Journal of Mathematical Extension
Vol. 8, No. 3, (2014), 79-86

(α, β)-Monotone Variational Inequalities Over Arbitrary Product Sets

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Abstract. The purpose of this paper is to introduce some new concept and extend the usual ones are introduced for variational inequality problems over arbitrary product sets. Our result can be viewed as an extension of the results obtained by Igor V. Konnov [Relatively monotone variational inequalities over product sets, Operation research letters 28(2001), 21-26]. Moreover the results of this paper can be consider as a new version of the main results in [1].

AMS Subject Classification: 46A32; 46M05; 41A17
Keywords and Phrases: (α, β)-Monotone, produce set, locally convex, variational inequality

1. Introduction

It is advised to give each section and subsection a unique label. In recent years variational inequality have been generalized and extended in various different directions in abstract; see ref.[11, 12]. Moreover many authors have investigated vector variational inequalities in abstract spaces; see ref.[7, 8, 9, 14]. The development of efficient methods for proving existence of solution is one the most interesting and important in variational inequalities theory and equilibrium type problem arising in operation research, economics, mathematical, physics

Received: March 2014; Accepted: August 2014
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and other fields. It is well known that most of such problems arising game theory, transportion and network economics have a decomposable structure i.e. thay can be formulated as variational inequalities over Cartesian produce sets; see e.g. Nagurney [13] and Ferris and Pang [6]. The most existence results for such variational inequalities established under either compactness of the feasible set in the norm topology or monotonicity-type assumption regardless of the decomposable structure of the variational inequalities; see [2, 10]. In fact Bianchi [2] considered the corresponding extension of P-mapping and noticed that thay are not sufficient to derive existence results with the help of Fans lemma.

In this paper we present (α, β)-monoton concept, which is suitable for variational inequalities on arbitrary product set of locally convex spaces and our results extend theorems of Konnov [12]. Moreover our results is a new version of the result obtained in [1].

Throughout this paper, let I be any set indexes, $\langle I \rangle$ denote the set of all nonempty finite subsete of I and let P denotes the set of all positive vector in $l^\infty(I)$ i.e. $P = \{(u_i)_{i\in I} \in l^\infty(I) : \ u_i > 0 \ \forall i \in I\}$, $l^\infty(I) = \{(u_i)_{i\in I} : \ \exists \ c > 0 \ , |u_i| < c \ \ \forall i \in I\}$. Note that if in the set P, we have $u_i \geqslant 0$ then P is a cone; see [15].

2. Basic Definition

At first, we define some notations that will be used in the following.

For each $i \in I$, let X_i be a locally convex spaces and X_i^* its dual.
Set $X = \prod_{i\in I} X_i$, so that for each $x \in X$, we have $x = (x_i)_{i\in I}$ where $x_i \in X_i$.
We define the map $<,>: \ X^* \times X \to \mathbb{R}$ by $< f, x >= f(x)$ and

$$\ll, \gg: \ \prod_{i\in I} X_i^* \times X \to \mathbb{R} \cup \{+\infty\}$$

by

$$\ll g, x \gg = < g, x >^+ - < g, x >^-$$

where $x \in X, \ g \in \prod_{i\in I} X_i^*$ and
$< g, x >^+ = \sup_{J\in\langle I\rangle} \{\sum_{j\in J} < g_j, x_j >: < g_j, x_j > \geqslant 0 \ \ \forall j \in J\}$,
$< g, x >^- = < -g, x >^+$.

We define the vector space X_w^* as follows :

$$X_w^* = \{g \in \prod_{i\in I} X_i^* \ : \ (g, x) \in D^{e'}_{\ll,\gg} \ : \ \forall x \in \prod_{i\in I} X_i\},$$

where $D^{e'}_{\ll,\gg} = \{(g,x) \in (\prod_{i\in I} X_i^*) \times X :\ll g,x \gg < \infty\}$.
It is clear that $D^{e'}_{\ll,\gg} \neq \emptyset$, $X_w^* \neq \emptyset$.

Let K_i be nonempty subset of X and let $K = \prod_{i\in I} K_i$, next for each $i \in I$, let $G : K \to X_w^*$ be a mapping, now we define $G_i : K \to X_i^*$ by $G_i = P_i o G$, where $P_i : X_w^* \to X_i^*$ is defined to be $P_i\Big((g_j)_{j\in J}\Big) = g_i$.
We note that $G(x) = (G_i(x))_{i\in I}$ and

$$\ll G(x), y - x \gg = \sum_{i\in I} < G_i(x), y_i - x_i >< \infty.$$

In this paper we study variational inequality problem as following:

a) The SyVIP(G,K) consist of finding $x^* \in K$ such that

$$< G_i(x^*), y_i - x_i^* > \geqslant 0, \qquad \forall y_i \in K_i \ , i \in I.$$

We denote the solution set of the SyVIP(G,K) by $S_{SyVIP}(G,K)$.

b) For every given $u = (u_i)_{i\in I} \in P$, the $VIP(G,K,u)$ consist of finding $x^* \in K$ such that

$$\ll \big(u_i G_i(x^*)\big)_{i\in I}, y - x^* \gg = \sum_{i\in I} u_i < G_i(x^*), y_i - x_i^* > \geqslant 0, \quad \forall y_i \in K_i \ , i \in I.$$

We denote the solution set of the VIP(G,K,u) by $S_{VIP}(G,K,u)$.

c) The dual VIP(G,K,u) (abbreviated DVIP(G,K,u)) consist of finding $x^* \in K$ such that

$$\ll \big(u_i G_i(y)\big)_{i\in I}, y - x^* \gg = \sum_{i\in I} < u_i G_i(y), y_i - x_i^* > \geqslant 0, \qquad \forall y_i \in K_i \ , i \in I.$$

We denote by $S_{DVIP}(G,K,u)$ the solution set of the DVIP(G,K,u).

Definition 2.1. *For each $u = (u_i)_{i\in I} \in l^\infty$, the mapping $G : K \to X_w^*$ is said to be u-hemicontinuous, if for any $x, y \in K$, the mapping $g : [0,1] \to \mathbb{R}$ by $g(\lambda) = \sum_{i\in I} u_i < G_i(x + \lambda(x-y)), y_i - x_i >$ is continuous.*

We note that for each $\lambda \in [0,1]$, $g(\lambda) < \infty$.

Definition 2.2. (See [6]) *Let $\alpha, \beta \in l^\infty(I)$, the mapping $G : K \to X_w^*$ is said to be*

a) (α, β)-monotone, if for all $x, y \in K$, we *have*

$$\ll \beta G(x) - \alpha G(y), x - y \gg \geqslant 0,$$

and strictly (α, β)-monotone, if the inequality is strict for all $x \neq y$.

b) (α, β)-psedumonotone, if for all $x, y \in K$, we have

$$\ll \alpha G(x), y - x \gg \geqslant 0 \implies \ll \beta G(y), y - x \gg \geqslant 0,$$

and strictly (α, β)-psedumonotone, if the second inequality is strict for all $x \neq y$.

c) (α, β)-psedumonotone-like, if for all $x, y \in K$, we have

$$\ll \alpha G(x), y - x \gg > 0 \implies \ll \beta G(y), y - x \gg \geqslant 0,$$

and strictly (α, β)-psedumonotone-like, if the second inequality is strict for all $x \neq y$

Lemma 2.3. *Let $\alpha, \beta \in P$ and $G : K \to X_w^*$ then*

a) $S_{SyVIP}(G, K) = S_{VIP}(G, K, \alpha)$;

b) $S_{DVIP}(G, K, \alpha) = S_{DVIP}(G, K, \beta)$;

c) $S_{VIP}(G, K, \alpha) = S_{VIP}(G, K, \beta)$.

Proof. by Definition 2.2 the desired result is obtained. □

Lemma 2.4. *Let $\alpha \in P$ and the mapping $G : K \to X_w^*$ be α-hemicontinuous, then*

$$S_{DVIP}(G, K, \alpha) \subseteq S_{VIP}(G, K, \alpha).$$

Proof. let $x^* \in S_{DVIP}(G, K, \alpha)$, thus

$$\sum_{i \in I} < \alpha_i G_i(y), y_i - x_i^* > \geqslant 0, \quad \forall y \in K.$$

Set $y = x^* + \lambda(y - x^*)$, therefore from α-hemicontinuous of G, we have $x^* \in S_{VIP}(G, K, \alpha)$. □

The proof of following lemma is parallel to the proof of lemma 2.4 and so is omitted.

Lemma 2.5. *Let $\alpha, \beta \in P$ and the mapping $G : K \to X_w^*$ be β-hemicontinuous and (α, β)-psedumonotone then*

$$S_{DVIP}(G, K, \beta) = S_{VIP}(G, K, \alpha).$$

Corollary 2.6. *Let the conditions of lemma 2.5 hold, then*

$$S_{DVIP}(G, K, \alpha) = S_{VIP}(G, K, \alpha) = S_{SyVIP}(G, K).$$

Definition 2.7. (See [5]) *A set-valued $F : E \to 2^E$ is called a KKM-mapping if, for every finite subset $\{x_1, x_2, ..., x_n\}$ of E, $co\{x_1, x_2, ..., x_n\} \subseteq \bigcup_{i=1}^n F(x_i)$, where "co" denotes the convexhull.*

Lemma 2.8. (See [4]) *Let E be a Hausdorff topological vector space and $F : E \to 2^E$ be a KKM-mapping such that for any $x \in E, F(x)$ is closed and $F(x_0)$ contained in a compact set $D \subseteq E$ for some $x_0 \in E$. Then $\bigcap_{x \in E} F(x) \neq \emptyset$.*

3. Main Results

In this section we obtain a new version of Konnov's results in [12] without assuming the finiteness on I. Hence it is the extension of [12].

Theorem 3.1. *suppose that $\alpha, \beta \in P, X$ locally convex space, $K \subseteq X$ is nonempty weakly compact and let the mapping $G : K \to X_w^*$ be β-hemicontinuous and (α, β)-psedumonotone then $S_{VIP}(G, K, \alpha) \neq \emptyset$.*

Proof. Define set-valued mapping $H, T : K \to 2^K$ by

$$T(y) = \{x \in K \ : \ \sum_{i \in I} < \alpha_i G_i(x), y_i - x_i > \geqslant 0\}$$

$$H(y) = \{x \in K \ : \ \sum_{i \in I} < \beta_i G_i(y), y_i - x_i > \geqslant 0\}.$$

We denote T is KKM-mapping. Let $\{y^1, y^2, ..., y^n\}$ be any finite subset of K and $z \in co\{y^1, y^2, ..., y^n\}$ then $z = \sum_{j=1}^n \lambda_j y^j$, for some $\lambda_j \geqslant 0, j = 1, 2, ..., n$ and $\sum_{j=1}^n \lambda_j = 1$. On the contrary, if $z \notin \bigcup_{j=1}^n T(y^j)$, then

$$\sum_{i \in I} \alpha_i < G_i(z), y_i^j - z_i > < 0 \qquad \forall j = 1, 2, ..., n.$$

Therefore, $0 = \sum_{i\in I} \alpha_i < G_i(z), z_i - z_i > < 0$, is a contradiction, hence T is a KKM-mapping. Since $\overline{T(y)}^w \subseteq K$, by lemma 2.8 $\bigcap_{y\in K} \overline{T(y)}^w \neq \emptyset$. Since G is (α,β)-psedumonotone we have $T(y) \subseteq H(y)$, that is clear $H(y)$ is weakly closed, therefore $\bigcap_{y\in K} H(y) \neq \emptyset$, that is

$$S_{DVIP}(G, K, \alpha) \neq \emptyset$$

But Lemma 2.5. implies that $S_{VIP}(G, K, \alpha) \neq \emptyset$. $\square$

Corollary 3.2. *Suppose that $\alpha, \beta \in P, X$ locally convex space, $K \subseteq X$ is nonempty weakly compact and let the mapping $G : K \to X_w^*$ be β-hemicontinuous, and stictly (α,β)-psedumonotone then $VIP(G, K, \alpha)$ has a uniqe solution.*

Proof. Theorem 3.1 implies that $S_{VIP}(G, K, \alpha) \neq \emptyset$. Assume that, for contradiction, there exist $x^1 \neq x^2$ such that $x^1, x^2 \in S_{VIP}(G, K, \alpha)$. For any $y \in K$, we have $\sum_{i\in I} \alpha_i < G_i(x^1), x_i^2 - x_i^1 > \geqslant 0$, since G is strictly (α,β)-psedumonotone, so $\sum_{i\in I} \beta_i < G_i(x^2), x_i^1 - x_i^2 > < 0$ thus $x^2 \notin S_{VIP}(G, K, \beta) = S_{VIP}(G, K, \alpha)$. $\square$

Corollary 3.3. Suppose that $\alpha, \beta \in P$, X locally convex space and the mapping $G : K \to X_w^*$ be β-hemicontinuous and stictly (α,β)-psedumonotone and let there exist a weakly compact subset E of X, and a piont $e \in E \cap K$ such that $\sum_{i\in I} \alpha_i < G_i(x), e_i - x_i > < 0 \quad \forall x \in K \setminus E$ then $S_{VIP}(G, K, \alpha) \neq \emptyset$.

Proof. From the proof of theorem 3.1 and under the above assumption we have $T(e) \subseteq E$, thus $\overline{T(e)}^w$ is weakly compact, so by review of the proof of theorem 3.1 and lemma 2.8 we have $S_{VIP}(G, K, \alpha) \neq \emptyset$. $\square$
Next theorem shows that our results generalized the main result of V. Konnov [12]. Of course our results obtained by this new method is a new version of results obtained in [1].

Theorem 3.4. *Suppose that $|I| = n < \infty$(i.e. n be the number of elements I) and $\{X_i\}_{i\in I}$ be finite family of locally convex spaces. Then*

$$\prod_{i\in I} X_i^* = X^* = X_w^*$$

and

$$\ll (f_i)_{i\in I}), (x_i)_{i\in I}) \gg = \sum_{i=1}^{n} < f_i, x_i > = < f, x >,$$

where $x \in X = \prod_{i=1}^n X_i$, $f \in X^*$.

Proof. For each $f \in X^*$, we define $< f, \overline{x_i} > = < f_i, x_i >$
where $\overline{x_i} = (0, ..., x_i, 0, ...), f_i \in {X_i}^*$. Now we define $\Gamma : X^* \to X_w^*$ by $\Gamma(f) =$

$(f_i)_{i\in I}$.
It is easy to see that Γ is homeomorphism (i.e. Γ is linear, onto, one to one, continuous and inverse of Γ exists). $\square$

Acknowledgements:
The authors would like to thank the referees for helpful comments.

References

[1] E. Allevi, A. Gnudi, and I. V. Konnov, Generalized vector variational inequalities over countable product of sets, *Journal of Global Optimization*, 30 (2004), 155-167.

[2] M. Bianchi, *Pseudo P-Monotone Operators and Variational Inequalities*, Universita Cattolica del Sa Cuox, Milan, (1993).

[3] G. Y. Chen, Existence of solution for a vector variational inequality, An extension of Hartman-Stampcchia thoery, *J. Optim. Theory Appl.*, 74 (1992), 445-456.

[4] J. E. Crouzeix, L. Martinez, and M. Volle, *Generalized Convexity, Generalized Monotonicity,* Kluwer Academic Publishers, (1998), 257-275.

[5] K. Fan, A generalization of Tychonoff fixed point theorem, *Math. Ann.*, 142 (1961), 305-310.

[6] A. P. Farajzadeh, A. Karamian, and S., Plubtieng, On the translations of quasimonotone maps and monotonicity *J. Ineq. Appl.,* (2012), 192-197.

[7] M. Ferris and J. S. Pang, Engineering and economics applications of complementarity problems, *SIAM Rev.,* 39 (1997), 669-713.

[8] F. Giannessi, *Vector Variational Inequalities and Vector Vatiational Equilibrium*, Kluwer Academic Publisher, (2000).

[9] F. Giannessi, *Theorem of Alternative Programs and Complementarity Problems, In Variational Inequalities and Complmentarity Problems*, Wiley, New York, 1980 .

[10] F. Giannessi and A. Maugeri, *Variational Inequalities and Network Equilibrium Plenum,* New York, 1995.

[11] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications,* Academic Press, New York, 1980 .

[12] V. Konnov, Relatively monotone variational inequalities over product sets, *Oper. Res. Letts.*, 28 (2001), 21-26.

[13] A. Nagurney, Network Economics, *A Variational Inequality Approach, Kluwer Academic Publishers Group,* Dordrecht, 1993 .

[14] W. Oettli, A remark on vector valued equibria and generalized monotonicity, *Acta Math. Vietnam,* 22 (1997), 213-221.

[15] P. Zangenehmehr, A. P. Farajzadeh, and S. M. Vaezpour, On fixed point theorems for monotone increasing vector valued mappings via scalarizing, *Positivity*, (2014), 1-8.

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