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Some Graph Parameters of the Zero-divisor Graphs of Finite Commutative Rings

F. Movahedi*
Golestan University

M. H. Akhbari
Islamic Azad University

Abstract. In this paper, some graph parameters of the zero-divisor graph $\Gamma(R)$ of a finite commutative ring R for $R \simeq \mathbb{Z}_p \times \mathbb{Z}_{p^2}$ and $R \simeq \mathbb{Z}_p \times \mathbb{Z}_{2p}$ where $p > 2$ a prime, are investigated. The graph $\Gamma(R)$ is a simple graph whose vertex set is the set of non-zero zero-divisors of a commutative ring R with non-zero identity and two vertices u and v are adjacent if and only if $uv = vu = 0$.
In this paper, we study some of the topological indices such as graph energies, the Zagreb indices and the domination parameters of graphs $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{p^2})$ and $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{2p})$.

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1 Introduction

Suppose that $G = (V, E)$ is a simple graph in which V and E are the vertex and edge sets, respectively. The set $N_G(u) = \{v \in V | uv \in E\}$

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*Corresponding Author

and $|N_G(u)|$ are called the neighborhood of vertex $u \in V$ and the degree of vertex u in graph G, respectively. The degree of vertex u is denoted by $deg(u)$. The isolated vertex and pendant vertex are the vertices with degrees zero and 1 in graph G, respectively. The first Zagreb index $M_1(G)$ and the second Zagreb index $M_2(G)$ are defined as follows[13]

$$M_1 = M_1(G) = \sum_{v \in V(G)} d(v)^2, \qquad M_2 = M_2(G) = \sum_{vu \in E(G)} d(u)d(v), \quad (1)$$

for a graph G.

A topological index in a graph G is the Randić index, which is defined in [22] as follows

$$R(G) = \sum_{uv \in E(G)} \frac{1}{d(u)d(v)}. \qquad (2)$$

An $n \times n$ matrix is called the adjacency matrix of the graph G in which $a_{ij} = 1$ if two vertices v_i and v_j are adjacent and $a_{ij} = 0$ otherwise. One can consider the eigenvalues of $A(G)$ as the eigenvalues of graph G [14]. The energy of a graph G is defined as $E(G) = \sum_{i=1}^{n} |\lambda_i|$ in which $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of the graph G [10].

For a graph G, its line graph L_G is a graph such that each vertex of the L_G represents an edge of G, and two vertices of L_G are adjacent if and only if their corresponding edges have a common vertex in G [14]. The energy of the line graph of a given graph is called the edge energy of a graph G and is denoted by $EE(G)$ [3].

If we assume that $D(G) := (d_{ii})$ is the diagonal matrix of order $n \times n$ such that $d_{ii} = deg(v_i)$ for any $v_i \in V(G)$, then the matrix $L(G) = D(G) - A(G)$ is called the Laplacian matrix of the graph G. In [11], it is defined the Laplacian energy of G as follows

$$LE = LE(G) = \sum_{i=1}^{n} \left|\mu_i - \frac{2m}{n}\right|,$$

where $\mu_1, \mu_2, \ldots, \mu_n$ are the eigenvalues of the matrix $L(G)$.

More about Laplacian energies and the energy of a line graph can be seen in [7, 8, 11, 12].

A dominating set of a given graph G is a set $D \subseteq V$ such that every vertex of $V \setminus D$ is adjacent to at least one vertex of D. The number of vertices in the smallest dominating set is called a minimum dominating set and is denoted by $\gamma(G)$ [15]. A dominating set D of the graph G is a total dominating set if every vertex has, at least, a neighbor in D. The total domination number $\gamma_t(G)$ is the number of elements of the smallest total dominating set [15]. The dominating set D is a connected dominating set if the induced subgraph $\langle D\rangle$ is connected. The connected domination number $\gamma_c(G)$ is the number of elements of the smallest size among all connected dominating sets of G [15].

Suppose that $f : V \to \{0, 1, 2\}$, and (V_0, V_1, V_2) is the ordered divisions of V such that $V_i = \{v \in V | f(v) = i\}$ and $|V_i| = n_i$, for $i = 0, 1, 2$. A $1-1$ correspondence between the functions $f : V \to \{0, 1, 2\}$ and the ordered divisions (V_0, V_1, V_2) of $V(G)$. So, one can consider $f = (V_0, V_1, V_2)$. For a function $f = (V_0, V_1, V_2)$, if $V_2 \succ V_0$, where $\succ$ means that the set V_2 dominates the set V_0, then f is a Roman dominating function. The weight of f is $f(V) = \sum_{v\in V(G)} f(v) = 2n_2 + n_1$ [5]. The Roman domination number, denoted $\gamma_R(G)$, equals the minimum weight of a Roman dominating function of G.

For a graph G with edge set $\{e_1, e_2, \dots, e_m\}$, a subset $F \subseteq E$ is the edge dominating set if every edge in $E\backslash F$ is adjacent to some edges in F. The edge domination number, denoted by γ', is the number of edges of the smallest edge dominating set of G [9]. Note that the minimum edge dominating set of G is the minimum dominating set of L_G. A $m \times m$ matrix $A_F(G) := (a_{ij})$ is defined as follows is called the minimum edge dominating matrix of G,

$$A_F(G) := (a_{ij}) = \begin{cases} 1 & \text{if } e_i \textit{ and } e_j \textit{ are adjacent}, \\ 1 & \text{if } i = j \textit{ and } e_i \in F, \\ 0 & \textit{otherwise}. \end{cases}$$

The minimum edge dominating energy of G is introduced and studied in [1] as following

$$EE_F(G) = \sum_{i=1}^{m} |\lambda_i|,$$

where $\lambda_1, \lambda_2, \dots, \lambda_m$ are the eigenvalues of $A_F(G)$. Note the minimum edge dominating energy of graph G is a minimum dominating energy for its line graph L_G. For more results about the minimum edge dominating energy of a graph and its line graph, see [1, 18, 19, 20].

For a given ring R, $Z(R)$ denotes the set of all zero-divisors of R. The zero-divisor graph of R, denoted by $\Gamma(R)$, is a simple graph with vertex set $Z(R) \setminus \{0\}$ such that two distinct vertices $x, y \in V(\Gamma(R))$ are adjacent if and only if $xy = 0$ [2].

In this paper, we study some graph parameters, namely, the Zagreb indices and Randić index, graph energy, Laplacian energy, the minimum edge dominating energy, and some parameters of domination of $\Gamma(R)$ for the commutative rings $R \simeq \mathbb{Z}_p \times \mathbb{Z}_{p^2}$ and $R \simeq \mathbb{Z}_p \times \mathbb{Z}_{2p}$ where p is a prime number.

2 Preliminaries and Known Results

In this section, we state some previous results that will be required in the proof of the main results.

Lemma 2.1. [4, 17] *Let G be a graph with n vertices and m edges. Then*

$$2\sqrt{m} \leq E(G) \leq \sqrt{2nm}.$$

Lemma 2.2. [16] *For a graph G with n vertices,*

$$E(G) \leq \frac{n}{2}\big(1 + \sqrt{n}\big).$$

Lemma 2.3. [23, 24] *Let G be a graph with n vertices and m edges. Then*

$$\frac{4m}{n} \leq LE(G) \leq 4m\Big(1 - \frac{1}{n}\Big).$$

Lemma 2.4. [6] *Let G be a connected graph of order n and size $m \geq \frac{n}{2}$ with maximum degree Δ. Then*

$$2\Big(\Delta + 1 - \frac{2m}{n}\Big) \leq LE(G) \leq 4m - 2\Delta - \frac{4m}{n} + 2.$$

Lemma 2.5. [12] *Let G be a graph of order n with m edges. Then*

$$\sqrt{2M_1(G) - 4m} \leq E(L_G) \leq M_1(G) - 2m,$$

where L_G and $M_1(G)$ are the line graph and the first Zagreb index of graph G.

Lemma 2.6. [18] *Let G be a graph of size m. If F is the minimum edge dominating set of graph G, then*

$$EE_F(G) \leq M_1(G) - m,$$

where $M_1(G)$ is the first Zagreb index of graph G.

Lemma 2.7. [18] *For a connected graph G with n vertices and $m(\geq n)$ edges,*

$$EE_F(G) \geq 4(m - n + s) + 2p,$$

where p and s are the number of pendant vertices and isolated vertices in G, respectively.

3 Main Results

In this section, we give the results of some of the topological indices on the zero-divisor graphs $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{p^2})$ and $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{2p})$.

First, we investigate these topological indices of the zero-divisor graph $\Gamma(R)$ where $R \simeq \mathbb{Z}_p \times \mathbb{Z}_{p^2}$. This graph is a connected graph with the vertex set $Z(R^*) = \bigcup_{i=1}^{4} A_i$ where $R^* = R \setminus \{0\}$ such that [21],

$$\begin{aligned}
&A_1 = \{(u,0) : u \in \mathbb{Z}_p^*\}, \quad |A_1| = p-1,\\
&A_2 = \{(0,v) : v \in \mathbb{Z}_{p^2}^*,\ v \notin Z(\mathbb{Z}_{p^2}^*)\}, \quad |A_2| = p^2 - p,\\
&A_3 = \{(0,w) : w \in \mathbb{Z}_{p^2}^*,\ w \in Z(\mathbb{Z}_{p^2}^*)\}, \quad |A_3| = p-1,\\
&A_4 = \{(a,b) : a \in \mathbb{Z}_p^*,\ b \in \mathbb{Z}_{p^2}^* \text{ and } b \in Z(\mathbb{Z}_{p^2}^*)\}, \quad |A_4| = (p-1)^2.
\end{aligned} \tag{3}$$

Therefore, the order of graph $\Gamma\big(\mathbb{Z}_p \times \mathbb{Z}_{p^2}\big)$ equals $n = \sum_{i=1}^{4} \big|A_i\big| = 2p^2 - p - 1$. In [21], the degree of any vertex in the sets A_i for $i = 1, 2, 3, 4$ is obtained. Thus, we can show the degree sequence of $\Gamma\big(\mathbb{Z}_p \times \mathbb{Z}_{p^2}\big)$ as following

$$DS\big(\Gamma\big(\mathbb{Z}_p \times \mathbb{Z}_{p^2}\big)\big) = \Big\{(p^2-1)^{[p-1]}, (p^2-2)^{[p-1]}, (p-1)^{[2p^2-3p+1]}\Big\}. \quad (4)$$

Here we study some of the topological indices such as the Zagreb indices and Randić index on the zero-divisor graph $\Gamma(R)$ where $R \simeq \mathbb{Z}_p \times \mathbb{Z}_{p^2}$ for a prime number $p > 2$.

Theorem 3.1. *Let $\Gamma(R)$ be a zero-divisor graph of the commutative ring such that $R \simeq \mathbb{Z}_p \times \mathbb{Z}_{p^2}$ for a prime number $p > 2$. Then*

i) $M_1\big(\Gamma(R)\big) = 2p^5 - 13p^3 + 15p^2 - 4.$

ii) $M_2\big(\Gamma(R)\big) = \frac{1}{2}(p-1)(7p^5 - 14p^4 - 8p^3 + 28p^2 - 6p - 8).$

iii) $R\big(\Gamma(R)\big) = \frac{4p^5-2p^4-11p^3+2p^2+7p+2}{2(p+1)(p^2-2)^2}.$

Proof. We suppose that G is the zero-divisor graph $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{p^2})$ of order $n = 2p^2 - p - 1$.

i) According to the degree sequence (4) of graph $\Gamma\big(\mathbb{Z}_p \times \mathbb{Z}_{p^2}\big)$, we get

$$\begin{aligned} M_1(G) &= \sum_{i=1}^{n} d_i^2 \\ &= (p-1)(p^2-1)^2 + (p-1)(p^2-2)^2 + (2p^2-3p+1)(p-1)^2 \\ &= (p-1)(2p^4 + 2p^3 - 11p^2 + 4p + 4) \\ &= 2p^5 - 13p^3 + 15p^2 - 4. \end{aligned}$$

ii) According to the structure of zero-divisor graph $\Gamma\big(\mathbb{Z}_p \times \mathbb{Z}_{p^2}\big)$ in [21], the adjacency matrix of graph G is as follows

$$A(G) = \begin{bmatrix} \mathbf{0}_{p-1} & \mathbf{1}_{(p-1)\times(p^2-p)} & \mathbf{1}_{p-1} & \mathbf{0}_{(p-1)\times(p-1)^2} \\ \mathbf{1}_{(p^2-p)\times(p-1)} & \mathbf{0}_{p^2-p} & \mathbf{0}_{(p^2-p)\times(p-1)} & \mathbf{0}_{(p^2-p)\times(p-1)^2} \\ \mathbf{1}_{p-1} & \mathbf{0}_{(p-1)\times(p^2-p)} & J_{p-1} & \mathbf{1}_{(p-1)\times(p-1)^2} \\ \mathbf{0}_{(p-1)^2\times(p-1)} & \mathbf{0}_{(p-1)^2\times(p^2-p)} & \mathbf{1}_{(p-1)^2\times(p-1)} & \mathbf{0}_{(p-1)^2} \end{bmatrix},$$

in which $\mathbf{1}_i$, $\mathbf{0}_i$ and J_i denoting the all-ones matrix, the zero matrix, and a matrix with any elements on the main diagonal is equal to 0 and all other elements equal to 1, respectively, of order i.
Based on the above matrix, the edge set of the graph $\Gamma\big(\mathbb{Z}_p \times \mathbb{Z}_{p^2}\big)$ can be divided $E(G) = \overset{4}{\underset{i=1}{\cup}} E_i$ such that

$$\begin{aligned}
&E_1 = \big\{uv \in E : u \in A_1,\ v \in A_2\big\}, \quad \big|E_1\big| = (p-1)(p^2-p),\\
&E_2 = \big\{uv \in E : u \in A_1,\ v \in A_3\big\}, \quad \big|E_2\big| = (p-1)^2,\\
&E_3 = \big\{uv \in E : u \in A_3,\ v \in A_3,\ u \neq v\big\}, \quad \big|E_3\big| = \frac{(p-1)(p-2)}{2},\\
&E_4 = \big\{uv \in E : u \in A_3,\ v \in A_4\big\}, \quad \big|E_4\big| = (p-1)^3.
\end{aligned}$$

Consequently, the number of edges of the graph $\Gamma\big(\mathbb{Z}_p \times \mathbb{Z}_{p^2}\big)$ is equal to

$$|E| = \sum_{i=1}^{4} |E_i| = \frac{1}{2}(p-1)(4p^2-3p-2).$$

Therefore, from the definition 1, we get

$$\begin{aligned}
M_2(G) &= \sum_{uv \in E} d(u)d(v)\\
&= \sum_{i=1}^{4} \big|E_i\big| \big\{d(u)d(v) : uv \in E_i\big\}\\
&= (p-1)(p^2-p)\Big((p^2-1)(p-1)\Big) + (p-1)^2\Big((p^2-1)(p^2-2)\Big)\\
&+ \frac{(p-1)(p-2)}{2}\Big((p^2-2)^2\Big) + (p-1)^3\Big((p^2-2)(p-1)\Big)\\
&= \frac{1}{2}(p-1)(7p^5-14p^4-8p^3+28p^2-6p-8).
\end{aligned}$$

iii) Similar to the case (ii) and the definition 2, we have

$$\begin{aligned}
R(G) &= \sum_{uv\in E} \frac{1}{d(u)d(v)} \\
&= \sum_{i=1}^{4} |E_i| \{ \frac{1}{d(u)d(v)} : uv \in E_i \} \\
&= \frac{(p-1)(p^2-p)}{(p^2-1)(p-1)} + \frac{(p-1)^2}{(p^2-1)(p^2-2)} \\
&+ \frac{(p-1)(p-2)}{2(p^2-2)^2} + \frac{(p-1)^3}{(p^2-2)(p-1)} \\
&= \frac{4p^5-2p^4-11p^3+2p^2+7p+2}{2(p+1)(p^2-2)^2}.
\end{aligned}$$

□

The following results, some of the graph energies such as the graph energy, Laplacian energy, edge energy and the minimum edge dominating energy of the zero-divisor graph $\Gamma(R)$ where $R \simeq \mathbb{Z}_p \times \mathbb{Z}_{p^2}$ for a prime number $p > 2$.

In the Theorem 3.2, we get a lower bound and upper bound for $E\big(\Gamma(R)\big)$ where $R \simeq \mathbb{Z}_p \times \mathbb{Z}_{p^2}$.

Theorem 3.2. *Let $\Gamma(R)$ be a zero-divisor graph of a commutative ring $R \simeq \mathbb{Z}_p \times \mathbb{Z}_{p^2}$ for a prime number $p > 2$. Then*

$$\sqrt{(2p-2)\alpha} \leq E\big(\Gamma(R)\big) \leq \alpha(p-1)\sqrt{(2p+1)\alpha},$$

where $\alpha = 4p^2 - 3p - 2$.

Proof. Suppose that G is the zero-divisor graph $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{p^2})$ for $p > 2$ where p is a prime.

Since the graph G has $n = 2p^2 - p - 1$ vertices and $m = \frac{1}{2}(p-1)(4p^2-3p-2)$ edges, using Lemma 2.1 we get

$$\begin{aligned}
E(G) &\geq 2\sqrt{m} \\
&= 2\sqrt{\frac{1}{2}(p-1)(4p^2-3p-2)} \\
&= \sqrt{2(p-1)(4p^2-3p-2)}.
\end{aligned}$$

With considering $\alpha = 4p^2 - 3p - 2$, the lower bound is achieved.

For the upper bound, by applying Lemma 2.1, we have

$$\begin{aligned} E(G) &\leq \sqrt{2nm} \\ &= \sqrt{(p-1)^2(2p+1)(4p^2-3p-2)} \\ &= (p-1)\sqrt{(2p+1)(4p^2-3p-2)}. \end{aligned}$$

With putting $\alpha = 4p^2 - 3p - 2$, the result follows. □

Theorem 3.3. *Let $\Gamma(R)$ be a zero-divisor graph of a commutative ring $R \simeq \mathbb{Z}_p \times \mathbb{Z}_{p^2}$ for a prime number $p > 2$. Then*

$$\alpha \leq LE\big(\Gamma(R)\big) \leq \alpha(2p^2 - p - 2),$$

in which $\alpha = \frac{2(4p^2-3p-2)}{(2p+1)}$.

Proof. Assume that G is the zero-divisor graph $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{p^2})$ of order $n = 2p^2 - p - 1$ and size $m = \frac{1}{2}(p-1)(4p^2 - 3p - 2)$ for $p > 2$ where p a prime.

For the lower bound, we apply Lemma 2.3 and we have

$$\begin{aligned} LE(G) &\geq \frac{4m}{n} \\ &= \frac{2(p-1)(4p^2-3p-2)}{2p^2-p-1} \\ &= \frac{2(p-1)(4p^2-3p-2)}{(p-1)(2p+1)} \\ &= \frac{2(4p^2-3p-2)}{2p+1}. \end{aligned}$$

With putting $\alpha = \frac{2(4p^2-3p-2)}{(2p+1)}$ the result follows.

For the upper bound, using Lemma 2.3, we get

$$\begin{aligned} LE(G) &\leq 4m\Big(1-\frac{1}{n}\Big)\\ &= 4m-\frac{4m}{n}\\ &= 2(p-1)(4p^2-3p-2)-\frac{2p(4p^2-3p-2)}{2p+1}\\ &= \frac{2(4p^2-3p-2)}{2p+1}(2p^2-p-2), \end{aligned}$$

where by putting $\alpha = \frac{2(4p^2-3p-2)}{(2p+1)}$ in the above relation, the result is completed. □

Theorem 3.4. *Let $\Gamma(R)$ be a zero-divisor graph of a commutative ring $R \simeq \mathbb{Z}_p \times \mathbb{Z}_{p^2}$ for a prime number $p > 2$. Then*

$$\sqrt{2\alpha(p-1)} \leq E\big(L_{\Gamma(R)}\big) \leq \alpha(p-1),$$

where $\alpha = 2p^4 + 2p^3 - 15p^2 + 7p + 6$.

Proof. Theorem 3.1(i) and Lemma 2.5, give us the results. □

In the next theorem, we study the minimum edge dominating energy of the zero-divisor graph $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{p^2})$.

Theorem 3.5. *Let $\Gamma(R)$ be a zero-divisor graph of a commutative ring $R \simeq \mathbb{Z}_p \times \mathbb{Z}_{p^2}$ for a prime number $p > 2$. Then*

$$2(p-1)\alpha \leq EE_F\big(\Gamma(R)\big) \leq \frac{1}{2}(p-1)\big(\alpha(p-2)+\beta\big),$$

where $\alpha = 4p^2 - 7p - 4$ and $\beta = 4p^4 - 11p^2 + p + 2$.

Proof. Let G be the zero-divisor graph $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{p^2})$ for $p > 2$ where p a prime.

According to the structure of the zero-divisor graph G, the number of isolated vertices and pendant vertices are zero. Since graph G has

$n = 2p^2 - p - 1$ vertices and $m = \frac{1}{2}(p-1)(4p^2-3p-2)$ edges, by applying Lemma 2.7 we have

$$\begin{aligned} EE_F(G) &\geq 4(m-n) \\ &= 4\big(\frac{1}{2}(p-1)(4p^2-3p-2) - (2p^2-p-1)\big) \\ &= \big(2(p-1)(4p^2-3p-2) - 2(p-1)(2p+1)\big) \\ &= 2(p-1)\big(4p^2-7p-4\big). \end{aligned}$$

With considering $\alpha = 4p^2 - 7p - 4$, the result is obtained for the lower bound.

For the upper bound, by applying Lemma 2.6 and Theorem 3.1(i), we have

$$\begin{aligned} EE_F(G) &\leq M_1(G) - m \\ &= (2p^5 - 13p^3 + 15p^2 - 4) - \frac{1}{2}(p-1)(4p^2-3p-2) \\ &= \frac{1}{2}(p-1)\big(4p^4 + 4p^3 - 26p^2 + 11p + 10\big) \\ &= \frac{1}{2}(p-1)\big(4p^4 + 4p^3 - 18p^2 - 3p + 2 - 2\alpha\big), \end{aligned}$$

in which $\alpha = 4p^2 - 7p - 4$.

By simplifying the right side of the above relation, we get

$$\begin{aligned} EE_F(G) &\leq \frac{1}{2}(p-1)\big(4p^4 + p(\alpha - 11p + 1) + 2 - 2\alpha\big) \\ &= \frac{1}{2}(p-1)\big(\alpha(p-2) + 4p^4 - 11p^2 + p + 2\big). \end{aligned}$$

With putting $\beta = 4p^4 - 11p^2 + p + 2$, the result follows. $\square$

In the follwing results, we study some parameters of the domination of the zero-divisor graph $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{p^2})$.

Theorem 3.6. *Let $\Gamma(R)$ be a zero-divisor graph of a commutative ring $R \simeq \mathbb{Z}_p \times \mathbb{Z}_{p^2}$ for a prime number $p > 2$. Then $\gamma\big(\Gamma(R)\big) = 2$ and the minimum dominating set of $\Gamma(R)$ is the set $\{x, y\} \in \{A_1, A_3\}$.*

Proof. Let G be the zero-divisor graph $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{p^2})$ for $p > 2$ where p is a prime. We suppose that D is a minimum dominating set of graph G. According to the proof of Theorem 3.1(ii) and since the vertex set of graph G is $V(G) = \cup_{i=1}^{4} A_i$ such that the sets A_i are defined in (3), each vertex in A_3 is adjacent to all other vertices in A_1 and A_4. Also, any two vertices of A_3 are adjacent to each other. Therefore, by selecting one vertex, called x, in A_3 all vertices of $V(G) \setminus A_2$ dominate. Thus, $x \in D$.

It is sufficient to dominate vertices of the set A_2. To do this, we select an arbitrary vertex, called y in A_1 that is adjacent to all vertices of A_2. So, $y \in D$. Therefore, $|D| = 2$. □

Theorem 3.7. *Let $\Gamma(R)$ be a zero-divisor graph of a commutative ring $R \simeq \mathbb{Z}_p \times \mathbb{Z}_{p^2}$ for a prime number $p > 2$. Then*

i) $\gamma_t\big(\Gamma(R)\big) = 2$,

ii) $\gamma_c\big(\Gamma(R)\big) = 2$,

iii) $\gamma_R\big(\Gamma(R)\big) = 4$.

Proof. Let G be the zero-divisor graph $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{p^2})$ for $p > 2$ where p is a prime. According to the proof of Theorem 3.6 and the definition of a total dominating set, by selecting the vertices $x \in A_1$ and $y \in A_3$, the set $D = \{x, y\}$ is the total dominating set of graph G. Therefore, $\gamma_t(G) = 2$. Also, based on the definition of a connected dominating set, the set D is a connected dominating set.

We consider the function $f : V \longrightarrow \{0, 1, 2\}$ such that $f(x) = f(y) = 2$ in which $x \in A_1$ and $y \in A_3$ and $f(u) = 0$ for $u \in V(G) \setminus \{x, y\}$. In this case, all the vertices are dominated by the set $\{x, y\}$ and $\gamma_R(G) = 4$. □

Now, we consider the zero-divisor graph $\Gamma(R)$ where $R \simeq \mathbb{Z}_p \times \mathbb{Z}_{2p}$ for $p > 2$. In [21], the vertex set of graph $\Gamma\big(\mathbb{Z}_p \times \mathbb{Z}_{2p}\big)$ is partitioned as

$V = \overset{4}{\underset{i=1}{\cup}} B_i$ where B_i's are as follows

$$\begin{aligned} &B_1 = \{(u,0) : u \in \mathbb{Z}_p^*\}, \quad |B_1| = p-1, \\ &B_2 = \{(0,v) : v \in \mathbb{Z}_{2p}^*,\ v \notin Z(\mathbb{Z}_{2p}^*)\}, \quad |B_2| = p-1, \\ &B_3 = \{(0,w) : w \in \mathbb{Z}_{2p}^*,\ w \in Z(\mathbb{Z}_{2p}^*)\}, \quad |B_3| = p, \\ &B_4 = \{(a,b) : a \in \mathbb{Z}_p^*,\ b \in \mathbb{Z}_{2p}^* \textit{ and } b \in Z(\mathbb{Z}_{2p}^*)\}, \quad |B_4| = p^2-p. \end{aligned} \tag{5}$$

Therefore, the order of graph $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{p^2})$ equals $n = \sum_{i=1}^{4} |B_i| = p^2+2p-2$. Based on the obtained degree of any vertices of the sets B_i for $i = 1,2,3,4$ in [21], the degree sequence of $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{p^2})$ is as following

$$DS(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{2p})) = \Big\{(2p-1)^{[2(p-1)]}, (p-1)^{[2(p-1)]}, (p^2-1)^{[1]}, (1)^{[(p-1)^2]}\Big\}. \tag{6}$$

In the first, some of the topological indices such as the Zagreb indices and Randić index on the zero-divisor graph $\Gamma(R)$ where $R \simeq \mathbb{Z}_p \times \mathbb{Z}_{2p}$ for a prime number $p > 2$ are obtained.

Theorem 3.8. *Let $\Gamma(R)$ be a zero-divisor graph of a commutative ring such that $R \simeq \mathbb{Z}_p \times \mathbb{Z}_{2p}$ for a prime number $p > 2$. Then*

i) $M_1(\Gamma(R)) = (p-1)(p^3+11p^2-12p+2)$.

ii) $M_2(\Gamma(R)) = (p-1)^2(5p^2-2p-1)$.

iii) $R(\Gamma(R)) = \frac{p(9p^2-8p+1)}{(2p-1)^2(p+1)}$.

Proof. Let G be the zero-divisor graph $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{2p})$ of order $n = p^2+2p-2$.

i) According to the degree sequence (6) of graph $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{2p})$, we get

$$\begin{aligned} M_1(G) &= \sum_{i=1}^{n} d_i^2 \\ &= 2(p-1)(2p-1)^2 + 2(p-1)(p-1)^2 + (p^2-1)^2 + (p-1)^2 \\ &= p^4 + 10p^3 - 23p^2 + 14p - 2 \\ &= (p-1)(p^3+11p^2-12p+2). \end{aligned}$$

ii) According to the structure of the zero-divisor graph $\Gamma\big(\mathbb{Z}_p \times \mathbb{Z}_{2p}\big)$ in [21], the adjacency matrix of graph G is as follows

$$A(G) = \begin{bmatrix} \mathbf{0}_{p-1} & \mathbf{1}_{p-1} & \mathbf{1}_{(p-1)\times p} & \mathbf{0}_{(p-1)\times(p^2-p)} \\ \mathbf{1}_{p-1} & \mathbf{0}_{p-1} & \mathbf{0}_{(p-1)\times p} & \mathbf{0}_{(p-1)\times(p^2-p)} \\ \mathbf{1}_{p\times(p-1)} & \mathbf{0}_{p\times(p-1)} & M_1 & M_2 \\ \mathbf{0}_{(p^2-p)\times(p-1)} & \mathbf{0}_{(p^2-p)\times(p-1)} & M_2^T & \mathbf{0}_{p^2-p} \end{bmatrix},$$

in which $\mathbf{1}_i$ and $\mathbf{0}_i$ denote the all-ones matrix and the zero matrix of order i, respectively. Also, the matrix M_2^T is the transpose of the matrix M_2 and the matrices M_1 of order $p \times p$ and M_2 of order $p \times (p^2 - p)$ are as following.

$$M_1 = \begin{bmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & 0 \end{bmatrix},$$

and

$$M_2 = \begin{bmatrix} 1 & \dots & 1 & 0 & \dots & 0 \\ 0 & \dots & 0 & 1 & \dots & 1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & \dots & 1 \end{bmatrix}.$$

Based on the above matrix, the edge set of graph $\Gamma\big(\mathbb{Z}_p \times \mathbb{Z}_{2p}\big)$ can be partitioned $E(G) = \bigcup\limits_{i=1}^{4} E_i$ such that

$$\begin{aligned} &E_1 = \big\{uv \in E : u \in B_1,\ v \in B_2\big\}, \\ &E_2 = \big\{uv \in E : u \in B_1,\ v \in B_3\big\}, \\ &E_3 = \big\{uv \in E : u, v \in B_3,\ u = (0, p) \ and\ v = (0, 2m)\big\}, \\ &E_4 = \big\{uv \in E : u \in B_3,\ v \in B_4,\ u = (0, p) \ and\ v = (a, 2m)\big\}, \\ &E_5 = \big\{uv \in E : u \in B_3,\ v \in B_4,\ u = (0, 2m) \ and\ v = (a, p)\big\}, \end{aligned}$$

in which $1 \leq m \leq p-1$ and $a \in Z(P^*)$.

One can easily obtain the cardinality of each set E_i for $1 \leq i \leq 5$. Therefore, $\left|E_1\right| = (p-1)^2$, $\left|E_2\right| = p(p-1)$, $\left|E_3\right| = p-1$, $\left|E_4\right| = (p-1)^2$ and $\left|E_5\right| = (p-1)^2$. Consequently, the number of edges of the graph $\Gamma\big(\mathbb{Z}_p \times \mathbb{Z}_{2p}\big)$ is equal to

$$|E| = \sum_{i=1}^{5} |E_i| = 2(p-1)(2p-1).$$

Therefore, using the definition 1, we get

$$\begin{aligned}
M_2(G) &= \sum_{uv\in E} d(u)d(v) \\
&= \sum_{i=1}^{5} \left|E_i\right| \big\{d(u)d(v) : uv \in E_i\big\} \\
&= (p-1)^2\Big((2p-1)(p-1)\Big) + p(p-1)\Big((2p-1)^2\Big) \\
&+ p(p-1)\Big((2p-1)(p^2-1)\Big) + (p-1)\Big((p^2-1)(2p-1)\Big) \\
&+ (p-1)^2\Big(p^2-1\Big) + (p-1)^2\Big((2p-1)(p-1)\Big) \\
&= (p-1)^2(5p^2-2p-1).
\end{aligned}$$

iii) Similar to the case (ii) and the definition 2, we have

$$\begin{aligned}
R(G) &= \sum_{uv\in E} \frac{1}{d(u)d(v)} \\
&= \sum_{i=1}^{5} \left|E_i\right| \big\{\frac{1}{d(u)d(v)} : uv \in E_i\big\} \\
&= \frac{(p-1)^2}{(2p-1)(p-1)} + \frac{p(p-1)}{(2p-1)^2} + \frac{p(p-1)}{(2p-1)(p^2-1)} \\
&+ \frac{(p-1)}{(p^2-1)(2p-1)} + \frac{(p-1)^2}{(p^2-1)} + \frac{(p-1)^2}{(2p-1)(p-1)} \\
&= \frac{p\big(9p^2-8p+1\big)}{(2p-1)^2(p+1)}.
\end{aligned}$$

□

Theorem 3.9. *Let $\Gamma(R)$ be a zero-divisor graph of a commutative ring $R \simeq \mathbb{Z}_p \times \mathbb{Z}_{2p}$ for a prime number $p > 2$. Then*

$$2\sqrt{\alpha(\alpha+1)} \leq E\big(\Gamma(R)\big) \leq \frac{1}{16}\Big(\alpha(\alpha+8)+4\Big)\Big(2+\sqrt{\alpha(\alpha+8)+4}\Big),$$

where $\alpha = 2(p-1)$.

Proof. Let G be the zero-divisor graph $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{2p})$ for $p > 2$ where p is a prime. According to the above discussions, the graph G has $n = p^2 + 2p - 2$ vertices and $m = 2(p-1)(2p-1)$ edges. So using Lemma 2.1 we obtain

$$\begin{aligned} E(G) &\geq 2\sqrt{m} \\ &= 2\sqrt{2(p-1)(2p-1)} \\ &= 2\sqrt{2(p-1)\big(2(p-1)+1\big)}. \end{aligned}$$

With considering $\alpha = 2(p-1)$, the lower bound is obtained.

For the upper bound, by applying Lemma 2.2, we have

$$\begin{aligned} E(G) &\leq \frac{n}{2}\big(1+\sqrt{n}\big) \\ &= \frac{p^2+2p-2}{2}\Big(1+\sqrt{p^2+2p-2}\Big) \\ &= \frac{p^2+2(p-1)}{2}\Big(1+\sqrt{p^2+2(p-1)}\Big) \\ &= \frac{\Big(\frac{2(p-1)}{2}+1\Big)^2+2(p-1)}{2}\Bigg(1+\sqrt{\Big(\frac{2(p-1)}{2}+1\Big)^2+2(p-1)}\Bigg). \end{aligned}$$

With putting $\alpha = 2(p-1)$, the result follows. □

Theorem 3.10. *Let $\Gamma(R)$ be a zero-divisor graph of a commutative ring $R \simeq \mathbb{Z}_p \times \mathbb{Z}_{2p}$ for a prime number $p > 2$. Then*

$$\alpha \leq LE\big(\Gamma(R)\big) \leq \alpha(2p^2-p-2),$$

in which $\alpha = \frac{2(4p^2-3p-2)}{(2p+1)}$.

Proof. Let G be the zero-divisor graph $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{2p})$ of order $n = p^2 + 2p - 2$ and size $m = 2(p-1)(2p-1)$ for $p > 2$ where p a prime.

For lower bound using Lemma 2.3, we have

$$\begin{aligned} LE(G) &\geq \frac{4m}{n} \\ &= \frac{8(p-1)(2p-1)}{p^2+2p-2} \\ &= \frac{2(p-1)(4p^2-3p-2)}{(p-1)(2p+1)} \\ &= \frac{2(4p^2-3p-2)}{2p+1}. \end{aligned}$$

With putting $\alpha = \frac{2(4p^2-3p-2)}{(2p+1)}$ the result follows.

For the upper bound, using Lemma 2.3, we get

$$\begin{aligned} LE(G) &\leq 4m\Big(1 - \frac{1}{n}\Big) \\ &= 4m - \frac{4m}{n} \\ &= 2(p-1)(4p^2-3p-2) - \frac{2p(4p^2-3p-2)}{2p+1} \\ &= \frac{2(4p^2-3p-2)}{2p+1}(2p^2-p-2), \end{aligned}$$

where by putting $\alpha = \frac{2(4p^2-3p-2)}{(2p+1)}$ in the above relation, the result is completed. □

Theorem 3.11. *Let $\Gamma(R)$ be a zero-divisor graph of a commutative ring $R \simeq \mathbb{Z}_p \times \mathbb{Z}_{2p}$ for a prime number $p > 2$. Then*

$$\frac{2\alpha}{\beta} \leq LE\big(\Gamma(R)\big) \leq \frac{\alpha}{\beta} + (p-1)^2,$$

in which $\alpha = p^4 + 2p^3 - 10p^2 + 12p - 4$ and $\beta = p^2 + 2p - 2$.

Proof. Let G be the zero-divisor graph $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{2p})$ of order $n = p^2 + 2p - 2$ and size $m = 2(p-1)(2p-1)$ for $p > 2$ where p a prime.

According to the degree sequence of graph G in (6), the maximum degree in graph G is $\Delta = P^2 - 1$. Therefore, using Lemma 2.4, for the lower bound we obtain

$$\begin{aligned} LE(G) &\geq 2\Big(\Delta + 1 - \frac{2m}{n}\Big) \\ &= 2\Big(p^2 - 1 + 1 - \frac{4(p-1)(2p-1)}{p^2+2p-2}\Big) \\ &= \frac{p^4 + 2p^3 - 10p^2 + 12p - 4}{p^2 + 2p - 2)}. \end{aligned}$$

With putting $\alpha = p^4 + 2p^3 - 10p^2 + 12p - 4$ and $\beta = p^2 + 2p - 2$ the result follows.

For the upper bound, using Lemma 2.4, we get

$$\begin{aligned} LE(G) &\leq 4m - 2\Delta - \frac{4m}{n} + 2 \\ &= 8(p-1)(2p-1) - 2(p^2-1) - \frac{8(p-1)(2p-1)}{p^2+2p-2} + 2 \\ &= \frac{7p^4 + 2p^3 - 40p^2 + 48p - 16}{p^2 + 2p - 2} \\ &= \frac{(p^4 + 2p^3 - 10p^2 + 12p - 4) + (p-1)^2(p^2+2p-2)}{p^2+2p-2}, \end{aligned}$$

where by putting with putting $\alpha = p^4 + 2p^3 - 10p^2 + 12p - 4$ and $\beta = p^2 + 2p - 2$ in the above relation, the result follows. □

Theorem 3.12. *Let $\Gamma(R)$ be a zero-divisor graph of a commutative ring $R \simeq \mathbb{Z}_p \times \mathbb{Z}_{2p}$ for a prime number $p > 2$. Then*

$$\sqrt{2\alpha(p-1)} \leq E\big(L_{\Gamma(R)}\big) \leq \alpha(p-1),$$

where $\alpha = p^3 + 11p^2 - 20p + 6$.

Proof. Theorem 3.8(i) and Lemma 2.5, give us the results. □

Theorem 3.13. *Let $\Gamma(R)$ be a zero-divisor graph of a commutative ring $R \simeq \mathbb{Z}_p \times \mathbb{Z}_{2p}$ for a prime number $p > 2$. Then*

$$4(p-2)(3p-2) \leq EE_F\big(\Gamma(R)\big) \leq (p-1)^2(p^2 + 12p - 4).$$

Proof. Let G be the zero-divisor graph $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{2p})$ for $p > 2$ where p a prime. According to the structure of the zero-divisor graph G, the number of isolated vertices and pendant vertices are zero. Since graph G has $n = p^2 + 2p - 2$ vertices and $m = 2(p-1)(2p-1)$ edges, by applying Lemma 2.7 we have

$$\begin{aligned} EE_F(G) &\geq 4(m-n) \\ &= 4\big(2(p-1)(2p-1) - (p^2+2p-2)\big) \\ &= 4\big(3p^2 - 8p + 4\big) \\ &= 4(p-2)(3p-2). \end{aligned}$$

For the upper bound, by applying Lemma 2.6 and Theorem 3.8(i), we have

$$\begin{aligned} EE_F(G) &\leq M_1(G) - m \\ &= (p-1)(p^3 + 11p^2 - 12p + 2) - 2(p-1)(2p-1) \\ &= (p-1)\big(p^3 + 11p^2 - 16p + 4\big) \\ &= (p-1)^2(p^2 + 12p - 4). \end{aligned}$$

□

Finally, we investigate the domination number, the total domination number, the connected domination number, and the Roman domination number of the zero-divisor graph $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{2p})$.

Theorem 3.14. *Let $\Gamma(R)$ be a zero-divisor graph of the commutative ring $R \simeq \mathbb{Z}_p \times \mathbb{Z}_{2p}$ for a prime number $p > 2$. Then*

i) $\gamma\big(\Gamma(R)\big) = 3$,

ii) $\gamma_t\big(\Gamma(R)\big) = 3$,

iii) $\gamma_c\big(\Gamma(R)\big) = 3$,

iv) $\gamma_R\big(\Gamma(R)\big) = 6$.

Proof. Let G be the zero-divisor graph $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{2p})$ for $p > 2$ where p is a prime. Let D be the minimum dominating set of graph G.

i) Similar to the proof of Theorem 3.6, the subsets B_i's of the vertex set of graph G and the subsets of the edge set in (5), one can select the vertex $x = (0, p) \in B_3$ to dominate all of the vertices of B_1 and the vertices $(0, 2m) \in B_3$ and the vertices $(a, 2m) \in B_4$) for $1 \leq m \leq p-1$ and $a \in Z(P^*)$. For dominating the remaining vertices in sets B_3 and b_4, we select one of the vertices of the form $(0, 2m) \in B_3$, called y. Finally, it is sufficient to dominate vertices of set B_2. To do this, we select an arbitrary vertex z in B_1 that is adjacent to all vertices of B_2. Therefore, $|D| = 3$.

ii) According to the definition of the total dominating set and the proof of case (i), since the vertex $z \in B_1$ is adjacent to any two vertices $x = (0, p)$ and $y = (0, 2m)$, thus D is the minimum total dominating set in G and $\gamma_t(G) = 3$.

iii) Based on selecting the set D in the case (i) and the definition of the connected dominating set, the result is completed.

iv) We consider the function $f : V \longrightarrow \{0, 1, 2\}$ such that for the set D in case (i), $f(x) = f(y) = f(z) = 2$ in which $x, y \in B_3$ and $z \in B_1$ and $f(u) = 0$ for $u \in V(G) \setminus \{x, y, z\}$. In this case, all the vertices are dominated by the set $\{x, y\}$ and $\gamma_R(G) = 6$.

□

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Fateme Movahedi
Department of Mathematics
Faculty of Sciences
Assistant Professor of Mathematics
Golestan University
Gorgan, Iran

E-mail: f.movahedi@gu.ac.ir

Mohammad Hadi Akhbari
Department of Mathematics
Assistant Professor of Mathematics
Estahban Branch, Islamic Azad University
Estahban, Iran
E-mail: hadi.akhbari@iau.ac.ir