{y}_r, \quad r = 1, \cdots, s, \\ & \sum_{j=1}^{n} \lambda_j = 1, \quad \lambda_j \geq 0, \quad j = 1, \cdots, n. \end{aligned} \tag{5}$$

Where $\sum_{r=1}^{s} \bar{y}_{ro}$ is the amount of revenue observed from DMU_o means the unit under evaluation. Suppose that $\bar{y}_r^*$, $r = 1, \cdots, s$, is an optimal solution of model (5), in this case, we define the amount of revenue efficiency corresponding to DMU_o, i.e. the unit under evaluation as follows.

$$\tau_o^{RE} = \frac{\sum_{r=1}^{s} \bar{y}_{ro}}{\sum_{r=1}^{s} \lambda_j \bar{y}_r^*}. \tag{6}$$

It should be noted that $\sum_{r=1}^{s} \bar{y}_r^*$ is the maximum revenue obtaining of model (5). DMU_o means that the under-evaluation unit is called revenue efficient in evaluation with model (5) if $\tau_o^{RE} = 1$ otherwise DMU_o is called revenue inefficient.

3 Our proposed approach

In this section, we first present a novel FDEA models for calculating fuzzy cost and revenue efficiency. Consider n $DMUs$ $DMU_j = (\tilde{X}_j, \tilde{Y}_j)$, $j = 1, \cdots, n$ that consume fuzzy input vector $\tilde{X}_j = (\tilde{x}_{1j}, \cdots, \tilde{x}_{mj})^T$, in order to produce fuzzy output vector $\tilde{Y}_j = (\tilde{y}_{1j}, \cdots, \tilde{y}_{sj})^T$.
$\tilde{x}_{ij}$, $i = 1, \cdots, m$, $\tilde{y}_{rj}$, $r = 1, \cdots, s$, represent the $i-th$ and $r-th$ components of the fuzzy input and output vector corresponding to DMU_j,$j = 1, \cdots, n$ respectively. Suppose we denote the unit under evaluation with $DMU_o = (\tilde{X}_o, \tilde{Y}_o)$. Also assume that we show the fuzzy price vector corresponds to the inputs and outputs of $DMU_j = (\tilde{X}_j, \tilde{Y}_j)$, with $\tilde{c}_j = (\tilde{c}_{1j}, \cdots, \tilde{c}_{mj})^T$, $\tilde{p}_j = (\tilde{p}_{1j}, \cdots, \tilde{p}_{sj})^T$, $j = 1, \cdots, n$, respectively.
Each of the components of the vectors are fuzzy numbers. In this paper, we assume that all of fuzzy numbers are non-negative trapezoidal fuzzy numbers as follows.

$$\begin{aligned}
\tilde{x}_{ij} &= (x_{ij}^{(1)}, x_{ij}^{(2)}, x_{ij}^{(3)}, x_{ij}^{(4)})^T \in TF_+^m(F), \quad i = 1, \cdots, m, \quad j = 1, \cdots, n, \\
\tilde{c}_{ij} &= (c_{ij}^{(1)}, c_{ij}^{(2)}, c_{ij}^{(3)}, c_{ij}^{(4)})^T \in TF_+^m(F), \quad i = 1, \cdots, m, \quad j = 1, \cdots, n, \\
\tilde{y}_{rj} &= (y_{rj}^{(1)}, y_{rj}^{(2)}, y_{rj}^{(3)}, y_{rj}^{(4)})^T \in TF_+^s(F), \quad r = 1, \cdots, r, \quad j = 1, \cdots, n, \\
\tilde{p}_{rj} &= (p_{rj}^{(1)}, p_{rj}^{(2)}, p_{rj}^{(3)}, p_{rj}^{(4)})^T \in TF_+^s(F), \quad r = 1, \cdots, r, \quad j = 1, \cdots, n.
\end{aligned} \tag{7}$$

We put

$$\begin{array}{c}\tilde{\bar{x}}_{ij} = \tilde{c}_{ij} \times \tilde{x}_{ij} = (c_{ij}^{(1)}, c_{ij}^{(2)}, c_{ij}^{(3)}, c_{ij}^{(4)})^T \times (x_{ij}^{(1)}, x_{ij}^{(2)}, x_{ij}^{(3)}, x_{ij}^{(4)}) = \\ (c_{ij}^{(1)} x_{ij}^{(1)}, c_{ij}^{(2)} x_{ij}^{(2)}, c_{ij}^{(3)} x_{ij}^{(3)}, c_{ij}^{(4)} x_{ij}^{(4)}) = (\bar{x}_{ij}^{(1)}, \bar{x}_{ij}^{(2)}, \bar{x}_{ij}^{(3)}, \bar{x}_{ij}^{(4)}), \\ \tilde{\bar{y}}_{rj} = \tilde{p}_{rj} \times \tilde{y}_{rj} = (p_{rj}^{(1)}, p_{rj}^{(2)}, p_{rj}^{(3)}, p_{rj}^{(4)})^T \times (y_{rj}^{(1)}, y_{rj}^{(2)}, y_{rj}^{(3)}, y_{rj}^{(4)}) = \\ (p_{rj}^{(1)} y_{rj}^{(1)}, p_{rj}^{(2)} y_{rj}^{(2)}, p_{rj}^{(3)} y_{rj}^{(3)}, p_{rj}^{(4)} y_{rj}^{(4)}) = (\bar{y}_{rj}^{(1)}, \bar{y}_{rj}^{(2)}, \bar{y}_{rj}^{(3)}, \bar{y}_{rj}^{(4)}),\end{array} \tag{8}$$

We also define the vectors of fuzzy input-cost variable and fuzzy output-revenue variable as follows.

$$\begin{array}{c}\tilde{\bar{x}} = \tilde{c}^T \times \tilde{x} = (c^{(1)}x^{(1)}, c^{(2)}x^{(2)}, c^{(3)}x^{(3)}, c^{(4)}x^{(4)}) = \\ (\bar{x}^{(1)}, \bar{x}^{(2)}, \bar{x}^{(3)}, \bar{x}^{(4)}) \in TF_+^m(F), \\ \tilde{\bar{y}} = \tilde{p}^T \times \tilde{y} = (p^{(1)}y^{(1)}, p^{(2)}y^{(2)}, p^{(3)}y^{(3)}, p^{(4)}y^{(4)}) = \\ (\bar{y}^{(1)}, \bar{y}^{(2)}, \bar{y}^{(3)}, \bar{y}^{(4)}) \in TF_+^s(F).\end{array} \tag{9}$$

We now consider the two production technologies dependent to fuzzy data available as follows.

$$\begin{array}{c}T_{\tilde{\bar{x}},\tilde{y}}^{FDEA} = \{(\tilde{\bar{x}}, \tilde{y}) \mid (\tilde{\bar{x}}, \tilde{y}) \in TR_+^{m+s}, \sum_{j=1}^{n} \lambda_j \tilde{\bar{x}}_j \leq \tilde{\bar{x}}, \\ \sum_{j=1}^{n} \lambda_j \tilde{y}_j \geq \tilde{y}, \sum_{j=1}^{n} \lambda_j = 1\}.\end{array} \tag{10}$$

In the set (10), we used the input-cost vector instead of the input vector.

$$\begin{array}{c}T_{\tilde{x},\tilde{\bar{y}}}^{FDEA} = \{(\tilde{x}, \tilde{\bar{y}}) \mid (\tilde{x}, \tilde{\bar{y}}) \in TR_+^{m+s}, \sum_{j=1}^{n} \lambda_j \tilde{x}_j \leq \tilde{x}, \\ \sum_{j=1}^{n} \lambda_j \tilde{\bar{y}}_j \geq \tilde{\bar{y}}, \sum_{j=1}^{n} \lambda_j = 1\}.\end{array} \tag{11}$$

In the set (11), we used fuzzy revenue-output vector instead of output vector. We now present the fuzzy cost efficiency evaluation model based on the $T^{FDEA}_{\tilde{\bar{x}},\tilde{y}}$ set as follows.

$$\begin{aligned}
\min \quad & \Big(\frac{\sum_{i=1}^{m}\bar{x}_i^{(1)}}{\sum_{i=1}^{m}\bar{x}_{io}^{(4)}}+\frac{\sum_{i=1}^{m}\bar{x}_i^{(2)}}{\sum_{i=1}^{m}\bar{x}_{io}^{(3)}}+\frac{\sum_{i=1}^{m}\bar{x}_i^{(3)}}{\sum_{i=1}^{m}\bar{x}_{io}^{(2)}}+\frac{\sum_{i=1}^{m}\bar{x}_i^{(4)}}{\sum_{i=1}^{m}\bar{x}_{io}^{(1)}}\Big)\\
s.t. \quad & \sum_{j=1}^{n}\lambda_j\bar{x}_{ij}^{(k)} \le \bar{x}_i^{(k)}, \quad i=1,\cdots,m, \quad k=1,2,3,4,\\
& \sum_{j=1}^{n}\lambda_j y_{rj}^{(k)} \ge y_{ro}^{(k)}, \quad r=1,\cdots,s, \quad k=1,2,3,4,\\
& \sum_{j=1}^{n}\lambda_j = 1, \quad \lambda_j \ge 0, \quad r=1,\cdots,s,\\
& 0 \le \bar{x}_i^{(1)}, \quad \bar{x}_i^{(k-1)} \le \bar{x}_i^{(k)}, \quad i=1,\cdots,m, \quad k=2,3,4.
\end{aligned} \tag{12}$$

Suppose $(\bar{x}^{(1)^*},\bar{x}^{(2)^*},\bar{x}^{(3)^*},\bar{x}^{(4)^*})$ where $\bar{x}^{(k)^*}=(\bar{x}_1^{(k)^*},\cdots,\bar{x}_m^{(k)^*})$, $k=1,2,3,4$, is an optimal solution obtained from model (12). We suppose that $\tilde{\bar{X}}_o \neq 0$. In this case, we define the fuzzy cost score corresponding to $D\tilde{M}U_0=(\tilde{\bar{X}}_o,\tilde{Y}_o)$ as follows.

$$\begin{aligned}
&\tilde{\delta}_o^{FCE}=(\delta_o^{FCE^1},\delta_o^{FCE^2},\delta_o^{FCE^3},\delta_o^{FCE^4})=\\
&\Big(\frac{\sum_{i=1}^{m}\bar{x}_i^{(1)^*}}{\sum_{i=1}^{m}\bar{x}_{io}^{(4)}},\frac{\sum_{i=1}^{m}\bar{x}_i^{(2)^*}}{\sum_{i=1}^{m}\bar{x}_{io}^{(3)}},\frac{\sum_{i=1}^{m}\bar{x}_i^{(3)^*}}{\sum_{i=1}^{m}\bar{x}_{io}^{(2)}},\frac{\sum_{i=1}^{m}\bar{x}_i^{(4)^*}}{\sum_{i=1}^{m}\bar{x}_{io}^{(1)}}\Big)
\end{aligned} \tag{13}$$

It should be noted that the actual observed fuzzy cost vector is as follows.

$$\Big(\sum_{i=1}^{m}\bar{x}_{io}^{(1)},\sum_{i=1}^{m}\bar{x}_{io}^{(2)},\sum_{i=1}^{m}\bar{x}_{io}^{(3)},\sum_{i=1}^{m}\bar{x}_{io}^{(4)}\Big) \tag{14}$$

Also, we suppose that the minimum fuzzy cost vector obtained from model (12) is as follows.

$$\Big(\sum_{i=1}^{m}\bar{x}_i^{(1)^*},\sum_{i=1}^{m}\bar{x}_i^{(2)^*},\sum_{i=1}^{m}\bar{x}_i^{(3)^*},\sum_{i=1}^{m}\bar{x}_i^{(4)^*}\Big) \tag{15}$$

Definition 3.1. *$D\tilde{M}U_0 = (\tilde{\bar{X}}_o, \tilde{Y}_o)$ is called fully fuzzy cost efficient in model (12), if*

$$\tilde{\delta}_o^{FCE} = (\delta_o^{FCE^1}, \delta_o^{FCE^2}, \delta_o^{FCE^3}, \delta_o^{FCE^4}) = (1,1,1,1). \tag{16}$$

Definition 3.2. *$D\tilde{M}U_0 = (\tilde{\bar{X}}_o, \tilde{Y}_o)$ is called fuzzy cost efficient in model (12), if the minimum fuzzy cost vector obtained and the actual observed fuzzy cost vector corresponding to $D\tilde{M}U_0 = (\tilde{\bar{X}}_o, \tilde{Y}_o)$ are equal, or in this case, we will have $\delta_o^{FCE^2} = \delta_o^{FCE^3} = 1$.*
Otherwise, $D\tilde{M}U_0 = (\tilde{\bar{X}}_o, \tilde{Y}_o)$ is called fuzzy cost inefficient in evaluation to model (12).

Theorem 3.3. *The model (12) is always feasible.*

Proof. It can be easily shown that $\lambda_j = 0$, $j = 1, \cdots, n$, $j \neq o$ and $\lambda_o = 1$, $\bar{x}_i^{(k)} = \bar{x}_{io}^{(k)}$, $i = 1, \cdots, m$, $k = 1,2,3,4$. is a feasible solution for model (12), and the proof is completed. $\square$
We now propose the fuzzy revenue efficiency evaluation model based on the $T_{\tilde{x},\tilde{\bar{y}}}^{FDEA}$ set as follows.

$$\begin{aligned}
\max \quad & \left(\frac{\sum_{r=1}^{s} \bar{y}_r^{(1)}}{\sum_{r=1}^{s} \bar{y}_{ro}^{(4)}} + \frac{\sum_{r=1}^{s} \bar{y}_r^{(2)}}{\sum_{r=1}^{m} \bar{y}_{ro}^{(3)}} + \frac{\sum_{r=1}^{s} \bar{y}_r^{(3)}}{\sum_{r=1}^{m} \bar{y}_{ro}^{(2)}} + \frac{\sum_{r=1}^{s} \bar{y}_r^{(4)}}{\sum_{r=1}^{m} \bar{y}_{ro}^{(1)}}\right) \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij}^{(k)} \leq x_i^{(k)}, \quad i = 1, \cdots, m, \quad k = 1,2,3,4, \\
& \sum_{j=1}^{n} \lambda_j \bar{y}_{rj}^{(k)} \geq \tilde{y}_{ro}^{(k)}, \quad r = 1, \cdots, s, \quad k = 1,2,3,4, \\
& \sum_{j=1}^{n} \lambda_j = 1, \quad \lambda_j \geq 0, \quad r = 1, \cdots, s, \\
& 0 \leq \bar{y}_r^{(1)}, \quad \bar{y}_r^{(k-1)} \leq \bar{y}_r^{(k)}, \quad r = 1, \cdots, s, \quad k = 2,3,4.
\end{aligned} \tag{17}$$

Suppose $(\bar{y}^{(1)^*}, \bar{y}^{(2)^*}, \bar{y}^{(3)^*}, \bar{y}^{(4)^*})$ where $\bar{y}^{(k)^*} = (\bar{y}_1^{(k)^*}, \cdots, \bar{y}_s^{(k)^*})$, $k = 1,2,3,4$, is an optimal solution obtained from model (17). We suppose that $\tilde{\bar{Y}}_o \neq 0$. In this case, we define the fuzzy revenue score corresponding to $D\tilde{M}U_0 = (\tilde{X}_o, \tilde{\bar{Y}}_o)$ as follows.

$$\tau_o^{FRE} = (\tau_o^{FRE^1}, \tau_o^{FRE^2}, \tau_o^{FRE^3}, \tau_o^{FRE^4}) = \left(\frac{\sum_{r=1}^{s} \bar{y}_{ro}^{(1)}}{\sum_{r=1}^{s} \bar{y}_r^{(4)^*}}, \frac{\sum_{r=1}^{s} \bar{y}_{ro}^{(2)}}{\sum_{r=1}^{s} \bar{y}_r^{(3)^*}}, \frac{\sum_{r=1}^{s} \bar{y}_{ro}^{(3)}}{\sum_{r=1}^{s} \bar{y}_r^{(2)^*}}, \frac{\sum_{r=1}^{s} \bar{y}_{ro}^{(4)}}{\sum_{r=1}^{s} \bar{y}_r^{(1)^*}}\right) \tag{18}$$

It should be noted that the actual observed fuzzy revenue vector is as follows.

$$\left(\sum_{r=1}^{s} \bar{y}_{ro}^{(1)}, \sum_{r=1}^{s} \bar{y}_{ro}^{(2)}, \sum_{r=1}^{s} \bar{y}_{ro}^{(3)}, \sum_{r=1}^{s} \bar{y}_{ro}^{(4)}\right) \tag{19}$$

Also, we suppose that the maximum fuzzy revenue vector obtained from model (17) is as follows.

$$\left(\sum_{r=1}^{s} \bar{y}_r^{(1)^*}, \sum_{r=1}^{s} \bar{y}_r^{(2)^*}, \sum_{r=1}^{s} \bar{y}_r^{(3)^*}, \sum_{r=1}^{s} \bar{y}_r^{(4)^*}\right) \tag{20}$$

Definition 3.4. *$D\tilde{M}U_0 = (\tilde{X}_o, \tilde{\bar{Y}}_o)$ is called fully fuzzy revenue efficient in model (17), if*

$$\tilde{\tau}_o^{FRE} = (\tau_o^{FRE^1}, \tau_o^{FRE^2}, \tau_o^{FRE^3}, \tau_o^{FRE^4}) = (1, 1, 1, 1). \tag{21}$$

Definition 3.5. *$D\tilde{M}U_0 = (\tilde{X}_o, \tilde{\bar{Y}}_o)$ is called fuzzy revenue efficient in model (17), if the maximum fuzzy revenue vector obtained and the actual observed fuzzy cost vector corresponding to $D\tilde{M}U_0 = (\tilde{X}_o, \tilde{\bar{Y}}_o)$ are equal, or in this case, we will have $\tau_o^{FCE^2} = \tau_o^{FCE^3} = 1$.*
Otherwise, $D\tilde{M}U_0 = (\tilde{X}_o, \tilde{\bar{Y}}_o)$ is called fuzzy revenue inefficient in evaluation to model (17).

Theorem 3.6. *The model (17) is always feasible.*

Proof. It can be easily shown that $\lambda_j = 0,\ j = 1, \cdots, n,\ j \neq o$ and $\lambda_o = 1,\ \bar{y}_r^{(k)} = \bar{y}_{ro}^{(k)},\ r = 1, \cdots, s,\ k = 1, 2, 3, 4.$ is a feasible solution for model (17), and the proof is completed. □

Table 1: Fuzzy input-output data.

DMU	Input1	Input2	Output1	Output2
1	(49.4,53,56.6)	(40.5,45,49.5)	(70,77.5,85)	(28.8,35.4,42)
2	(20,25,30)	(41.6,46.5,51.4)	(41.5,49.6,57.7)	(28.9,34.7,40.5)
3	(11.5,18,24.5)	(11.9,15.7,19.5)	(21.6,26.5,31.4)	(29.4,37.6,45.8)
4	(12.1,18,23.9)	(20.1,25.5,30.9)	(34.2,37.5,40.8)	(40.9,47.5,54.1)
5	(29.1,32,34.9)	(20.3,25,29.7)	(59.2,64,68.8)	(72.9,76.4,79.9)
6	(50.8,56,61.2)	(41.6,45.1,48.6)	(32.6,35.3,38)	(3742.5,48)
7	(17.6,24,30.4)	(13.6,17.5,21.4)	(75.6,82.9,90.2)	(33.2,38.5,43.8)
8	(71.2,78,84.8)	(18.5,23.9,29.3)	(60,66,72)	(38.2,47.4,56.6)
9	(45.5,52,58.5)	(13.7,19.8,25.9)	(51.1,56.5,61.9)	(51.3,56,60.7)
10	(44.6,49,53.4)	(16.3,20.6,24.9)	(38,46.5,55)	(32.1,38,43.9)

4 Numerical examples

In this section, we use two numerical examples to illustrate the proposed approach. In the first numerical example, we obtain the fuzzy cost efficiency scores corresponding to the DMUs based on the two approaches presented in this paper to calculate the fuzzy cost efficiency, and then in the second numerical example, the fuzzy revenue efficiency scores corresponding to the DMUs. The proposed approach in this paper is used to calculate fuzzy cost-efficient targets and fuzzy revenue efficient targets.

4.1 Numerical example 1

In this section, we use a numerical example provided by Puri and Yadav [28] to illustrate the approaches presented in this paper to calculate fuzzy cost efficiency. They considered 10 DMUs with two fuzzy inputs and outputs. In their observations, they used triangular fuzzy numbers as (a^1, a^2, a^3) which a^1, a^2, a^3, are the lower bound and the center and upper bound of the fuzzy number a , respectively. Input and output data and the corresponding price of inputs are fuzzy triangular numbers. Tables (1), (2) shows the fuzzy input and output data sets and the price of fuzzy inputs for each of the 10 DMUs.

Table 2: Fuzzy input prices for DMUs.

DMU	Input price1	Input price2	DMU	Input price1	Input price2
1	(4.7,5,5.3)	(4.5,5,5.5)	6	(5.4,6,6.6)	(3.25,4,4.75)
2	(3.55,6.5)	(4.5,6,7.5)	7	(1.6,2,2.4)	(1.5,2,2.5)
3	(7.28,8.8)	(6.3,7,7.7)	8	(1.8,3,4.2)	(3,3.9,4.8)
4	(7.2,9,10.8)	(5,5.5,6)	9	(4.8,5,5.2)	(89.8,11.6)
5	(2.8,3,3.2)	(1.75,2,2.25)	10	(2,2.9,3.8)	(2,2.6,3.2)

Table 3: Fuzzy cost efficiency and Minimum cost values of Model (12)

DMU	Fuzzy cost efficiency	The minimum cost
1	(0.0849, 0.1694, 0.3051)	(48.5600, 83.0000 ,126.4600)
2	(0.0837, 0.2054, 0.4917)	(48.5600, 83.0000 ,126.4600)
3	(0.1431, 0.3406, 0.8198)	(52.3520, 86.4903 ,129.3434)
4	(0.1535, 0.3341, 0.7532)	(68.0886 ,100.9751 ,141.3094)
5	(0.6555, 1.0000, 1.5256)	(117.0050 ,146.0000 ,178.5050)
6	(0.0890, 0.1749, 0.3236)	(56.5231, 90.3296 ,132.5151)
7	(0.3840, 1.0000, 2.6042)	(48.5600, 83.0000 ,126.4600)
8	(0.1466, 0.3219, 0.7890)	(72.8286 ,105.3380 ,144.9136)
9	(0.1333, 0.2478, 0.4598)	(80.6021 ,112.4931 ,150.8246)
10	(0.1725, 0.4251, 1.0394)	(48.7496, 83.1745 ,126.6042)

Table 4: Fuzzy observed cost

DMU	Fuzzy observed cost	DMU	Fuzzy observed cost
1	(414.43,490,572.23)	6	(409.52,516.4,634.77)
2	(257.2,404,580.5)	7	(48.56,83,126.46)
3	(157.77,253.9,365.75)	8	(183.66,327.21,496.8)
4	(187.62,302.25,443.52)	9	(328,454.04,604.64)
5	(117.005,146,178.505)	10	(121.8,195.66,282.6)

The second and third columns of Table (3) show the fuzzy cost efficiency scores corresponding to each of the DMUs and the minimum fuzzy cost obtained from model (12) and the actual observed fuzzy cost corresponding to each of the DMUs, respectively. As can be seen, units 5 and 7 are the only efficient fuzzy cost units based on model (12). Because according to the third and fourth columns of Table (3), Table (4) and the scores of the minimum fuzzy cost obtained from model (12) and the actual observed fuzzy cost corresponding to these units are equal, and this is not the case for other units. For units 5 and 7, the fuzzy number center corresponding to the fuzzy cost efficiency vector is equal to one. According to the definition (3.3), as can be seen, the lower and upper bounds values of fuzzy cost efficiency vector corresponding to units 5 and 7 are larger than the lower and upper bound values of fuzzy cost vector corresponding to other units, in other words the fuzzy cost vector corresponding to units 5 and 7 dominate the fuzzy cost vector corresponding to other units. Also, the fuzzy cost vector corresponding to unit 7 dominate the fuzzy cost vector corresponding to unit 3, which means that unit 7 is more efficient than unit 5. We now compare the results of the approach presented in this paper with the results of previous approaches including Paryab et al. [26], Puri and Yadav [28], Pourmahmoud and Bafekr Sharak [27]. At first, we compare the results of model (12) and the approach presented by Pourmahmoud and Bafekr Sharak [27] . The approach results provided by Pourmahmoud and Bafekr Sharak [27] are listed in Table (5). As can be seen in the paper of Pourmahmoud and Bafekr Sharak [27], they called a unit under evaluation cost efficient if the minimum cost efficiency score obtained of the cost efficiency evaluation model be located in the interval where the lower and upper bounds are the lower and upper bounds of the fuzzy number corresponding to observed cost for each given value of in the a-level-based proposed approach by them, i.e. $\sum_{i=1}^{m} \bar{x}_{io}^{(l)} \leq \sum_{i=1}^{m} \bar{x}_{i}^{(*)} \leq \sum_{i=1}^{m} \bar{x}_{io}^{(u)}$.

The only unit that has in this situation is unit 7, and they introduced unit 7 as the only fuzzy cost-efficient unit. DMU5 is cost efficient unit for values of parameter $\alpha \geq 0.5$. Table (5) shows the fuzzy cost efficiency scores for different values of parameter α. As can be seen the main problem in the approach provided by Pourmahmoud and Bafekr

Table 5: The results of proposed approach by Pourmahmoud and Bafekr Sharak [27].(The minimum cost)

DMU	$\alpha = 0$	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1$
1	47.1957	56.9116	67.3748	78.6338	90.741
2	40.3121	49.7968	60.455	71.2224	81.5065
3	40.8664	51.2774	63.1726	74.2971	83.3777
4	53.6162	66.2176	80.2769	92.7912	102.2966
5	89.0938	108.2455	128.9526	146.0742	157.5243
6	49.2923	60.5982	73.1634	84.3073	92.7416
7	50.2105	60.5019	71.5549	83.419	96.1475
8	50.6227	63.4584	78.039	92.8902	106.8484
9	65.1464	79.5764	95.3832	109.3622	120.174
10	43.8598	54.1376	65.6503	76.7667	86.7213

Sharak [27] is that they do not obtain a unique value for cost efficiency corresponding to each DMU, and for different values of parameter , the fuzzy cost efficiency scores are different and the fuzzy cost efficiency scores not uniquely specified. The condition for that a unit to be fuzzy cost efficient is that the proposed model introduce this unit fuzzy cost efficient for all values of parameter α. On the other hand, to determine whether a DMU is fuzzy cost efficient, the minimum fuzzy cost vector obtained from the proposed model must be compare with the actual observed fuzzy cost vector, which is not suitable from a computational point of view. However, model (12) presented in this paper uniquely determines the fuzzy cost efficiency scores corresponding to each DMU and easily according to the definition (3.1) if the actual observed fuzzy cost vector and the minimum fuzzy cost vector is obtained of model (12) to be equal then the unit under evaluation is fuzzy cost efficient. Also the cost efficiency scores obtained from the approach provided by Pourmahmoud and Bafekr Sharak [27] is a crisp number corresponding for all values of parameter α, but if the values of the inputs and outputs and the price corresponding to them are fuzzy numbers, we expect the cost efficiency score to be presented as a fuzzy number. The approach presented in this paper presents the fuzzy cost efficiency score as a fuzzy

Table 6: Results of cost efficiency by Paryab et al. [26] approach.

DMU	$\alpha = 0$	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1$
1	0.7091	0.6259	0.5498	0.4799	0.4191
2	1.2639	1.0738	0.9118	0.773	0.6439
3	1.8936	1.5354	1.2607	1.0446	0.8111
4	2.0817	1.7417	1.4719	1.2536	1
5	1.4009	1.2836	1.1787	1.0846	1
6	0.4538	0.4153	0.3799	0.3473	0.3064
7	2.0946	1.7371	1.4443	1.2021	1
8	0.5709	0.4943	0.431	0.3767	0.329
9	0.7499	0.6733	0.6055	0.5452	0.4749
10	0.6849	0.5941	0.5179	0.4526	0.3962

number. Now compare the approach presented in Paryab et al. [26] to calculate the fuzzy cost efficiency and the approach presented in this paper in the form of model (12). Paryab et al. [26] used the α-level-based approach to obtain cost efficiency scores. Similar to the approach presented in Pourmahmoud and Bafekr Sharak [27], in the approach presented in Paryab et al. [26], the cost efficiency scores obtained corresponding to each of parameter are a crisp number, and the efficiency scores obtained are not unique and by changing the values the cost efficiency scores change. The approach presented by Paryab, Tavana and Shiraz [26] introduces units 4, 5 and 7 as cost efficient units according to Table (6), while unit 4 in evaluation with other approaches means Puri and Yadav [28], Pourmahmoud and Bafekr Sharak [27] and the approach presented in this paper (model 12) is a fuzzy cost inefficient unit. Also the approach presented by Paryab et al. [26] provides cost efficiency score as a crisp number while the values of inputs and outputs and their corresponding prices are fuzzy numbers, we expect the cost efficiency scores to be a fuzzy number, but the cost efficiency scores in the article Paryab et al. [26] are presented as a crisp number, but in this paper, the amount of cost efficiency is presented as a fuzzy number corresponding to each of the DMUs. The results of the approach presented by Puri and Yadav [28] and the model (12) presented in this paper in-

Table 7: Results of cost efficiency by Puri and Yadav [28] approach.

DMU	Fuzzy cost efficiency	DMU	Fuzzy cost efficiency
1	(0.0786,0.1584,0.2909)	6	(0.0932,0.1582,0.2655)
2	(0.0766,0.1736,0.4002)	7	(0.3840,1,2.6042)
3	(0.1290,0.2830,0.6486)	8	(0.1177,0.2944,0.7813)
4	(0.1478,0.3010,0.6473)	9	(0.1347,0.2397,0.4304)
5	(0.6555,1,1.5256)	10	(0.1775,0.3851,0.8822)

troduce units 5 and 7 as a fuzzy cost-efficient unit, but the fuzzy cost vector corresponds to each of the DMUs are different. Because the form of the objective function and constraints in model (12) and the model presented by Puri and Yadav [28] to calculate the fuzzy cost efficiency are different. In the case of the approach proposed by Puri and Yadav [28], the multiples λ_j as intensity vector in the constraints of the model presented by Puri and Yadav [28] are also fuzzy numbers. But in the present paper, the multiples λ_j in the constraints of the proposed model in this paper for calculating the fuzzy cost efficiency, i.e. model (12), are variables with exact values. According to the form of the objective function and constraints in the model presented by Puri and Yadav [28] and model (12) in the present paper, the model (12) presented in this paper is more appropriate compared to the model presented by Puri and Yadav [28] from a computational point of view. Table (8) show the fuzzy cost inputs and outputs targets obtaining of model (12) corresponding to all DMUs.

4.2 Numerical example 2

In this section, we use a numerical example to illustrate the approaches presented in this paper to calculate fuzzy revenue efficiency. In this example we consider 8 DMUs with one fuzzy input and one output. The price corresponding to the outputs is also a fuzzy number. We considered the data to be triangular fuzzy numbers, which are a special form of trapezoidal fuzzy numbers. We used triangular fuzzy numbers as (a^1, a^2, a^3) which a^1, a^2, a^3, are the lower bound and the center and upper bound of the fuzzy number $\tilde{a}$ respectively. Input and output data

Table 8: Inputs and outputs targets obtained of model (12).

DMU	Input1	Input2
1	(78.1792,101.2015,128.6181)	(60.0805, 81.5820,107.1305)
2	(28.1600, 48.0000, 72.9600)	(20.4000, 35.0000, 53.5000)
3	(31.1140, 50.6593, 75.1052)	(21.2380, 35.8310, 54.2382)
4	(43.3732, 61.6953, 84.0075)	(24.7154, 39.2798, 57.3019)
5	(81.4800, 96.0000,111.6800)	(35.5250, 50.0000, 66.8250)
6	(34.3634, 53.5845, 77.4648)	(22.1597, 36.7452, 55.0503)
7	(28.1600, 48.0000, 72.9600)	(20.4000, 35.0000, 53.5000)
8	(47.0657, 65.0194, 86.6890)	(25.7629, 40.3186, 58.2247)
9	(53.1214, 70.4709, 91.0865)	(27.4807, 42.0222, 59.7380)
10	(28.3077, 48.1330, 73.0673)	(20.4419, 35.0416, 53.5369)
DMU	Ouput1	Output2
1	(74.2271, 81.5761, 88.9251)	(32.1213, 37.7400, 43.3587)
2	(75.6000, 82.9000, 90.2000)	(33.2000, 38.5000, 43.8000)
3	(74.6914, 81.8529, 89.0144)	(35.3994, 40.5997, 45.8000)
4	(70.9208, 77.5075, 84.0942)	(44.5271, 49.3136, 54.1000)
5	(59.2000, 64.0000, 68.8000)	(72.9000, 76.4000, 79.9000)
6	(73.6920, 80.7011, 87.7102)	(37.8188, 42.9094, 48.0000)
7	(75.6000, 82.9000, 90.2000)	(33.2000, 38.5000, 43.8000)
8	(69.7850, 76.1986, 82.6122)	(47.2765, 51.9382, 56.6000)
9	(67.9224, 74.0521, 80.1817)	(51.7853, 56.2427, 60.7000)
10	(75.5546, 82.8476, 90.1407)	(33.3100, 38.6050, 43.9000)

Table 9: Fuzzy input-output data and fuzzy output prices for DMUs.

DMU	Fuzzy input	Fuzzy output	Fuzzy output price
1	(1, 3, 4)	(2, 3, 4)	(3,4.5,6)
2	(3.5, 4, 4.5)	(1.5, 2.5, 3.5)	(2,4.5,7)
3	(3, 4.5, 6)	(5, 6,7)	(7,7.5,8)
4	(6, 6.5, 7)	(2.75, 4, 5.25)	(0.5,1.5,2.5)
5	(5, 7, 9)	(4.5, 5, 5.5)	(8,8.5,9)
6	(7.5, 8, 8.5)	(3, 3.5, 4)	(1,3,5)
7	(9, 10, 11)	(5.5,6, 6.5)	(6,6.5,7)
8	(5.5, 6, 6.5)	(0.5, 2, 3.5)	(4,5,6)

and the corresponding price of outputs are fuzzy triangular numbers. Table (9) shows the fuzzy input and output data and the fuzzy output price corresponding to each of 8 DMUs. First, we evaluate the revenue efficiency of DMUs based on model (17) and obtain the fuzzy revenue efficiency scores. The results are given in the second column of Table (10). The Tables (10), (11) show the actual observed fuzzy revenue efficiency scores and the maximum revenue from model (17). According to definition (3.4), due to the fuzzy revenue efficiency scores obtained from model (17), units 1 and 3 are fuzzy revenue efficiency units in evaluation with model (17). Because according to the definition (3.4) the actual observed fuzzy revenue efficiency vectors from these units are equal to the maximum revenue vector derived from model (17).
Table (12) show the fuzzy revenue targets obtaining of model (17) corresponding to all DMUs.

5 Conclusion

One of the important issues in DEA is evaluating the cost and revenue efficiency of DMUs in the presence of variable prices of inputs and outputs and provides very important information about the performance of DMUs to the decision-maker. Cost efficiency shows the ability of a DMU to generate its current output with the least cost. Also, revenue efficiency shows the ability of a DMU to generate current input with

Table 10: Fuzzy revenue efficiency and Fuzzy observed revenue of model (17)

DMU	Fuzzy revenue efficiency	Fuzzy observed revenue
1	(0.25, 1, 4)	(6, 13.5, 24)
2	(0.0938, 0.5263, 1.8491)	(3, 11.25, 24.5)
3	(0.625, 1, 1.6)	(35, 45, 56)
4	(0.0246, 0.1333, 0.375)	(1.3750, 6, 13.125)
5	(0.6429, 0.9444, 1.4143)	(36, 42.5, 49.5)
6	(0.0536, 0.2333, 0.5714)	(3, 10.5, 20)
7	(0.5893, 0.8667, 1.3)	(33, 39, 45.5)
8	(0.0357, 0.2222, 0.6)	(2, 10, 21)

Table 11: Maximum revenue values of model (17)

DMU	Maximum revenue values	DMU	Maximum revenue values
1	(6, 13.5, 24)	5	(35, 45, 56)
2	(13.25, 21.375, 32)	6	(35, 45, 56)
3	(35, 45, 56)	7	(35, 45, 56)
4	(35, 45, 56)	8	(35, 45, 56)

Table 12: Targets obtained of model (17).

DMU	Input1	Output1
1	(2, 3, 4)	(6, 13.5, 24)
2	(2.25, 3.375, 4.5)	(13.25, 21.375, 32)
3	(3, 4.5, 6)	(35, 45, 56)
4	(3, 4.5, 6)	(35, 45, 56)
5	(5, 7, 9)	(36, 42, 49.5)
6	(3, 4.5, 6)	(35, 45, 56)
7	(3, 4.5, 6)	(35, 45, 56)
8	(3, 4.5, 6)	(35, 45, 56)

maximum revenue. The present paper obtains cost and revenue efficiency models in the presence of fuzzy inputs and outputs along with the price corresponding to those which are fuzzy numbers, and as seen in the numerical example section, the results of the models presented in this paper with the result of the previous approaches presented to calculate the cost and revenue efficiency is consistent. Fuzzy cost and revenue efficiency scores for each DMU are presented as a fuzzy number whose components are in the interval $[0, 1]$, which are consistent with the definition of cost efficiency in traditional DEA models. Using the proposed approach in this paper, we can easily determine whether the unit under evaluation is a fuzzy cost-efficient unit or not, and the need for additional calculations in previous approaches no longer needed. In the previous approaches, we must compare vectors observed fuzzy cost and the minimum fuzzy cost vector in the efficiency evaluation model which were not computationally appropriate. Also, in revenue efficiency evaluation models, we no longer need to compare the observed fuzzy revenue vector with the maximum fuzzy revenue vector derived from the fuzzy revenue efficiency model to identify the fuzzy revenue efficient unit. The new models presented to calculate cost efficiency and fuzzy revenue can be easily used because these models have a linear structure and we can easily solve them using optimization software. The proposed new models present the fuzzy cost and revenue efficiency targets corresponding to the inefficient fuzzy cost and revenue DMUs and can be suggested to managers as a suitable fuzzy benchmark. As future work, we can solve the models presented in this paper with other methods of solving fuzzy models such as α-level set approach and compare the results presented in this paper with the results of those methods. We can also develop the above models to calculate fuzzy profit efficiency based on the concepts of cost efficiency and fuzzy revenue presented in this paper, and also develop the models presented in the paper for other fuzzy data structures in DEA such as fuzzy network structures.

References

[1] M. Arana-Jim´enez, M.C. S´anchez-Gil, and S. Lozano, Efficiency Assessment and Target Setting Using a Fully Fuzzy DEA Approach,

International Journal of Fuzzy Systems , 22 (4) (2020a), 1056-1072.

[2] M. Arana-Jim´enez, M.C. S´anchez-Gil, and S. Lozano, A fuzzy DEA slacks-based approach, *Journal of Computational and Applied Mathematics*, 404 (2022), 113180.

[3] S. Barak, J. Heidary Dahooei, A novel hybrid fuzzy DEA-Fuzzy MADM method for airlines safety evaluation, *Journal of Air Transport Management* , 73 (2018), 134-149.

[4] B. Bhardwaj, and A. Kumar, A note on "A new algorithm to solve fully fuzzy linear programming problems using the MOLP problem", *Applied mathematical modelling*, 39 (19) (2015), 5982-5985.

[5] A. Charnes, W. W. Cooper and E. Rhodes, Measuring efficiency of decision making units,*European Journal of Operational Research*, 2 (6) (1978), 429–44.

[6] W. W. Cooper, L. M. Seiford, and K. Tone, Data Envelopment Analysis: A Comprehensive Text with Models, Applications, References and DEA Solver Software,*2nd edition, Springer, New York*, (2007).

[7] D. Dubois, and H. Prade, Fuzzy Sets and Systems: Theory and Applications, *Academic Press: New York*, (1980).

[8] A. Emrouznejad, M. Tavana, and A. Hatami-Marbini, The state of the art in fuzzy data envelopment analysis,*Studies in Fuzziness and Soft Computing*, 309 (2014) 1-45.

[9] R. Ezzati, E. Khorram, and R. Enayati, A new algorithm to solve fully fuzzy linear programming problems using the MOLP problem, *Applied Mathematical Modelling*, 39 (2015), 3183–3193.

[10] R. Fare, S. Grosskopf, and C. A. K. Lovell, The measurement of efficiency of production, *Boston: Kluwer Nijhoff*, (1985).

[11] R. Fare, and S. Grosskopf, Resolving a strange case of efficiency, *Journal of the Operational Research Society*, 57 (2006), 1366–1368.

[12] M. J. Farrell, The measurement of productive efficiency, *Journal of the Royal Statistical Society, Series A*, 129 (1957), 253–351.

[13] H. Fukuyama, and W. L. Weber, Economic inefficiency measurement of input spending when decision-making units faces different input prices, *Journal of the Operational Research Society*, 55 (2004), 1102–1110.

[14] M.R. Ghasemi, J. Ignatius, S. Lozano, A. Emrouznejad, and A. Hatami-Marbini, A fuzzy expected value approach under generalized data envelopment analysis, *Knowledge-Based Systems*, 89 (2015) 148-159.

[15] A. Hatami-Marbini, A. Emrouznejad, and M. Tavana, A taxonomy and review of the fuzzy data envelopment analysis literature: two decades in the making, *European Journal of Operational Research*, 214 (3) (2011), 457-472.

[16] F. Hosseinzadeh Lotfi, T. Allahviranloo, M. Alimardani Jondabeh, and L. Alizadeh, Solving a full fuzzy linear programming using lexicography method and fuzzy approximate solution, *Applied Mathematical Modelling*, 33 (2009), 3151–3156.

[17] A. Hosseinzadeh, and S.A. Edalatpanah, A new approach for solving fully fuzzy linear programming by using the lexicography method, *Advances in fuzzy systems*, 2016 (2016), 1-6.

[18] M. Izadikhah, and A. Khoshroo, Energy management in crop production using a novel fuzzy data envelopment analysis model, *RAIRO-Operations Research*, 52 (2018), 595-617.

[19] G. R. Jahanshahloo, M. Soleimani-Damaneh, and A. Mostafaee, A simplified version of the DEA cost efficiency model, *European Journal of Operational Research*, 184 (2008), 814–815.

[20] C. Kao, and S.T. Liu, Fuzzy efficiency measures in data envelopment analysis, *Fuzzy Sets and Systems*, 113 (3) (2000), 427–437.

[21] A. Kumar, J. Kaur, P. Singh, A new method for solving fully fuzzy linear programming problems, *Applied Mathematical Modelling*, 35 (2011), 817–823.

[22] P.J. Korhonen, A. Dehnokhalaji, N. Nasrabadi, A lexicographic radial projection onto the efficient frontier in Data Envelopment Analysis, *European Journal of Operational Research*, 265 (2018), 1005-1012.

[23] S. Lozano, Process efficiency of two-stage systems with fuzzy data, *Fuzzy Sets and Systems*, 234 (2014), 36-49.

[24] S.H. Nasseri, F.H. Lotfi, and M. Soltanifar, A primal-dual method for linear programming problems with fuzzy variables, *European Journal of Industrial Engineering*, 4 (2010), 189-209.

[25] J. C. Paradi, and H. Zhu, A survey on bank branch efficiency and performance research with data envelopment analysis, *Omega*, 41 (2013), 61–79.

[26] K. Paryab, M. Tavana, and R.K. Shiraz, Convex and non-convex approaches for cost efficiency models with fuzzy data, *International journal of data mining modeling and management*, 7 (3) (2015), 213–238.

[27] J. Pourmahmoud, N. Bafekr Sharak, Measuring Cost Efficiency with New Fuzzy DEA Models, *International journal fuzzy systems*, 20 (1) (2018), 155-162.

[28] J. Puri, and S.P. Yadav, A fully fuzzy DEA approach for cost and revenue efficiency measurements in the presence of undesirable outputs and its application to the banking sector in India, *International journal fuzzy systems*, 18 (2) (2016), 212–226.

[29] J. L. Ruiz, and I. Sirvent, Fuzzy cross-efficiency evaluation: a possibility approach, *Fuzzy Optimization and Decision Making*, 16 (2017), 111-126.

[30] S. Saati, and A. Memariani, SBM model with fuzzy input-output levels in DEA, *Australian Journal of Basic and Applied Sciences*, 3, 2 (2009), 352-357.

[31] B. K. Sahoo, M. Mehdiloozad, and K. Tone, Cost, revenue and profit efficiency measurement in DEA: A directional distance function approach, *European Journal of Operational Research*, 237 (2014), 921–931.

[32] L. Stefanini, and M. Arana-Jime´nez, Karush-Kuhn—Tucker conditions for interval and fuzzy optimization in several variables under total and directional generalized differentiability,*Fuzzy Sets and Systems*, 362 (2019) 1-34.

[33] K. Tone, A Strange case of the cost and allocative efficiencies in DEA, *Journal of the Operational Research Society*, 53 (2002), 1225–1231.

[34] L. Wang, X. Liu, Y. Wang, A two-stage granular consensus model for minimum adjustment and minimum cost under Pythagorean fuzzy linguistic information, *Applied Soft Computing* , 125 (2022), 109110.

[35] J. Zhu, Quantitative Models for Performance Evaluation and Benchmarking: Data Envelopment Analysis with Spreadsheets and DEA Excel Solver, *Kluwer Academic Publishers, Boston*, (2002).

[36] W. Zhu, M. Wan, Y. Zhou, and W. Pan, Fuzzy computation of teaching performance based on data envelopment analysis method, *Cognitive Systems Research*, 52 (2018), 351–358.

[37] G. Zou, Y. Huang, W. Chen, L. Wu, Sh. Wen, Life-cycle cost model and evaluation system for power grid assets based on fuzzy membership degree, *Global Energy Interconnection*, 4 (4) (2021), 434-440.

Javad Gerami
Department of Mathematics
Associate Professor of Mathematics
Shiraz branch, Islamic Azad University.
Shiraz, Iran.
E-mail: Geramijavad@gmail.com