Journal of Mathematical Extension
Vol. 17, No. 3, (2023) (3)1-32
ISSN: 1735-8299
URL: https://doi.org/10.30495/JME.2023.2405
Original Research Paper

Class of Modified Two-Stage Procedure in a Autoregressive Process

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Abstract. In this paper, we first discuss the class of modified two-stage procedure for estimation of the autoregressive parameter in a first-order autoregressive model (AR(1)). We prove the significant properties of the modified two-stage procedure, including asymptotic efficiency, asymptotic risk efficiency, and asymptotic consistency for the point and the interval estimation based on least-squares estimators. Then, the introduced class is generalized to the p-order autoregressive model (AR(p)) and is checked for their asymptotic properties. Also, we conduct comprehensive Monte Carlo simulation studies to test the properties of the proposed procedure based on least-squares estimators and Yule-Walker estimators in practice. Finally, a real-time series is provided to investigate the applicability of the class of modified two-stage variables.

AMS Subject Classification: 62L12 ;62M10;62L10; 62L15.
Keywords and Phrases: Modified two-stage procedure, Autoregressive process, Asymptotic risk efficiency, Asymptotic efficiency, Asymptotic consistency.

Received: May 2022; Accepted: January 2023
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1 Introduction

The time-series data includes various sources and applications and has been receiving increasing attention in recent years. Some examples of time series are first expressed, daily sales data for demand prediction, yearly macroeconomic data for long-term political planning, sensor data from a smartwatch for analyzing a workout session, the steps in a Data Science cycle for accessing to transforming, modeling, evaluating, and deploying time series data. Also, the application of time-series analysis is currently one of the most active topics in statistics. Due to the increasing development and application of time series data, many studies have been deduced in this field and the branch of statistics called Time Series analysis. It could also be that the time-series data is not available at regular intervals and can only be collected from random event points. In some cases, the analysis cannot be examined because of the unknown sample size. Therefore, we cannot effectively use time-series data to model, parameter estimation, forecast, and predict real practical problems. A sequential procedure is one of the proposed approaches to overcome this problem and determine the required sample size in practice when the sample size is unknown in advance.

The sequential procedures offer a strategy for the best decision for stopping the sampling procedure that is an alternative to the best-fixed sample size procedure for such conditions. For the first time, an alternative inferential method is introduced for point and interval estimation of an unknown population mean by Anscombe [1], Robbins[39] and Chow and Robbins [8] that the required sample size is determined by sequentially stopping rules. The sequential procedures are distinguished by different stopping strategies in order to determine the sample size so that different names are appropriately assigned. The most widely used sequential procedures are purely sequential, two-stage, modified two-stage and three-stage procedures. Many researchers have dealt with how sequential procedures can be used to tackle some unsolved problems in point and interval estimation that we give a glimpse of several of them. Stein[46, 47] was first to introduce a two-stage procedure that used this procedure for the problem of constructing a fixed-width confidence interval and hypothesis testing by considering the mean in a normal population.

Mukhopadhyay [31] studied a two-stage procedure for constructing a confidence interval of the mean in a normal population under the condition, assuming the variance is unknown. Sriram [42, 43] investigated a parameter estimating of a first-order autoregressive model via a purely sequential sampling scheme. Basawa, McCormick, and Sriram [2] became interested in examining a sequential sampling procedure for dependent observations and, as a result of their study, estimated the autoregressive parameters in a non-explosive first-order autoregressive process with Weibull errors. Fakhre-Zakeri and Lee [10] have estimated the mean vector parameters in a multivariate linear process within the framework of a purely sequential procedure. Mukhopadhyay and Sriram [36] considered the problem of point estimation of a linear combination of means of p-independent non-explosive first-order autoregressive models via purely sequential. Lee [24] published his review about sequential point and confidence interval estimation in a p-the order autoregressive model. Mukhopadhyay and Duggan [33] with the idea of reducing the weaknesses of the two-stage procedure, modified a two-stage procedure to estimate the mean in a normal population for a confidence interval with a known positive lower bound for variance.

Basu and Das [3] extended purely sequential estimation of autoregressive parameters to multiple pth-order autoregressive models. Lee and Sriram [26] generalized their studies of a purely sequential procedure to the nonlinear threshold AR(1) model and examined the performance of the procedure for point estimation. Sriram [44], by presenting a supplementary study of a purely sequential procedure, investigated fixed-size confidence regions in single and multiple first-order threshold autoregressive models. Lee [25] studied a purely sequential procedure in a stochastic regression model. Kashkovsky and Konev [22] considered the sequential procedure for first-order RCA models and suggested a strongly consistent sequential estimator of coefficients of a univariate p-order RCA model. Gombay [12] applied a sequential procedure for estimating a confidence interval of time-series observations to check the performance of a sequential procedure. In the case of the unknown variance, Mukhopadhyay and Zacks [37] studied a two-stage and purely sequential procedure by using a modified Linex loss function in a normal distribution. Karmakar and Mukhopadhyay [17, 18], reviewed a sequential procedure for estimating parameters in a single and multivariate random coefficient autoregressive pth-order model.

Under the squared error loss plus the linear cost of sampling, Hu and Mukhopadhyay [13] studied the minimum risk point estimation (MRPE) problem for an unknown normal mean by defining a new class of purely sequential MRPE methodologies when the variance remains unknown. Mukhopadhyay and Bapat [32] designed purely sequential bounded-risk methodologies to estimate an unknown mean of negative binomial distribution under different forms of loss functions including customary and modified Linex loss as well as squared error loss. Mahmoudi, Khalifeh, and Nekoukhou [27] demonstrated the application of a two-stage procedure to estimate a parameter in a stress-strength model. Sriram and Samadi [45] reviewed the performance of a purely sequential procedure in an AR(1) model that was previously studied by Sriram [43]. Mukhopadhyay and Zhuang [38] designed two-stage and purely sequential methodologies for testing hypotheses regarding the difference of mean values from two independent (or dependent) normal populations when their variances are unknown and unequal. Joshi and Bapat [16] proposed improved accelerated sequential procedures to estimate the unknown mean of an inverse Gaussian distribution when the scale parameter is unknown. Khalifeh, Mahmoudi, and Chaturvedi [20] discussed the challenge of constructing a confidence interval for a parameter via a two-stage procedure in the case of an exponential distribution. Hu and Zhuang [15] proposed an innovative and general class of modified two-stage sampling schemes by assuming the squared error loss. Chaturvedi, Bapat and Joshi [7] considered developing sequential procedures for point estimating the mean of an Inverse Gaussian (IG) distribution when the population coefficient of variation (CV) is known.

Chaturvedi, Bapat, and Joshi [6] studied a two-stage procedure and a purely sequential procedure for the generalized positive exponential family of distributions for point and interval estimation. Also, Sajjadipanah, Mahmoudi, and Zamani [40] discussed a two-stage procedure for point and interval estimation in AR(1) model. Furthermore, they compared the performance of this procedure with the purely sequential procedure. Sajjadipanah, Mahmoudi, and Zamani [41] investigated a modified two-stage procedure for the mean of the autoregressive model for point and interval estimation. Hu and Pham [14] proposed an alternative three-stage sampling procedure with termination defined via Ginis mean difference for the problem of minimum risk point estimation for an unknown normal mean when the variance is unknown. Mukhopadhyay, Hu, and Wang [35] considered sequential minimum risk point estimation (MRPE) problems for two independent normal populations, with all parameters, assumed unknown, under the squared error loss (SEL) plus the linear cost of sampling. Mahmoudi, Nemati, and Khalifeh [29] investigated a two-stage sequential sampling procedure to estimate the mean of an exponential distribution under the modified square error loss function. Bishnoi and Mukhopadhyay [5] developed a new class of purely sequential methodologies under an assumption that the population distribution belongs to a location-scale family. Malinovsky and Zacks [28] proposed two-stage and purely sequential procedures to estimate the unknown parameter of a binomial distribution with unknown parameter p. Christensen and Sohr [9] investigated both discrete time and continuous time stopping problems for general Markov processes on the real line with general linear costs as they naturally arise in many problems in sequential decision making.

As mentioned earlier, the purely sequential procedure has been investigated in some linear and nonlinear time series models. Also, the two-stage procedure has recently been studied in the time series linear model because of its simplicity of implementation, operational savings, and simpler execution. Also, the stein two-stage procedure and the purely sequential procedure share the same asymptotic first-order properties that the two-stage procedure is operationally much more convenient because of sampling at most two batches. But, the stein two-stage procedure is oversampling on average, even asymptotically. Indeed, a major drawback of the two-stage procedure in confidence interval estimation is that for a very small fixed-width confidence interval, the procedure ended up with a large sample at the second stage regardless of the fixed value of the initial sample size. This weakness of the procedure encouraged scholars to seek an appropriate two-stage strategy that the modified two-stage procedure is the best option. Researchers offer a modified two-stage procedure based on the simplicity of two-stage procedure, more accurate than the two-stage procedure. In other words, the simplicity of execution and more accuracy in determining the sample size are important considerations to evaluate the performance of modified two-stage procedure due to the importance of time and the cost of sampling. Also, in the two-stage procedure, gathering one observation at-a-time beyond a pilot sample size and following along a stopping rule may be operationally inconvenient. These intuitive reasons, therefore, encouraged the authors to explore the behavior of the modified two-stage procedure in the linear model.

Among the sampling procedures that have been presented so far, the modified two-stage procedure and the three-stage procedure are very similar according to the relevant stopping rule. The simplicity in the execution to determine sample size even compared to similar procedures in terms of the stopping rule encouraged the authors to investigate the properties of this procedure. To check the performance of the procedure, we were looking for an idea to provide a suitable stopping rule based on the strategy of the procedure. As mentioned, Mukhopadhyay and Zacks [33] modified a two-stage stopping rule with a known positive lower bound for variance, which is asymptotic second-order bounds. Furthermore, Hu and Zhuang [15] examined the variables introduced by Mukhopadhyay and Zacks [33] and the modified two-stage as a class of modified two-stage variables in independent data. The idea considered by these authors encouraged us to investigate the class of modified two-stage analogy with the best-fixed sample size in terms of different modified two-stage variables. Three methods least squares, Yule-Walker, and Burgs can be mentioned among the common methods of the estimation of the autoregressive parameters. Also considering that the Yule-Walker estimators and the least-squares estimators are asymptotically the same for large samples. We examine the performance of the procedure based on the Yule-Walker estimators considering that it has not been studied so far.

The objective of this study is to prove the properties of the procedure, such as asymptotic efficiency, asymptotic risk efficiency, and asymptotic consistency. Also, the performance of this procedure has been investigated using Monte Carlo simulation for an autoregressive model. Finally, we evaluated the performance of the procedure with real data.

The rest of the paper is segmented organized as follows. In Section 2, we first review the AR(1) model, derive the main properties of the point, and fixed-width confidence interval estimation of the procedure. Also, the results are discussed and are generalized to AR(p) model. In Section 3, comprehensive simulation studies are presented to validate our theoretical results and the functionality of the proposed modified two-stage procedure. Finally, in Section 4, numerical studies with an application of real-time series data is considered to illustrate the applicability of the modified two-stage procedure.

2 A Broader Class of Modified Two-Stage

A first-order autoregressive model with $\mid \beta \mid < 1$ takes the form,

$$X_i = \beta X_{i-1} + \varepsilon_i, \quad i = 1, 2, ...,$$

where the sequence of $\{\varepsilon_i, i \geq 1\}$ is assumed iid random variables with an unknown distribution F with $\mathbb{E}[\varepsilon_i] = 0$, $\mathbb{E}[\varepsilon_i^2] = \sigma^2 \in (0, \infty)$. Moreover, the initial variable X_0 is supposed to be independent of $\{\varepsilon_i, i \geq 1\}$ with $\mathbb{E}[{X_0}^2] < \infty$. As we know, the least-squares estimator of β is given by

$$\hat{\beta}_n = \frac{\sum_{i=1}^n X_i X_{i-1}}{\sum_{i=1}^n X_{i-1}^2}.$$

The loss function by assuming the reciprocal of the cost of estimation error (A) is given by

$$L_n(\hat{\beta}_n, \beta) \;\; = \;\; An^{-1} \sum_{i=1}^n X_{i-1}^2 (\hat{\beta}_n - \beta)^2 + n.$$

It is well known that the asymptotic distribution of $\hat{\beta}_n$ when $\mid \beta \mid < 1$ is

$$(\sum_{i=1}^n X_{i-1}^2)^{1/2} (\hat{\beta}_n - \beta) \xrightarrow{d} N(0, \sigma^2). \qquad (1)$$

Sriram [43] proved that $\left\{ (\sum_{i=1}^n X_{i-1}^2)^{1/2} (\hat{\beta}_n - \beta), n \geqslant 1 \right\}$ is uniformly integrable, under certain regularity conditions. The risk function of $\hat{\beta}_n$ can be written due to uniformly integrable property along with the asymptotic normality result

$$\begin{aligned} R_n &= \mathbb{E}\left[L_n(\hat{\beta}_n, \beta) \right] = An^{-1}(\sigma^2 + o(1)) + n \\ &= An^{-1}\sigma^2 + n + o(n^{-1}), \end{aligned}$$

where σ is known. The best fixed sample size is calculated by ignoring the $o(n^{-1})$ term which is $n_A \approx A^{1/2}\sigma$. Then, the associated minimum risk is approximately yielded $R_{n_A} \approx 2A^{1/2}\sigma$.

It is clear if σ is unknown, the best-fixed sample size procedure cannot be used. So, we have to look for an alternative method to this end, we investigate the modified two-stage procedure based on two different stopping rules. Also, the stopping rules are called the traditional modified two-stage and the modified two-stage with known lower bound for variance that the results of the studies are presented in subsections 2.1 and 2.2. Both point as well as confidence interval estimation techniques are considered.

2.1 Traditional modified two-stage procedure

In this subsection, we review the performance of a traditional modified two-stage procedure for point and interval estimation. The first step is determining the pilot sample size for defining the stopping variable for point estimation. According to the procedure strategy, the pilot sample size is defined as follows

$$m = \max\{m_0, [A^{1/2(1+\gamma)}] + 1\},$$

where $\gamma \in (1/2, \infty)$ and $m_0 \geq 2$ are a fixed integer. Now, due to the pilot sample size, we determine the final sample size analogy with n_A

$$N_m = \max\{m, \lfloor A^{1/2}\hat{\sigma}_m \rfloor + 1\}, \tag{2}$$

where $\lfloor x \rfloor$ denotes the largest integer smaller than x. To remind, the least-squares estimator of $\hat{\sigma}_n^2$ is defined as $n^{-1}\sum_{i=1}^{n}(X_i - \hat{\beta}_n X_{i-1})^2 = n^{-1}\sum_{i=1}^{n}\varepsilon_i^2 - n^{-1}Q_n$ such that $Q_n = (\sum_{i=1}^{n} X_{i-1}^2)(\hat{\beta}_n - \beta)^2$, for any n. One of the main results of this subsection is Theorem 2.2, which demonstrates the properties of the procedure as $A \to \infty$. Before presenting the theorem, it is necessary to express the following lemma. The Lemma 2.1 is needed for proving the asymptotic efficiency property of the procedure and indicates the rate of convergence on the tail behavior of the stopping variable N_m.

Lemma 2.1. *Assume that* $\mathbb{E} \mid \varepsilon_1 \mid^{4s} < \infty$ *and* $\mathbb{E} \mid X_0 \mid^{4s} < \infty$ *for* $s \geq 1$. *In addition if* $m_0 = o(A^{1/2})$ *for* $\gamma \in (1/2, \infty)$. *Then for any* $0 < \theta < 1$,

$$P\left(N_m < (1-\theta)A^{1/2}\sigma\right) = O\left(A^{-s/2(1+\gamma)}\right),$$

$$P\left(N_m > \left[(1+\theta)A^{1/2}\sigma\right] + 1\right) = O\left(A^{-s/2(1+\gamma)}\right).$$

Proof. The argument of proof is similar to Lemma 1 [40] and is omitted. □

Theorem 2.2. *Suppose for* $s > 1$ *that* $\mathbb{E} \mid \varepsilon_1 \mid^{4s} < \infty$, $\mathbb{E} \mid X_0 \mid^{4s} < \infty$ *and* $m_0 = o(A^{1/2})$ *for* $\gamma \in (1/2, \infty)$. *Then as* $A \to \infty$,

$$\frac{N_m}{n_A} \xrightarrow{\text{a.s}} 1, \tag{3}$$

$$\mathbb{E}\left[\frac{N_m}{n_A}\right] \to 1, \quad \text{(asymptotic efficiency)}, \tag{4}$$

$$\frac{R_{N_m}}{R_{n_A}} \to 1, \quad \text{(asymptotic risk efficiency)}. \tag{5}$$

Proof. From (2) note that

$$\hat{\sigma}_m A^{1/2} \leq N_m \leq \hat{\sigma}_m A^{1/2} + m I_{(N_m = m)}.$$

The assertion of equation(3) follows from this inequality, definition of N_m and $\hat{\sigma}_m \xrightarrow{a.s} \sigma^2$, as $A \to \infty$. Then, equation (4) is yielded by the expectation of the above inequality and in view of equation(3). Finally, assertion of equation (5) is similar to Theorem 2.1 [40] and the proof is completed. □

In this part, we want to construct a fixed-width ($2d$) confidence interval of β with the confidence coefficient at least $1-\alpha$ based on the traditional modified two-stage stopping variable. By assuming two pre - assigned constants $\alpha \in (0,1)$ and $d > 0$, the best fixed sample size is achieved $k_d = \lfloor d^{-2}(1-\beta^2) z^2_{(1-\alpha)/2} \rfloor$. Also, $I_{k_d} = \left[\hat{\beta}_{k_d} - d, \hat{\beta}_{k_d} + d\right]$ denotes the fixed-width confidence interval based on the best fixed sample size for β. Note that

$$\begin{aligned} P\left(\beta \in I_{k_d}\right) &= P\left(\mid \hat{\beta}_{k_d} - \beta \mid < d\right) \\ &= P\left(\frac{\sqrt{k_d} \mid \hat{\beta}_{k_d} - \beta \mid}{\sqrt{1-\beta^2}} < \frac{\sqrt{k_d} d}{\sqrt{1-\beta^2}}\right) \\ &\longrightarrow 1-\alpha, \quad as\ d \to 0. \end{aligned}$$

As we know, if β is unknown, the best fixed-sample size (k_d) is unknown. Thus to determine the final sample size, we employ the proposed procedure. The traditional modified two-stage stopping rule for $\gamma \in (1/2, \infty)$ and $m_0 \geq 2$ analogy with k_d is defined as

$$m^d = \max\{m_0, [(z_{(1-\alpha)/2}/d)^{2/(1+\gamma)}] + 1\},$$

$$N_m^d = \max\{m^d, \lfloor z^2_{(1-\alpha)/2} d^{-2}(1 - \hat{\beta}^2_{m^d})\rfloor + 1\}. \tag{6}$$

The second main result of this subsection is presented under Theorem 2.3. Results show asymptotically efficient and asymptotically consistent properties, as $d \to 0$. It should be noted that $I_{N_m^d}$ is the confidence interval of β based on the stopping variable N_m^d.

Theorem 2.3. *Assume for $s > 1$, $\mathbb{E}\mid \varepsilon_1 \mid^{4s} < \infty$, $\mathbb{E}\mid X_0 \mid^{4s} < \infty$ and $m_0 = o(d^{-2})$. Then as $d \to 0$,*

$$\frac{N_m^d}{k_d} \xrightarrow{a.s} 1, \tag{7}$$

$$\mathbb{E}\left[\frac{N_m^d}{k_d}\right] \to 1, \quad \text{(asymptotic efficiency)} \tag{8}$$

$$P\left(\beta \in I_{N_m^d}\right) \to 1-\alpha, \quad \text{(asymptotic consistency)} \tag{9}$$

Proof. From (6) note that

$$(1 - \hat{\beta}^2_{m^d}) z^2_{(1-\alpha)/2} d^{-2} \leq N_m^d \leq (1 - \hat{\beta}^2_{m^d}) z^2_{(1-\alpha)/2} d^{-2} + m^d I_{(N_m^d = m^d)}.$$

Obviously, by division of above inequality by k_d and taking the limit as $d \to 0$ yields equation (7) since $\hat{\beta}^2_{m^d} \xrightarrow{a.s} \beta^2$ and $O(d^{2/(1+\gamma)})$ as well as

$I_{(N_m^d=m^d)} \xrightarrow{a.s} 0$. Similarly, taking the expectation and the limit as $d \to 0$ yields equation (8). The proof of equation (9) is similar to Theorem 2.2 [40] and we refuse to mention it again. □

2.2 Modified two-stage procedure with known positive lower bound for variance

In this subsection, the performance of the proposed procedure is examined with more information about the process, and the appropriate stopping rule is also provided. To this end, we assume $\sigma > \sigma_l > 0$ while σ and σ_l are unknown and known respectively. To consider the stopping rule, we first define the pilot sample size due to σ_l

$$m_1 = \max\{m_0, [\sigma_l A^{1/2}] + 1\},$$

Then, the final sample size is given by

$$T_{m_1} = \max\{m_1, \lfloor A^{1/2}\hat{\sigma}_{m_1}\rfloor + 1\}, \tag{10}$$

As before, in order to present one of the main results of this subsection, Theorem 2.5, we express a practical and important lemma.

Lemma 2.4. *Assume that* $\mathbb{E} \mid \varepsilon_1 \mid^{4s} < \infty$ *and* $\mathbb{E} \mid X_0 \mid^{4s} < \infty$ *for* $s \geq 1$. *In addition if* $A^{1/2(1+\eta)} \leqslant m_1$ *for some* $\eta > 0$ *and* $m_0 = o(A^{1/2})$. *Then for any* $0 < \theta < 1$,

$$P\left(T_{m_1} < (1-\theta)A^{1/2}\sigma\right) = O\left(A^{-s/2(1+\eta)}\right),$$

$$P\left(T_{m_1} > \left[(1+\theta)A^{1/2}\sigma\right] + 1\right) = O\left(A^{-s/2(1+\eta)}\right).$$

Proof. The argument is similar to the proof of Lemma 1 [40], and we refuse to review it again. □

Theorem 2.5. *Suppose for* $s > 1$ *that* $\mathbb{E} \mid \varepsilon_1 \mid^{4s} < \infty$, $\mathbb{E} \mid X_0 \mid^{4s} < \infty$, $A^{1/2(1+\eta)} \leqslant m_1$ *for some* $\eta \in (0, (s+1)/2 - 1)$ *and* $m_0 = o(A^{1/2})$. *Then as* $A \to \infty$,

$$\frac{T_{m_1}}{n_A} \xrightarrow{\text{p}} 1, \tag{11}$$

$$\mathbb{E}\left[\frac{T_{m_1}}{n_A}\right] \to 1, \quad \text{(asymptotic efficiency)} \tag{12}$$

$$\frac{R_{T_{m_1}}}{R_{n_A}} \to 1, \quad \text{(asymptotic risk efficiency)}, \tag{13}$$

Proof. From (10) note that

$$\hat{\sigma}_{m_1} A^{1/2} \leq T_{m_1} \leq \hat{\sigma}_{m_1} A^{1/2} + m_1 I_{(T_{m_1}=m_1)}.$$

Clearly, $m_1/n_A \to \sigma_l/\sigma$ and $\hat{\sigma}_m \xrightarrow{a.s} \sigma^2$ as $A \to \infty$, divide throughout above inequality by n_A. The assertion of equation (11) is followed immediately from Lemma 2.4, as $A \to \infty$,

$$m_1 I_{(T_m=m_1)} \xrightarrow{\text{p}} 0$$

In view of equation (11), equation(12) is achived. Finally, the argument of proof equation (13) is similar to Theorem 2.1 [40] and the proof is complete. □

In the following, we intend to construct a fixed-width ($2d$) confidence interval of β with the confidence coefficient at least $1-\alpha$. Also, $\alpha \in (0,1)$, $d > 0$, and k_d are assumed two pre-assigned constants and the best-fixed sample size, respectively. As before, we determine the final sample size via the modified two-stage strategy with the mentioned condition. After, the pilot sample size and the final sample size due to β_l known are defined as follows

$$m_2 = \max\{m_0, [z^2_{(1-\alpha)/2} d^{-2}(1-{\beta_l}^2)] + 1\},$$

$$T^d_{m_2} = \max\{m_2, \lfloor z^2_{(1-\alpha)/2} d^{-2}(1-\hat{\beta}^2_{m_2})\rfloor + 1\}. \tag{14}$$

The performance of the proposed procedure is presented under the theorem. The following important lemma is essential to prove the properties of the procedure.

Lemma 2.6. *Assume that* $\mathbb{E} \mid \varepsilon_1 \mid^{4s} < \infty$, $\mathbb{E} \mid X_0 \mid^{4s} < \infty$ *for* $s \geq 1$ *and* $m_0 = o(d^{-2})$. *So,*

$$P(T^d_{m_2} = m_2) = O\left(m_2^{-s}\right).$$

Proof.

$$P(T^d_{m_2} = m_2) = P\left(m_2^{-1}\sum\nolimits_{i=1}^{m_2} \varepsilon_i^2 - {m_2}^{-1}Q_{m_2} < m_2 d^2 z^2_{(1-\alpha)/2}\right)$$
$$\leq P\left(m_2^{-1}\sum\nolimits_{i=1}^{m_2} (1-\varepsilon_i^2) > (1-(m_2 d^2 z^2_{(1-\alpha)/2}))/2\right)$$
$$+ P\left(m_2^{-1}Q_{m_2} > (1-(m_2 d^2 z^2_{(1-\alpha)/2}))/2\right) = O\left(m_2^{-s}\right).$$

where the last equality is obtained from Theorem 1 of [24], the Markov inequality, and the Rosentall inequality [30]. □

Theorem 2.7. *Assume for $s > 1$, $\mathbb{E}|\ \varepsilon_1\ |^{4s} < \infty$, $\mathbb{E}|\ X_0\ |^{4s} < \infty$ and $m_0 = o(d^{-2})$. Then as $d \to 0$,*

$$\frac{T^d_{m_2}}{k_d} \xrightarrow{\text{a.s}} 1, \tag{15}$$

$$\mathbb{E}\left[\frac{T^d_{m_2}}{k_d}\right] \to 1, \quad \text{(asymptotic efficiency)}, \tag{16}$$

$$P\left(\beta \in I_{T^d_{m_2}}\right) \to 1-\alpha, \quad \text{(asymptotic consistency)}. \tag{17}$$

Proof. From (14), we have

$(1-\hat{\beta}^2_{m_2}) z^2_{(1-\alpha)/2} d^{-2} \leq T^d_{m_2} \leq (1-\hat{\beta}^2_{m_2}) z^2_{(1-\alpha)/2} d^{-2} + m_2 I_{(T^d_{m_2}=m_2)}$.

The equations (15) and (16) due to the following equation and similar to Theorem 2.3 are concluded. Also, equation (17) proves similar to Theorem 2.2 [40], so the proof is complete.

$$m_2 I_{(T^d_{m_2}=m_2)} \xrightarrow{\text{a.s}} 0.$$

□

2.3 Extention the class of modified two-stage procedure to AR(p)

In this subsection, the results obtained in the previous subsections are presented for the p-order autoregressive model for point and region estimation. For this purpose, we state the corresponding stopping rules and the resulting theorems.

A p-order autoregressive model (AR(p)) with $\mid \beta_i \mid< 1$ is given by,

$$X_i = \beta_1 X_{i-1} + \ldots + \beta_p X_{i-p} + \varepsilon_i, \quad i = 1, 2, ...,$$

where the sequence of independent and identically distributed random variables $\{\varepsilon_i, i \geq 1\}$ has an unknown distribution F. Also, let $\mathbb{E}[\varepsilon_i] = 0$ and $\mathbb{E}[\varepsilon_i^2] = \sigma^2 \in (0, \infty)$. The initial state $\boldsymbol{X}_0 = (X_0, \ldots, X_{-p+1})'$ is $\mathcal{F}_0$ measurable random vector with $\mathbb{E}(\boldsymbol{X}_0) = \mathbf{0}$ and $\mathbb{E}(X_i^2) < \infty, i = -p+1, ..., 0$ where $\mathcal{F}_0$ is independent of $\{\varepsilon_i, i \geq 1\}$. The least-squares estimators of $\boldsymbol{\beta}$ is given by

$$\hat{\boldsymbol{\beta}}_n = (\hat{\boldsymbol{\beta}}_{n1}, \ldots, \hat{\boldsymbol{\beta}}_{np})' = (\sum\nolimits_{i=1}^{n} \boldsymbol{X}_{i-1}\boldsymbol{X}'_{i-1})^{-1}(\sum\nolimits_{i=1}^{n} \boldsymbol{X}'_{i-1}\boldsymbol{X}_i),$$

where $\boldsymbol{X}_i = (X_i, \ldots, X_{i-p+1})'$. A positive-definite $p \times p$ matrix (Σ) is assumed that (i, j)-th entry is $\gamma(i - j)$. It should be noted that $\gamma(.)$ denotes the autocovariance function of the process $\{X_i, i \geq 1\}$. Also, we also know from [4] as $n \to \infty$,

$$n^{-1}(\sum\nolimits_{i=1}^{n} \boldsymbol{X}_{i-1}\boldsymbol{X}'_{i-1}) \xrightarrow{\text{a.s}} \Sigma.$$

It is well known that the asymptotic distribution of $\hat{\boldsymbol{\beta}}_n$ from [23] is

$$(\sum\nolimits_{i=1}^{n} \boldsymbol{X}_{i-1}\boldsymbol{X}'_{i-1})^{1/2}(\hat{\boldsymbol{\beta}}_n - \boldsymbol{\beta}) \xrightarrow{d} N(\mathbf{0}, \sigma^2 \boldsymbol{I}_p),$$

where I_p is the $p \times p$ identity matrix. Let $\boldsymbol{\varepsilon}_t = (\varepsilon_i, \ldots, \varepsilon_{i-p+1})'$. It is supposed that $\sum_{i=1}^{n} \boldsymbol{X}_{i-1}\boldsymbol{X'}_{i-1}$ and $\sum_{t=p}^{n} \boldsymbol{\varepsilon}_t \boldsymbol{\varepsilon'}_t$ are non singular for all sufficiently large n. The loss function for estimating of $\boldsymbol{\beta}$ is given by

$$\begin{aligned} L_n(\hat{\boldsymbol{\beta}}_n, \boldsymbol{\beta}) &= A\left[n^{-1}\left(\left(\hat{\boldsymbol{\beta}}_n - \boldsymbol{\beta}\right)'\left(\sum\nolimits_{i=1}^{n} \boldsymbol{X}_{i-1}\boldsymbol{X}'_{i-1}\right)\left(\hat{\boldsymbol{\beta}}_n - \boldsymbol{\beta}\right)\right)\right] + n \\ &= An^{-1}Q_n + n, \end{aligned}$$

which Q_n is considered as follows

$$\begin{aligned} Q_n &= \left(\hat{\boldsymbol{\beta}}_n - \boldsymbol{\beta}\right)'\left(\sum\nolimits_{i=1}^{n} \boldsymbol{X}_{i-1}\boldsymbol{X}'_{i-1}\right)\left(\hat{\boldsymbol{\beta}}_n - \boldsymbol{\beta}\right) \\ &= (\sum\nolimits_{i=1}^{n} \boldsymbol{X}_{i-1}\varepsilon_i)'(\sum\nolimits_{i=1}^{n} \boldsymbol{X}_{i-1}\boldsymbol{X}'_{i-1})^{-1}(\sum\nolimits_{i=1}^{n} \boldsymbol{X}_{i-1}\varepsilon_i). \end{aligned}$$

when σ is known, the risk function is achived $R_n = n^{-1}(Ap)\sigma^2 + n + o(n^{-1})$. The best fixed sample size is approximately obtained $n_A \approx (Ap)^{1/2}\sigma$ that the $o(n^{-1})$ term is ignored. Then, the corresponding minimum risk function is calculated $R_{n_A} \approx 2(Ap)^{1/2}\sigma$.

In the usual case, σ is unknown in advance so it is difficult to determine the best-fixed sample size in practice. Moreover, the least squares estimator $\hat{\sigma}_n$ is defined $\hat{\sigma}_n^2 = n^{-1}\sum_{i=1}^{n}\left(X_i - \hat{\beta}_1 X_{i-1} - \ldots - \hat{\beta}_p X_{i-p}\right)^2 = n^{-1}\sum_{i=1}^{n}\varepsilon_i^2 - n^{-1}Q_n$ for any n. As noted earlier, the traditional modified two-stage stopping rule and the modified two-stage stopping rule with known positive lower bound for variance (σ_l) are the proposed solution to overcome the limitation that are defined respectively as follows

$$m_3 = \max\{m_0, [(Ap)^{1/2(1+\gamma)}] + 1\},$$

$$N_{m_3} = \max\{m_3, \lfloor (Ap)^{1/2}\hat{\sigma}_{m_3} \rfloor + 1\},$$

and

$$m_4 = \max\{m_0, [\sigma_l (Ap)^{1/2}] + 1\},$$

$$T_{m_4} = \max\{m_4, \lfloor (Ap)^{1/2}\hat{\sigma}_{m_4} \rfloor + 1\}.$$

The introduced stopping rules demonstrate asymptotically risk efficient and asymptotically efficient that we mention these properties as theorems.

Lemma 2.8. *Assume that* $\mathbb{E} \mid \varepsilon_1 \mid^{4s} < \infty$, $\max\limits_{-p+1 \le j \le 0} \mathbb{E} \mid X_j \mid^{4s} < \infty$ *and* $m_0 = o(A^{1/2})$ *for* $s \ge 1$. *Then for any* $0 < \theta < 1$,

$$P\left(N_{m_3} < (1-\theta)(Ap)^{1/2}\sigma\right) = O\left(A^{-s/2(1+\eta)}\right),$$

$$P\left(N_{m_3} > \left[(1+\theta)(Ap)^{1/2}\sigma\right] + 1\right) = O\left(A^{-s/2(1+\eta)}\right).$$

Proof. The argument of proof is similar to Lemma 1 [40], which is omitted. □

Theorem 2.9. *Suppose for* $s > 1$ *that* $\mathbb{E} \mid \varepsilon_1 \mid^{4s} < \infty$, $\max\limits_{-p+1 \le j \le 0} \mathbb{E} \mid X_j \mid^{4s} < \infty$ *and* $m_0 = o(A^{1/2})$. *Then as* $A \to \infty$,

$$\frac{N_{m_3}}{n_A} \xrightarrow{\text{a.s}} 1,$$

$$\mathbb{E}\left[\frac{N_{m_3}}{n_A}\right] \to 1, \quad \text{(asymptotic efficiency)}$$

$$\frac{R_{N_{m_3}}}{R_{n_A}} \to 1, \quad \text{(asymptotic risk efficiency)}.$$

Proof. Simply by the same argument as Theorem 2.2 and we refuse to mention it. □

Lemma 2.10. *Assume that* $\mathbb{E} \mid \varepsilon_1 \mid^{4s} < \infty$, $\max_{-p+1 \leq j \leq 0} \mathbb{E} \mid X_j \mid^{4s} < \infty$, $m_0 = o(A^{1/2})$ *and* $A^{1/2(1+\eta)} \leqslant m_4$ *for some* $\eta \in (0, (s+1)/2 - 1)$ *for* $s \geq 1$. *Then for any* $0 < \theta < 1$,

$$P\left(T_{m_4} < (1-\theta)(Ap)^{1/2}\sigma\right) = O\left(A^{-s/2(1+\eta)}\right),$$

$$P\left(T_{m_4} > \left[(1+\theta)(Ap)^{1/2}\sigma\right] + 1\right) = O\left(A^{-s/2(1+\eta)}\right).$$

Proof. The argument of proof is similar to Lemma 1 [40], which is omitted. □

Theorem 2.11. *Suppose for* $s > 1$ *that* $\mathbb{E} \mid \varepsilon_1 \mid^{4s} < \infty$, $\max_{-p+1 \leq j \leq 0} \mathbb{E} \mid X_j \mid^{4s} < \infty$, $m_0 = o(A^{1/2})$ *and* $A^{1/2(1+\eta)} \leqslant m_4$ *for some* $\eta \in (0, (s+1)/2 - 1)$. *Then as* $A \to \infty$,

$$\frac{T_{m_4}}{n_A} \xrightarrow{\text{p}} 1,$$

$$\mathbb{E}\left[\frac{T_{m_4}}{n_A}\right] \to 1, \quad \text{(asymptotic efficiency)}$$

$$\frac{R_{T_{m_4}}}{R_{n_A}} \to 1, \quad \text{(asymptotic risk efficiency)}.$$

Proof. The proof is similar to Theorem 2.5, which is not mentioned. □

In the following, in order to construct a confidence set for $\boldsymbol{\beta}$ in p-dimensional Euclidean space $\mathbb{R}^p$ with the maximum diameter $2d$ ($d > 0$), we suppose the confidence region, based on the random sequence $\{X_i, i \geq 1\}$, has the coverage probability approximately equal to $1-\alpha$ ($0 < \alpha < 1$) as d tends to 0. An ellipsoidal confidence region of $\boldsymbol{\beta}$ is

$$S_n = \{\boldsymbol{z} : \left(\hat{\boldsymbol{\beta}}_n - \boldsymbol{z}\right)' \left(\sum_{i=1}^{n} \boldsymbol{X}_{i-1}\boldsymbol{X}'_{i-1}\right) \left(\hat{\boldsymbol{\beta}}_n - \boldsymbol{z}\right) \leq d^2 \lambda_{min} \left(\sum_{i=1}^{n} \boldsymbol{X}_{i-1}\boldsymbol{X}'_{i-1}\right) \},$$

where $\lambda_{min}(C)$ is assumed the smallest eigenvalue of a $p \times p$ matrix C. As in the previous subsections, the sample size required via a traditional modified two-stage stopping rule is define for the main study

$$m_5 = \max\{m_0, [(\chi_p^2(1-\alpha)/d)^{2/(1+\gamma)}] + 1\},$$

$$N_{m_5}^d = \max\{m_5, \lfloor d^{-2}\hat{\sigma}_{m_5}^2 \lambda_{min}^{-1} \left(m_5^{-1} \sum_{i=1}^{m_5} \boldsymbol{X}_{i-1}\boldsymbol{X'}_{i-1}\right) \chi_p^2(1-\alpha)\rfloor + 1\}.$$

Also, the modified two-stage stopping rule with a known positive lower bound for variance is considered as follows

$$m_6 = \max\{m_0, [\chi_p^2(1-\alpha)d^{-2}\sigma_l^2] + 1\},$$

$$T_{m_6}^d = \max\{m_6, \lfloor d^{-2}\hat{\sigma}_{m_6}^2 \lambda_{min}^{-1} \left(m_6^{-1} \sum_{i=1}^{m_6} \boldsymbol{X}_{i-1}\boldsymbol{X'}_{i-1}\right) \chi_p^2(1-\alpha)\rfloor + 1\}.$$

It can be shown that the introduced stopping rules are asymptotically efficient and asymptotically consistent. In the following, we will express properties as consequence of theorems.

Theorem 2.12. *Assume for $s > 1$, $\mathbb{E} \mid \varepsilon_1 \mid^{4s} < \infty$, $\max_{-p+1 \leq j \leq 0} \mathbb{E} \mid X_j \mid^{4s} < \infty$ and $m_0 = o(d^{-2})$. Then as $d \to 0$,*

$$\frac{N_{m_5}^d}{k_d} \xrightarrow{a.s} 1,$$

$$\mathbb{E}\left[\frac{N_{m_5}^d}{k_d}\right] \to 1, \quad \text{(asymptotic efficiency)}$$

$$P\left(\boldsymbol{\beta} \in S_{N_{m_5}^d}\right) \to 1-\alpha, \quad \text{(asymptotic consistency).}$$

Proof. The same is proved Theorem 2.3, which is not mentioned. □

Lemma 2.13. *Assume that* $\mathbb{E} \mid \varepsilon_1 \mid^{4s} < \infty$ *and* $\max_{-p+1 \leq j \leq 0} \mathbb{E} \mid X_j \mid^{4s} < \infty$ *for* $s \geq 1$*. So,*

$$P(T^d_{m_6} = m_6) = O\left(m_6^{-s}\right).$$

Proof. The proof is similar to Lemma 2.6, which is omitted. □

Theorem 2.14. *Assume for* $s > 1$, $\mathbb{E} \mid \varepsilon_1 \mid^{4s} < \infty$ *and* $\max_{-p+1 \leq j \leq 0} \mathbb{E} \mid X_j \mid^{4s} < \infty$ *and* $m_0 = o(d^{-2})$*. Then as* $d \to 0$*,*

$$\frac{T^d_{m_6}}{k_d} \xrightarrow{a.s} 1,$$

$$\mathbb{E}\left[\frac{T^d_{m_6}}{k_d}\right] \to 1, \quad \text{(asymptotic efficiency)},$$

$$P\left(\boldsymbol{\beta} \in S_{T^d_{m_6}}\right) \to 1 - \alpha, \quad \text{(asymptotic consistency)}.$$

Proof. Mention of proof is avoided because of its similarity to Theorem 2.7. □

In the following, the Yule-Walker estimators of AR(p) by assuming $p = 2$ are given by,

$$\hat{\theta}_1 = \frac{\sum_{i=2}^{n} X_i X_{i-1}}{\sum_{i=1}^{n} X_i^2}, \quad \hat{\theta}_2 = \frac{\sum_{i=3}^{n+1} X_i X_{i-2}}{\sum_{i=1}^{n} X_i^2}.$$

In all the research, the performance of sequential procedures based on the least squares estimator is considered. Based on the theorems presented in [43] and [24], it can be shown that $\{Q_n, n \geq 1\}$ is also uniformly integrable based on the Yule-Walker estimators, under certain regularity conditions. All the properties presented for the process based on the Yule-Walker estimator are also valid and we refrain from repeating these theorems. In the next section, we aim to investigate the performance of the procedures based on both estimators.

3 Simulation Study

In this section, we conduct Monte Carlo simulation studies to evaluate the performance of point and confidence region estimation. In this study, we compare the performance of the class of the modified two-stage procedure to the best-fixed sample size procedure for the two-order autoregressive model when $\varepsilon_i \sim N(0,1)$. For this purpose, we assess the performance of the point estimation for both stopping variables in terms of the ratio of the average modified stopping variable to the best-fixed sample size, the root of mean square error (RMSE) of estimators, the ratio of the modified two-stage risk function to the best-fixed sample size risk function. Also, we consider two different cases for m and A $10, 100$ and $500, 9000$, respectively. It should be noted that we considered the modified two-stage variable with $\sigma_l = 0.9$.

Moreover, the performance of the confidence region is computed in terms of the average modified two-stage stopping variables, the ratio of the average modified two-stage stopping variables to best fixed sample size, and the coverage probability of confidence region with 95% confidence coefficient for $m = 15$ and different d. In addition, we select the values of (β_1, β_2) based on the stationary conditions of AR(2) model which is given by the following triangular region

$$\begin{cases} \beta_1 + \beta_2 < 1 \\ \beta_2 - \beta_1 < 1 \\ \mid \beta_2 \mid < 1, \end{cases}$$

(for more details refer to [4]). Tables 1, 2, 3 and 4 report the simulation results of point and region estimation based on the least-squares and the Yule-Walker estimators (within parentheses), respectivility. All the computations are obtained using R software by 10,000 replications.

As shown in Tables 1 and 2, the stopping variables increase as A increases which is expected according to the definition of stopping variables. Also, the ratio of the stopping variables close to 1 by increasing A based on both estimators. The bias of estimators and the root of mean square error (RMSE) of estimators decrease as A increases which indicates the good performance of the procedure in the point estimation. Moreover, as we expected the ratios of the risk function to the best fixed-sample size function are approximately around 1 when A increases.

From Tables 3 and 4, the stopping variables increase as d decreases. Furthermore, the ratios of the stopping variables to the best fixed-sample size are close to 1 when d decreases based on both estimators. The coverages probability are close to 0.95 for different (β_1, β_2) with decreasing d, as we expected.

As we see, the simulation results confirm Theorems 2.2- 2.14 which demonstrate the good performance of the procedure. Based on the results, both variables perform well in point and region estimation. According to the conditions of the problem, any of them can be used. The results based on both estimators are also not significantly different, as we expected. According to the conditions of the problem and our goal in the research, we can get help from any of the estimators to determine the sample size.

Table 1: Point estimation of traditional modified two-stage procedure according to N_{m3}

$(m, n_A, A, \beta_1, \beta_2)$	$\widehat{\mathbb{E}}N_{m3}$	$\widehat{\mathbb{E}}\left[\frac{N_{m3}}{n_A}\right]$	$\widehat{\beta}_1$	$\widehat{\beta}_2$
$(10, 14.14214, 500, 0.1, 0.1)$	28.1693(27.9426)	0.8907(0.8836)	0.0964(0.0985)	0.1384(0.1375)
$(10, 14.14214, 500, 0.2, 0.1)$	28.5519(28.5762)	0.9028(0.9036)	0.1927(0.1978)	0.1621(0.1648)
$(10, 14.14214, 500, -0.1, -0.3)$	27.8552(26.9012)	0.8808(0.8506)	−0.0720(−0.0757)	−0.2304(−0.2470)
$(10, 14.14214, 500, 0.1, -0.5)$	25.3751(27.4491)	0.8024(0.8680)	0.0618(0.0666)	−0.4141(−0.4475)
$(10, 14.14214, 500, 0.2, -0.3)$	25.7811(27.0258)	0.8152(0.8546)	0.1392(0.1486)	−0.2120(−0.2300)
$(10, 14.14214, 500, -0.1, -0.7)$	27.6457(29.1613)	0.8742(0.9221)	−0.0560(−0.0612)	−0.6088(−0.6486)
$(10, 14.14214, 500, 0.2, -0.5)$	27.6574(27.3849)	0.8746(0.8659)	0.1285(0.1334)	−0.4075(−0.4272)
$(100, 113.1371, 9000, 0.1, 0.1)$	131.7921(131.6009)	0.9823(0.9808)	0.1076(0.1084)	0.1153(0.1168)
$(100, 113.1371, 9000, 0.2, 0.1)$	132.1405(132.3469)	0.9849(0.9864)	0.2172(0.2186)	0.1490(0.1511)
$(100, 113.1371, 9000, -0.1, -0.3)$	131.5322(131.4231)	0.9803(0.9795)	−0.0763(−0.0766)	−0.2813(−0.2839)
$(100, 113.1371, 9000, 0.1, -0.5)$	131.6605(131.7072)	0.9813(0.9816)	0.0659(0.0671)	−0.4807(−0.4863)
$(100, 113.1371, 9000, 0.2, -0.3)$	131.8926(131.7186)	0.9830(0.9817)	0.1502(0.1528)	−0.2584(−0.2606)
$(100, 113.1371, 9000, -0.1, -0.7)$	131.6686(131.8283)	0.9813(0.9825)	−0.0580(−0.0590)	−0.6811(−0.6853)
$(100, 113.1371, 9000, 0.2, -0.5)$	132.1382(131.9166)	0.9849(0.9832)	0.1319(0.1329)	−0.4610(−0.4655)

$(m, n_A, A, \beta_1, \beta_2)$	MSE $(\widehat{\beta}_1)$	MSE $(\widehat{\beta}_2)$	$\hat{R}_{\mathbb{E}N_{m3}}/R_{n_A}$
$(10, 14.14214, 500, 0.1, 0.1)$	0.2080(0.227)	0.2066(0.2286)	1.0275(1.0713)
$(10, 14.14214, 500, 0.2, 0.1)$	0.2056(0.2270)	0.2121(0.2343)	1.0704(1.1353)
$(10, 14.14214, 500, -0.1, -0.3)$	0.0257(0.1824)	0.0481(0.2406)	0.9159(0.9529)
$(10, 14.14214, 500, 0.1, -0.5)$	0.1558(0.1596)	0.2366(0.2398)	0.8950(0.9319)
$(10, 14.14214, 500, 0.2, -0.3)$	0.1794(0.1886)	0.2404(0.2431)	0.9567(0.9818)
$(10, 14.14214, 500, -0.1, -0.7)$	0.1195(0.1459)	0.2137(0.2370)	0.8885(0.9905)
$(10, 14.14214, 500, 0.2, -0.5)$	0.1516(0.1717)	0.2222(0.2486)	0.9369(0.9819)
$(100, 113.1371, 9000, 0.1, 0.1)$	0.0948(0.0971)	0.0901(0.0912)	1.0675(1.0835)
$(100, 113.1371, 9000, 0.2, 0.1)$	0.0952(0.0957)	0.1029(0.1039)	1.1946(1.2025)
$(100, 113.1371, 9000, -0.1, -0.3)$	0.0692(0.0688)	0.0883(0.0869)	0.9382(0.9198)
$(100, 113.1371, 9000, 0.1, -0.5)$	0.0608(0.0607)	0.0841(0.0797)	0.9134(0.9089)
$(100, 113.1371, 9000, 0.2, -0.3)$	0.0812(0.0807)	0.0964(0.0961)	1.0523(1.0442)
$(100, 113.1371, 9000, -0.1, -0.7)$	0.0032(0.0564)	0.0048(0.0685)	0.9546(0.9522)
$(100, 113.1371, 9000, 0.2, -0.5)$	0.0848(0.0847)	0.0734(0.0872)	1.1152(1.1079)

Table 2: Point estimation of modified two-stage procedure with lower bound for variance according to T_{m4}

$(m, n_A, A, \beta_1, \beta_2)$	$\widehat{\mathbb{E}}T_{m4}$	$\widehat{\mathbb{E}}\left[\frac{T_{m4}}{n_A}\right]$	$\widehat{\beta_1}$	$\widehat{\beta}_2$
$(10, 14.14214, 500, 0.1, 0.1)$	29.6541(31.2128)	0.9377(0.9870)	0.0982(0.1026)	0.1369(0.1364)
$(10, 14.14214, 500, 0.2, 0.1)$	29.8661(31.3382)	0.9444(0.9910)	0.1974(0.2036)	0.1634(0.1636)
$(10, 14.14214, 500, -0.1, -0.3)$	29.4846(31.0515)	0.9323(0.9819)	−0.0718(−0.0768)	−0.2414(−0.2542)
$(10, 14.14214, 500, 0.1, -0.5)$	29.3247(31.0397)	0.9273(0.9815)	0.0614(0.0666)	−0.4301(−0.4550)
$(10, 14.14214, 500, 0.2, -0.3)$	29.5605(31.1353)	0.9347(0.9845)	0.1429(0.1498)	−0.2211(−0.2325)
$(10, 14.14214, 500, -0.1, -0.7)$	29.2369(31.0891)	0.9245(0.9831)	−0.0570(−0.0605)	−0.6215(−0.6560)
$(10, 14.14214, 500, 0.2, -0.5)$	29.4371(31.0822)	0.9308(0.9828)	0.1280(0.1351)	−0.4113(−0.4352)
$(100, 113.1371, 9000, 0.1, 0.1)$	131.8521(132.4544)	0.9824(0.9872)	0.1071(0.1081)	0.1176(0.1159)
$(100, 113.1371, 9000, 0.2, 0.1)$	132.2057(133.0334)	0.9854(0.9915)	0.2162(0.2188)	0.1468(0.1468)
$(100, 113.1371, 9000, -0.1, -0.3)$	131.7924(132.4379)	0.9823(0.9871)	−0.0757(−0.0766)	−0.2801(−0.2827)
$(100, 113.1371, 9000, 0.1, -0.5)$	131.7315(132.7437)	0.9818(0.9894)	0.0662(0.0667)	−0.4789(−0.4843)
$(100, 113.1371, 9000, 0.2, -0.3)$	131.8378(132.7362)	0.9826(0.9893)	0.1503(0.1528)	−0.2595(−0.2625)
$(100, 113.1371, 9000, -0.1, -0.7)$	131.6685(132.5932)	0.9813(0.9882)	−0.0586(−0.0590)	−0.6792(−0.6855)
$(100, 113.1371, 9000, 0.2, -0.5)$	131.9912(132.7906)	0.9833(0.9897)	0.1315(0.1337)	−0.4590(−0.4646)

$(m, n_A, A, \beta_1, \beta_2)$	MSE $(\widehat{\beta}_1)$	MSE $(\widehat{\beta}_2)$	$\hat{R}_{\mathbb{E}T_{m4}}/R_{n_A}$
$(10, 14.14214, 500, 0.1, 0.1)$	0.2019(0.1966)	0.1339(0.1901)	1.0269(1.0435)
$(10, 14.14214, 500, 0.2, 0.1)$	0.1977(0.1946)	0.2073(0.2008)	1.0788(1.0949)
$(10, 14.14214, 500, -0.1, -0.3)$	0.1483(0.1466)	0.2012(0.1941)	0.8980(0.9219)
$(10, 14.14214, 500, 0.1, -0.5)$	0.1303(0.1238)	0.1992(0.1850)	0.8831(0.8912)
$(10, 14.14214, 500, 0.2, -0.3)$	0.1571(0.1530)	0.2092(0.2024)	0.9437(0.9590)
$(10, 14.14214, 500, -0.1, -0.7)$	0.1104(0.1035)	0.1886(0.1745)	0.8755(0.9079)
$(10, 14.14214, 500, 0.2, -0.5)$	0.1417(0.1355)	0.2063(0.1943)	0.9344(0.9415)
$(100, 113.1371, 9000, 0.1, 0.1)$	0.0969(0.0957)	0.0910(0.0900)	1.0797(1.0721)
$(100, 113.1371, 9000, 0.2, 0.1)$	0.0948(0.0947)	0.1024(0.1020)	1.1843(1.1831)
$(100, 113.1371, 9000, -0.1, -0.3)$	0.0692(0.0697)	0.0866(0.0878)	0.9242(0.9333)
$(100, 113.1371, 9000, 0.1, -0.5)$	0.0037(0.0610)	0.0066(0.0795)	0.9190(0.9135)
$(100, 113.1371, 9000, 0.2, -0.3)$	0.0818(0.0799)	0.0953(0.0950)	1.0481(1.0415)
$(100, 113.1371, 9000, -0.1, -0.7)$	0.0556(0.0560)	0.0692(0.0681)	0.9491(0.9485)
$(100, 113.1371, 9000, 0.2, -0.5)$	0.0854(0.0838)	0.0894(0.0864)	1.1258(1.0975)

4 Data Analysis

In this section, the number of sunspots data from 1700 to 1783 years is considered that [4] is modeled this data set by the linear AR(2) model by taking the square root of the data. We determine the initial sample size (m_0) by considering the available sample size. Then, a modified two-stage stopping variables are estimated, and if the initial sample size is insufficient, the difference between modified two-stage stopping variable and the initial sample size is generated at the second stage. The suggested procedure is compared with the two-stage procedure to investigate the performance of these procedures in execution and implementation based on the least-squares and the Yule-Walker estimators (within parentheses), respectively. It is noteworthy that we also evaluate the performance of the procedure as its accuracy is increased. The two-stage stopping rule variables for point and region estimation are

Table 3: Region estimation of traditional modified two-stage procedure according to N_{m5}^d

$(m, d, k_d, \beta_1, \beta_2)$	$\widehat{EN_{m5}^d}$	$\widehat{E\left[\frac{N_{m5}^d}{k_d}\right]}$	CP
$(15, 1.5, 5.6357, 0.1, 0.1)$	15.0012(15.0132)	4.1589(4.1622)	0.9872(0.9835)
$(15, 1.5, 5.0209, 0.2, 0.1)$	15.0073(15.0653)	3.7823(3.7970)	0.9813(0.9675)
$(15, 1.5, 4.0987, -0.1, -0.3)$	15.0007(15.0036)	4.6679(4.6690)	0.9999(0.9948)
$(15, 1.5, 5.3796, 0.1, -0.5)$	15.0030(15.0006)	5.7182(5.7184)	0.9997(0.9964)
$(15, 1.5, 2.7666, 0.2, -0.3)$	15.0061(15.0024)	4.3567(4.3574)	0.9996(0.9912)
$(15, 1.5, 2.7666, -0.1, -0.7)$	15.0011(15.0058)	8.4714(8.4746)	0.9996(0.9979)
$(15, 1.5, 2.7666, 0.2, -0.5)$	15.0062(15.0014)	5.3818(5.3823)	0.9923(0.9942)
$(15, 0.6, 5.6357, 0.1, 0.1)$	24.6132(25.7931)	1.0918(1.1441)	0.9388(0.9321)
$(15, 0.6, 5.0209, 0.2, 0.1)$	26.4796(28.3029)	1.0678(1.1413)	0.9395(0.9412)
$(15, 0.6, 4.0987, -0.1, -0.3)$	20.1778(20.6463)	1.0046(1.0280)	0.9601(0.9517)
$(15, 0.6, 5.3796, 0.1, -0.5)$	18.2996(18.8265)	1.1161(1.1483)	0.9612(0.9634)
$(15, 0.6, 2.7666, 0.2, -0.3)$	21.3654(22.0891)	0.9898(1.0265)	0.9572(0.9484)
$(15, 0.6, 2.7666, -0.1, -0.7)$	16.2214(16.9442)	1.4657(1.5310)	0.9684(0.9641)
$(15, 0.6, 2.7666, 0.2, -0.5)$	18.7326(19.2323)	1.0752(1.1040)	0.9653(0.9616)

Table 4: Region estimation of modified two-stage procedure with known positive lower bound for variance according to T_{m6}^d

$(m, d, k_d, \beta_1, \beta_2)$	$\widehat{ET_{m6}^d}$	$\widehat{E\left[\frac{T_{m6}^d}{k_d}\right]}$	CP
$(15, 1.5, 5.6357, 0.1, 0.1)$	15.0011(15.0224)	4.1589(4.1649)	0.9887(0.9774)
$(15, 1.5, 5.0209, 0.2, 0.1)$	15.0095(15.0508)	3.7830(3.7934)	0.9831(0.9689)
$(15, 1.5, 4.0987, -0.1, -0.3)$	15.0009(15.0020)	4.6699(4.6685)	0.9979(0.9946)
$(15, 1.5, 5.3796, 0.1, -0.5)$	15.0030(15.0032)	5.7298(5.7194)	0.9985(0.9969)
$(15, 1.5, 2.7666, 0.2, -0.3)$	15.0061(15.0024)	4.3522(4.3574)	0.9936(0.9899)
$(15, 1.5, 2.7666, -0.1, -0.7)$	15.0031(15.0077)	8.4524(8.4757)	0.9988(0.9986)
$(15, 1.5, 2.7666, 0.2, -0.5)$	15.0062(15.0009)	5.3818(5.3821)	0.9973(0.9947)
$(15, 0.6, 5.6357, 0.1, 0.1)$	24.5772(24.9607)	1.0902(1.1072)	0.9415(0.9343)
$(15, 0.6, 5.0209, 0.2, 0.1)$	26.1496(26.9017)	1.0545(1.0848)	0.9353(0.9211)
$(15, 0.6, 4.0987, -0.1, -0.3)$	20.3979(21.2373)	1.0156(1.0574)	0.9641(0.9623)
$(15, 0.6, 5.3796, 0.1, -0.5)$	18.5443(20.1138)	1.1310(1.2268)	0.9652(0.9699)
$(15, 0.6, 2.7666, 0.2, -0.3)$	21.5294(22.3594)	1.0005(1.0390)	0.9652(0.9539)
$(15, 0.6, 2.7666, -0.1, -0.7)$	16.8484(19.3975)	1.5224(1.5527)	0.9667(0.9781)
$(15, 0.6, 2.7666, 0.2, -0.5)$	19.0264(20.4243)	1.0918(1.1724)	0.9663(0.9662)

given by respectively.

$$N'_m = \max\{m, \lfloor (Ap)^{1/2}\hat{\sigma}_m \rfloor + 1\},$$

and

$$N'^{d}_{m} = \max\{m, \lfloor d^{-2}\hat{\sigma}^2_m \lambda^{-1}_{min}\left(m^{-1}\sum\nolimits_{i=1}^{m}\boldsymbol{X}_{i-1}\boldsymbol{X}'_{i-1}\right)\chi^2_p(1-\alpha)\rfloor + 1\}.$$

where $m > 2$ is fixed-initial sample size. The results are reported in Tables 5 and 6.

As shown in Table 5, the performance of the proposed stopping variables is not much different from the two-stage stopping variable. The stopping variables increase with A, the values of which do not differ much from each other. Also, the estimators based on both stopping variables are almost close to the two-stage estimators. As A increases, the Yule-Walker estimators approach the least-squares estimators but it has a larger sample size for the low value of A. As can be seen, the performance of the procedure based on the least-squares estimators is better in point estimation.

The results of region-stopping variables are reported in Table 6. From Table 6, stopping variables increase as d decreases and these variables are not much different from the two-stage variable. Also, the estimators of the procedures are close to each other for different values d. When d decreases, the procedure based on the Yule-Walker estimators performs better and has a smaller final sample size of course, the performance of both estimators is reasonable.

The results demonstrate the close performance of the modified two-stage and two-stage procedures in implementation. Indeed, with increasing process accuracy, performance has not diminished, which is what we expected. Therefore, by increasing the accuracy of execution and reducing the cost of sampling, the modified two-stage procedure can be used that has more performance, simplicity, and accuracy than the two-stage procedure.

Table 5: Point estimation of modified two-stage and two-stage procedures.

(m, A)	N_{m3}	T_{m4}	N'_m	$\widehat{\beta}_{1N_{m3}}$	$\widehat{\beta}_{1T_{m4}}$	$\widehat{\beta}_{1N'_m}$
$(3, 2)$	19(33)	19(33)	19(33)	1.0005(0.9703)	1.0005(0.9703)	1.0005(0.9703)
$(7, 9)$	24(26)	24(26)	24(26)	0.9800(1.1670)	0.9800(1.1670)	0.9800(1.1670)
$(20, 30)$	41(44)	41(44)	41(44)	0.9823(0.9705)	0.9823(0.9705)	0.9823(0.9705)
$(40, 60)$	72(76)	72(76)	72(76)	0.9845(0.9863)	0.9845(0.9863)	0.9845(0.9863)
(m, A)	$\widehat{\beta}_{2N_{m3}}$	$\widehat{\beta}_{2T_{m4}}$	$\widehat{\beta}_{2N'_m}$	$\widehat{\sigma}_{N_{m3}}$	$\widehat{\sigma}_{T_{m4}}$	$\widehat{\sigma}_{N'_m}$
$(3, 2)$	0.9368(0.9084)	0.9368(0.9084)	0.9368(0.9084)	29.1860(31.4746)	29.1860(31.4746)	29.1860(31.4746)
$(7, 9)$	0.9332(1.1508)	0.9332(1.1508)	0.9332(1.1508)	26.1477(62.7512)	26.1477(62.7512)	26.1477(62.7512)
$(20, 30)$	0.9336(0.9424)	0.9336(0.9424)	0.9336(0.9424)	42.8070(43.1162)	42.8070(43.1162)	42.8070(43.1162)
$(40, 60)$	0.9336(0.9424)	0.9336(0.9424)	0.9336(0.9424)	42.8070(43.1160)	42.8070(43.1160)	42.8070(43.1160)

Table 6: Region estimation of modified two-stage and two-stage procedures.

(m, d)	N^d_{m5}	N^d_p	N'^d_m	$\widehat{\beta}_{1N^d_{m5}}$	$\widehat{\beta}_{1T^d_{m6}}$	$\widehat{\beta}_{1N'^d_m}$
$(10, 7)$	10(10)	10(10)	10(10)	0.9692(0.9692)	0.9692(0.9692)	0.9692(0.9692)
$(10, 5)$	10(10)	10(10)	10(10)	0.9692(0.9692)	0.9692(0.9692)	0.9692(0.9692)
$(10, 3)$	20(15)	20(15)	20(15)	0.9415(1.1101)	0.9415(1.1101)	0.9415(1.1101)
$(10, 2)$	44(33)	44(33)	44(33)	0.9705(0.9703)	0.9705(0.9703)	0.9705(0.9703)
$(10, 1.5)$	78(59)	78(59)	78(59)	0.9473(1.0055)	0.9473(1.0055)	0.9473(1.0055)
(m, d)	$\widehat{\beta}_{2N^d_{m5}}$	$\widehat{\beta}_{2T^d_{m6}}$	$\widehat{\beta}_{2N'^d_m}$	$\widehat{\sigma}_{N^d_{m5}}$	$\widehat{\sigma}_{T^d_{m6}}$	$\widehat{\sigma}_{N'^d_m}$
$(10, 7)$	0.9206(0.9206)	0.9206(0.9206)	0.9206(0.9206)	19.9834(19.9834)	19.9834(19.9834)	19.9834(19.9834)
$(10, 5)$	0.9206(0.9206)	0.9206(0.9206)	0.9206(0.9206)	19.9834(19.9834)	19.9834(19.9834)	19.9834(19.9834)
$(10, 3)$	0.8571(1.0745)	0.8571(1.0745)	0.8571(1.0745)	21.9354(34.5843)	21.9354(34.5843)	21.9354(34.5843)
$(10, 2)$	0.9077(0.9084)	0.9077(0.9084)	0.9077(0.9084)	35.4501(31.4746)	35.4501(31.4746)	35.4501(31.4746)
$(10, 1.5)$	0.8740(0.9605)	0.8740(0.9605)	0.8740(0.9605)	38.9247(43.5317)	38.9247(43.5317)	38.9247(43.5317)

5 Discussion

In situations where we are faced with limited access to the sample, sequential procedures are suggested as a suitable alternative. Also, when the sample size is not known in advance, these procedures can be used to overcome the limitations in computing. Among the proposed procedures, it is very important to provide a suitable stopping rule based on the procedure strategy. In this paper, the stopping rule inspired by the best-fixed sample size is presented, under the condition of minimum risk function for point estimation. Considering the wide application of linear time series models, the aim of choosing a modified two-stage procedure between sequential procedures is simplicity in implementation and appropriate accuracy in point and interval estimation. The performance of the modified two-stage procedure is investigated and the results are expressed as theorems. The asymptotic properties are obtained the same as the properties of the purely sequential procedure and two-stage procedure ([43], [24], [40]).

The purely sequential procedure is the widely used procedure that provides the smallest sample size among the sequential procedures. Despite the good properties of this procedure, sampling has a high cost due to its stopping rule. Also, in terms of execution, it is time-consuming compared to other sequential procedures. In practice, these items are a weakness of this procedure, and in situations where cost and time are important to us, the two-stage procedure and the modified two-stage procedure are suitable. The two-stage procedure is also suitable, but if the initial sample size is not suitable, the procedure will suffer from overestimation. Moreover, the three-stage procedure strategy is presented inspired by the two-stage procedure. This procedure is also recommended in terms of implementation after the two-stage procedure and the modified two-stage procedure. If there are conditions for implementing the modified two-stage procedure, it is preferable to the mentioned procedures and is a suitable replacement procedure for determining the sample size.

Conclusions

In this paper, we investigate a modified two-stage in the autoregressive model by introducing a class of modified two-stage stopping variables. We present the important properties of this procedure under the theorems, which indicate the close efficiency compared to the best-fixed sample size procedure when A tends to ∞ and d tends to 0. These properties include asymptotically risk efficient, asymptotically efficient, and asymptotically consistent. Also, the result of simulation studies demonstrates the good performance of the procedure and the ability of the procedure to be substituted for the best-fixed sample size procedure based on the least-squares and the Yule-Walker estimators. As we expected, the performance of the procedure based on the Yule-Walker estimators is also good. Finally, we examined the performance of the procedure in implementation and execution in real data, which demonstrates the same performance to the two-stage procedure. In real data analysis, the least-squares estimators and the Yule-Walker estimators are suitable for the point and the region estimation, respectively.

This procedure has the ease of implementation of the two-stage procedure and increases the accuracy in execution. Also, the efficiency of this procedure is close to the best-fixed sample size procedure despite reducing the weaknesses of the two-stage procedure and reducing the sampling cost. Furthermore, according to the situation and available information, we can use each of the introduced stopping variables to determine the final sample size. Due to the importance of real-time-series data, these class of variables are a good suggestion for determining the sample size in point and region estimation.

Future Work

This work represents an alternative method for determining sample size. In the process of implementing and analyzing the present procedure, a minimum sample size of key points has been identified as worthy of further investigation. With regards to the application of time series data, the proper stopping rule, with the boundedness condition of the loss function in point estimation the fundamental questions still need to be resolved. Also, two-stage and modified two-stage procedures for non-linear time series models for example TAR(p) need to be investigated. However, further work is required to carry out simulation comparisons with different assumptions for the asymptotic properties of the procedure.

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