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Original Research Paper

On Biharmonic Hypersurfaces of Three Curvatures in Minkowski 5-Space

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Abstract. In this paper, we study the L_k-biharmonic Lorentzian hypersurfaces of the Minkowski 5-space $\mathcal{M}^5$, whose second fundamental form has three distinct eigenvalues. An isometrically immersed Lorentzian hypersurface, $\mathbf{x} : M_1^4 \to \mathcal{M}^5$, is said to be L_k-biharmonic if it satisfies the condition $L_k^2\mathbf{x} = 0$, where L_k is the linearized operator associated to the 1st variation of the mean curvature vector field of order $(k+1)$ on M_1^4. In the special case $k = 0$, we have L_0 is the well-known Laplace operator Δ and by a famous conjecture due to Bang-Yen Chen each Δ-biharmonic submanifold of every Euclidean space is minimal. The conjecture has been affirmed in many Riemanian cases. We obtain similar results confirming the L_k-conjecture on Lorentzian hypersurfaces in $\mathcal{M}^5$ with at least three principal curvatures.

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1 Introduction

The biharmonic maps on Euclidean spaces, as solutions of strongly elliptic semilinear differential equations of order four, appear in the theory of partial differential equations. From physical points of view, the biharmonic surfaces play interesting roles the theories of elastics and fluid mechanics. Also, the biharmonic Bezier surfaces play useful roles in computational geometry. In the homotopy class of Brower of degree ± 1, one cannot find a harmonic map from 2-torus into Euclidean 2-sphere, although, there exists a biharmonic one ([6]). A famous conjecture of Bang-Yen Chen states that each biharmonic submanifold of an Euclidean space is minimal. In [5], the conjecture has been confirmed on hypersurface of E^m whose second fundamental form has at most two distinct eigenvalues. Also, it has been affirmed on hypersurfaces of Euclidean 4-spaces in [8]. In [1], the subject is studied on submanifolds of Euclidean spaces. Also, Chen had introduced a nice connection between biharmonic hypersurfaces and the finite type ones.

Recently, some extensions of Chen's conjecture has been studied on some (semi-)Riemannian hypersurface of pseudo-Euclidean spaces. For instance, one may find some results on biharmonic Riemannian or Lorentzian hypersurfaces of $\mathcal{M}^4$ in [3, 4, 11]. In this paper, replacing Δ by L_k, we study L_k-conjecture on timelike hypersurfaces in $\mathcal{M}^5$ whose shape operator has at least three eigenvalues.

Now, we present the organization of paper. Section 2 is allocated to notations and concepts. In section 3, we show that if a L_k-biharmonic timelike hypersurface has diagonal shape operator with three distinct eigenvalues, the it is k-minimal. In section 4, we get same results on L_k-biharmonic timelike hypersurfaces with non-diagonal shape operator which has three possible csese. Also, in non-diagonal case, we show that if the kth mean curvature of a L_k-biharmonic timelike hypersurfaces is constant and one of its principal curvature is constant, then it is k-minimal.

2 Prerequisites

First, we recall prerequisite concepts and notations from [2, 9, 10, 12, 15]. By definition, the Minkowski 5-space $\mathcal{M}^5 = E^5_1$ is obtained from Euclidean 5-space E^5 by endowing with the following non-degenerate inner product $\langle \mathbf{v}, \mathbf{w} \rangle := -v_1w_1 + \Sigma_{i=2}^5 v_iw_i$, for every $\mathbf{v}, \mathbf{w} \in E^5$. For each non-zero vector $\mathbf{v} \in \mathcal{M}^5$, the value of $\langle \mathbf{v}, \mathbf{v} \rangle$ can be a negative, zero or positive number and the vector $\mathbf{v}$ is said to be time-like, light-like or space-like, respectively.

Every Lorentz hypersurface M^4_1 of $\mathcal{M}^5$ is defined by an isometric immersion $\mathbf{x} : M^4_1 \to \mathcal{M}^5$ such that induced metric on M^4_1 is Lorentzian. The Levi-Civita connections on M^4_1 and $\mathcal{M}^5$ (respectively) are denoted by $\tilde{\nabla}$ and $\bar{\nabla}$. We consider a unit normal vector field $\mathbf{n}$ which defines the second fundamental form S(i.e. the shape operator) on M^4_1.

In general, in each 4-dimensional Lorentz vector space V^4_1, a basis $\mathcal{B} := \{\mathbf{v}_1, \cdots, \mathbf{v}_4\}$ is named *orthonormal* if it satisfies $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = \epsilon_i \delta^j_i$ for $i, j = 1, \cdots, 4$, where $\epsilon_1 = -1$ and $\epsilon_i = 1$ for $i = 2, 3, 4$ (δ^j_i is the Kronecker delta). Also, $\mathcal{B}$ is named *pseudo-orthonormal* if $\langle \mathbf{v}_1, \mathbf{v}_1 \rangle = \langle \mathbf{v}_2, \mathbf{v}_2 \rangle = 0$, $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = -1$ and $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = \delta^j_i$, for $i = 1, 2, 3, 4$ and $j = 3, 4$.

According to an orthonormal or pseudo-orthonormal basis $\mathcal{B} := \{e_1, \cdots, e_4\}$ chosen on the tangent bundle of M^4_1, there are two possible matrix forms $\mathcal{G}_1 := \mathrm{diag}[-1, 1, 1, 1]$ and $\mathcal{G}_2 = \mathrm{diag}[\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right], 1, 1]$ for the (induced) Lorentz metric on M^4_1.

In the case $\mathcal{G}_1$ (with respect to an orthonormal basis), the fundamental form has two possible matrix forms $\mathcal{F}_1 = \mathrm{diag}[\lambda_1, \lambda_2, \lambda_3, \lambda_4]$ and

$$\mathcal{F}_2 = \mathrm{diag}[\left[\begin{smallmatrix} \kappa & \lambda \\ -\lambda & \kappa \end{smallmatrix}\right], \eta_1, \eta_2],$$

where $\lambda \neq 0$.

(Note that, $\kappa \pm i\lambda$ are two eigenvalues of $\mathcal{F}_2$).

In the case $\mathcal{G}_2$ (with respect to a pseudo-orthonormal basis), the fundamental form has two possible matrix forms $\mathcal{F}_3 = \mathrm{diag}[\left[\begin{smallmatrix} \kappa & 0 \\ 1 & \kappa \end{smallmatrix}\right], \lambda_1, \lambda_2]$ and $\mathcal{F}_4 = \mathrm{diag}[\left[\begin{smallmatrix} \kappa & 0 & 0 \\ 0 & \kappa & 1 \\ -1 & 0 & \kappa \end{smallmatrix}\right], \lambda]$.

Remark 2.1. In the case $\mathcal{G}_2$, we substitute $\mathcal{B} := \{e_1, e_2, e_3, e_4\}$ by a new orthonormal basis $\tilde{\mathcal{B}} := \{\tilde{e_1}, \tilde{e_2}, e_3, e_4\}$, where $\tilde{e_1} := \frac{1}{2}(e_1 + e_2)$ and

$\tilde{e_2} := \frac{1}{2}(e_1 - e_2)$. Then, we obtain $\tilde{\mathcal{F}_3} = \mathrm{diag}[\begin{bmatrix} \kappa+\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \kappa-\frac{1}{2} \end{bmatrix}, \lambda_1, \lambda_2]$ and $\tilde{\mathcal{F}_4} = \mathrm{diag}[\begin{bmatrix} \kappa & 0 & \frac{\sqrt{2}}{2} \\ 0 & \kappa & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & \kappa \end{bmatrix}, \lambda]$ (instead of $\mathcal{F}_3$ and $\mathcal{F}_4$, respectively).

Now, we define the principal curvatures κ_i's (for $i = 1, \cdots, 4$) of M_1^4 in non-diagonal cases, as follow:

In the case $S = \mathcal{F}_2$, we put $\kappa_1 = \kappa + i\lambda$, $\kappa_2 = \kappa - i\lambda$, and $\kappa_i := \eta_{i-2}$, for $i = 3, 4$.

When $S = \tilde{\mathcal{F}_3}$, we take $\kappa_1 = \kappa_2 := \kappa$, $\kappa_3 := \lambda_1$ and $\kappa_4 := \lambda_2$.

In case $S = \tilde{\mathcal{F}_4}$, we take $\kappa_i := \kappa$ for $i = 1, 2, 3$, and $\kappa_4 := \lambda$.

The characteristic polynomial of S on M_1^4 is of the form

$$Q(t) = \prod_{i=1}^{4}(t - \kappa_i) = \sum_{j=0}^{4}(-1)^j s_j t^{4-j},$$

where, $s_0 := 1$, $s_i := \sum_{1 \leq j_1 < \cdots < j_i \leq 4} \kappa_{j_1} \cdots \kappa_{j_i}$ for $i = 1, 2, 3, 4$.

For $k = 1, \cdots, 4$, the *kth mean curvature* H_k of M_1^4 is defined by $H_k = \frac{1}{\binom{4}{k}} s_k$. We put for convenience that, $H_0 = 1$. In usual, M_1^4 is named k*-minimal* if $H_{k+1} \equiv 0$,.

When M_1^4 has diagonal shape operator with constant eigenvalues, it is called *isoparametric*. In non-diagonal case, isoparametric means that the minimal polynomial of shape operator is constant. By Theorem 4.10 in [10], there is no isoparametric timelike hypersurface of $\mathcal{M}^5$ with complex principal curvatures.

The well-known Newton map $P_k : \chi(M_1^4) \to \chi(M_1^4)$ are defined by

$$P_0 = I, \quad P_k = s_k I - S \circ P_{k-1}, \qquad (j = 1, 2, 3, 4).$$

Newton map has another equivalent formula, $P_k = \sum_{i=0}^{k}(-1)^i s_{k-i} S^i$ which gives $P_4 = 0$. (see [2, 13]).

We will use the following notations

$$\mu_{i;k} = \sum_{1 \leq j_1 < \cdots < j_k \leq 4; j_l \neq i} \kappa_{j_1} \cdots \kappa_{j_k}, \qquad (i = 1, 2, 3, 4; \quad 1 \leq k \leq 3),$$

$$\mu_{i_1, i_2;k} = \sum_{1 \leq j_1 < \cdots < j_k \leq 4; j_l \neq i_1; j_l \neq i_2} \kappa_{j_1} \cdots \kappa_{j_k}, \quad (i = 1, 2, 3, 4; \quad 1 \leq k \leq 3).$$

According to possible matrix forms of S, the map P_j has different forms. In the diagonal case $S = \mathcal{F}_1$, where we have $P_j = diag[\mu_{1;j}, \cdots, \mu_{4;j}]$, for $j = 1, 2, 3$.

In three non-diagonal cases we have as follow.

In the case $S = \mathcal{F}_2$, we have

$$P_j = \mathrm{diag}\left[\begin{bmatrix} \kappa\mu_{1,2;j-1} + \mu_{1,2;j} & -\lambda\mu_{1,2;j-1} \\ \lambda\mu_{1,2;j-1} & \kappa\mu_{1,2;j-1} + \mu_{1,2;j} \end{bmatrix}, \mu_{3;j}, \mu_{4;j}\right].$$

When $S = \tilde{\mathcal{F}}_3$, we have

$$P_j(p) = \mathrm{diag}\left[\begin{bmatrix} \mu_{1,2;j} + (\kappa - \frac{1}{2})\mu_{1,2;j-1} & -\frac{1}{2}\mu_{1,2;j-1} \\ \frac{1}{2}\mu_{1,2;j-1} & \mu_{1,2;j} + (\kappa + \frac{1}{2})\mu_{1,2;j-1} \end{bmatrix}, \mu_{3;j}, \mu_{4;j}\right].$$

In the case $S = \tilde{\mathcal{F}}_4$, similarly $P_j(p)$ has the matrix form

$$\begin{bmatrix} u_j + 2\kappa u_{j-1} + (\kappa^2 - \frac{1}{2})u_{j-2} & -\frac{1}{2}u_{j-2} & -\frac{\sqrt{2}}{2}(u_{j-1} + \kappa u_{j-2}) & \\ \frac{1}{2}u_{j-2} & u_j + 2\kappa u_{j-1} + (\kappa^2 + \frac{1}{2})u_{j-2} & \frac{\sqrt{2}}{2}(u_{j-1} + \kappa u_{j-2}) & \\ \frac{\sqrt{2}}{2}(u_{j-1} + \kappa u_{j-2}) & \frac{\sqrt{2}}{2}(u_{j-1} + \kappa u_{j-2}) & u_j + 2\kappa u_{j-1} + \kappa^2 u_{j-2} & \\ & & & \mu_{4;j} \end{bmatrix},$$

where $u_3 = u_2 = 0$, $u_1 = \lambda$, $u_0 = 1$ and $u_{-1} = u_{-2} = 0$.

In all cases we have the following important identities ([2, 13]).

$$\begin{aligned} &\text{(i)} \quad s_{k+1} = \kappa_i\mu_{i;k} + \mu_{i;k+1}, \quad (1 \le i \le 4; 1 \le k \le 3) \\ &\text{(ii)} \quad \mu_{i;k+1} = \kappa_l\mu_{i,l;k} + \mu_{i,l;k+1}, \qquad (1 \le i, l \le 4,\ i \ne l) \end{aligned} \tag{1}$$

and

$$\begin{aligned} &\mu_{i,1} = 4H_1 - \lambda_i, \\ &\mu_{i,2} = 6H_2 - 4\lambda_i H_1 + \lambda_i^2, \quad (1 \le i \le 4), \\ &tr(P_k) = c_k H_k, \\ &tr(P_k \circ S) = c_k H_{k+1}, \\ &trS^2 = 4(4H_1^2 - 3H_2), \\ &tr(P_k \circ S^2) = \binom{4}{k+1}[4H_1H_{k+1} - (4-k-1)H_{k+2}], \end{aligned}$$

where $c_k = (4-k)\binom{4}{k} = (k+1)\binom{4}{k+1}$.

The *linearized operator* $L_k : \mathcal{C}^\infty(M_1^4) \to \mathcal{C}^\infty(M_1^4)$ is defined by

$$L_k(f) := tr(P_k \circ \nabla^2 f)$$

where $\langle \nabla^2 f(X), Y\rangle = \langle \nabla_X \nabla f, Y\rangle$ for every smooth vector fields X and Y on M_1^4.

Associated to the orthonormal frame $\{e_1, \cdots, e_4\}$ of tangent space on a local coordinate system in the hypersurface $x : M_1^4 \to \mathcal{M}^5$, for $k = 0, \cdots, 3$, $L_k(f)$ has an explicit expression as

$$L_k(f) = -\sum_{i=1}^{4} \epsilon_i \mu_{i;k}(e_i e_i f - \nabla_{e_i} e_i f). \tag{2}$$

For a Lorentzian hypersurface $\mathbf{x} : M_1^4 \to \mathcal{M}^5$, with a chosen (local) unit normal vector field $\mathbf{n}$, for an arbitrary vector $\mathbf{a} \in \mathcal{M}^5$ we use the decomposition $\mathbf{a} = \mathbf{a}^T + \mathbf{a}^N$ where $\mathbf{a}^T \in TM$ is the tangential component of $\mathbf{a}$, $\mathbf{a}^N \perp TM$, and we have the following formulae from [2, 13].

$$\begin{aligned}
&\nabla\langle \mathbf{x}, \mathbf{a}\rangle = \mathbf{a}^T,\\
&\nabla\langle \mathbf{n}, \mathbf{a}\rangle = -S\mathbf{a}^T,\\
&L_k\mathbf{x} = c_k H_{k+1}\mathbf{n},\\
&L_k\mathbf{n} = -\tbinom{4}{k+1}\nabla H_{k+1} - \tbinom{4}{k+1}[4H_1H_{k+1} - (4-k-1)H_{k+2}]\mathbf{n}.
\end{aligned}$$

Then, we get

$$\begin{aligned}
&(i) \quad L_1^2\mathbf{x} = 24[P_2\nabla H_2 - 9H_2\nabla H_2] + 12[L_1H_2 - 12H_2(2H_1H_2 - H_3)]\mathbf{n}\\
&(ii) \quad L_2^2\mathbf{x} = 24[P_3\nabla H_3 - 6H_3\nabla H_3] + 12[L_2H_3 - 4H_3(4H_1H_3 - H_4)]\mathbf{n}\\
&(iii) \quad L_3^2\mathbf{x} = -12H_4\nabla H_4 + 4(L_3H_4 - 4H_1H_4^2)\mathbf{n}
\end{aligned} \tag{3}$$

If a hypersurface $\mathbf{x} : M_1^4 \to \mathcal{M}^5$ satisfies the equation $L_k^2\mathbf{x} = 0$, then it is said to be L_k*-biharmonic*. Equivalently, $\mathbf{x}$ is L_k-biharmonic if and only if it satisfies conditions:

$$\begin{aligned}
&(i)\ L_kH_{k+1} = \binom{4}{k+1} H_{k+1}(4H_1H_{k+1} - (4-k-1)H_{k+2}),\\
&(ii)\ P_{k+1}\nabla H_{k+1} = 3(4-k)H_{k+1}\nabla H_{k+1}.
\end{aligned} \tag{4}$$

By (3)(i), a hypersurface $x : M_1^4 \to \mathcal{M}^5$ is L_1-biharmonic if and only if it satisfies conditions:

$$(i)\ L_1H_2 = 12H_2(2H_1H_2 - H_3),\ (ii)\ P_2\nabla H_2 = 9H_2\nabla H_2. \tag{5}$$

From (3)(ii) we get that a hypersurface $x : M_1^4 \to \mathcal{M}^5$ is L_2-biharmonic if and only if it satisfies conditions:

$$(i)\ L_2H_3 = 4H_3(4H_1H_3 - H_4),\ (ii)\ P_3\nabla H_3 = 6H_3\nabla H_3. \tag{6}$$

Finally, (3)(iii) implies that a hypersurface $x : M_1^4 \to \mathcal{M}^5$ is L_3-biharmonic if and only if it satisfies conditions:

$$(i)\ L_3H_4 = 4H_1H_4^2,\ (ii)\ \nabla H_4^2 = 0. \tag{7}$$

The structure equations on $\mathcal{M}^5$ are $d\omega_i = \sum\limits_{j=1}^{5} \omega_{ij} \wedge \omega_j$, $\omega_{ij} + \omega_{ji} = 0$ and $d\omega_{ij} = \sum\limits_{l=1}^{5} \omega_{il} \wedge \omega_{lj}$. Restricted on M, we have $\omega_5 = 0$ and then, $0 = d\omega_5 = \sum\limits_{i=1}^{4} \omega_{5,i} \wedge \omega_i$. So, by Cartan's lemma, there exist functions h_{ij} such that $\omega_{5,i} = \sum\limits_{j=1}^{4} h_{ij}\omega_j$ and $h_{ij} = h_{ji}$ Which give the second fundamental form of M, as $B = \sum\limits_{i,j} h_{ij}\omega_i\omega_j e_5$. The mean curvature H is given by $H = \frac{1}{4}\sum\limits_{i=1}^{4} h_{ii}$. Therefore, we obtain the structure equations on M as follow.

$$d\omega_i = \sum_{j=1}^{4} \omega_{ij} \wedge \omega_j, \qquad \omega_{ij} + \omega_{ji} = 0,$$

$$d\omega_{ij} = \sum_{k=1}^{4} \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l=1}^{3} R_{ijkl}\omega_k \wedge \omega_l,$$

for $i, j = 1, 2, 3$, and the Gauss equations $R_{ijkl} = (h_{ik}h_{jl} - h_{il}h_{jk})$, where R_{ijkl} denotes the components of the Riemannian curvature tensor of M.

Denoting the covariant derivative of h_{ij} by h_{ijk}, we have

$$dh_{ij} = \sum_{k=1}^{4} h_{ijk}\omega_k + \sum_{k=1}^{4} h_{kj}\omega_{ik} + \sum_{k=1}^{4} h_{ik}\omega_{jk},$$

and by the Codazzi equation we get $h_{ijk} = h_{ikj}$.

3 Diagonal shape operator

The first lemma can be proved by the same manner of similar one in [16].

Lemma 3.1. *Let M_1^4 be a Lorentzian hypersurface in $\mathcal{M}^5$ of type I with real principal curvatures of constant multiplicities. Then the distribution of the space of principal directions corresponding to the principal curvatures is completely integrable. In addition, if a principal curvature be of multiplicity greater than one, then it will be constant on each integral submanifold of the corresponding distribution.*

Proposition 3.2. *If $\boldsymbol{x} : M_1^4 \to \mathcal{M}^5$ is a L_k-biharmonic Lorentzian hypersurface with diagonal shape operator, constant $(k-1)$th and kth mean curvatures and non-constant $(k+1)$th mean curvature, then it has a non-constant principal curvature of multiplicity one.*

Proof. In the case $k = 3$, by condition (7)(ii), the 4th mean curvature is constant on M, which contradicts by assumption. So, it is enough to give proof for cases $k = 1, 2$. Using the assumptions, there exists an open connected subset $\mathcal{U}$ of M, on which we have $\nabla H_{k+1} \neq 0$. By conditions (5)(ii) and (6)(ii), $e_1 := \frac{\nabla H_{k+1}}{||\nabla H_{k+1}||}$ is an eigenvector of P_{k+1} with the corresponding eigenvalue $3(4-k)H_{k+1}$, on $\mathcal{U}$. Without loss of generality, we can take a suitable orthonormal local basis $\{e_1, e_2, e_3, e_4\}$ for the tangent bundle of M, consisting of the eigenvectors of the shape operator A such that $Ae_i = \lambda_i e_i$ and $P_{k+1}e_i = \mu_{i,k+1}e_i$, (for $i = 1, 2, 3, 4$) and then

$$\mu_{1,k+1} = 3(4-k)H_{k+1}. \tag{8}$$

By the polar decomposition $\nabla H_{k+1} = \sum_{i=1}^{4} e_i(H_{k+1})e_i$, we get

$$e_1(H_{k+1}) \neq 0, \quad e_2(H_{k+1}) = e_3(H_{k+1}) = e_4(H_{k+1}) = 0. \tag{9}$$

We continue the proof separately in two cases $k = 1$ and $k = 2$.

Case 1: $k = 1$. By (1) and (8) we have

$$H_2 = \frac{1}{3}\lambda_1(\lambda_1 - 4H). \tag{10}$$

Then, having assumed H to be constant, from (9) we get

$$e_1(\lambda_1) \neq 0, \quad e_2(\lambda_1) = e_3(\lambda_1) = e_4(\lambda_1) = 0, \tag{11}$$

which gives that λ_1 is non-constant. Now, putting $\nabla_{e_i} e_j = \sum_{k=1}^{4} \omega_{ij}^k e_k$ (for $i, j = 1, 2, 3, 4$), the identity $e_k < e_i, e_j >= 0$ gives $\epsilon_j \omega_{ki}^j = -\epsilon_i \omega_{kj}^i$ (for $i, j, k = 1, 2, 3, 4$). Furthermore, for distinct $i, j, k = 1, 2, 3, 4$, the Codazzi equation implies

$$e_i(\lambda_j) = (\lambda_i - \lambda_j)\omega_{ji}^j, \quad (\lambda_i - \lambda_j)\omega_{ki}^j = (\lambda_k - \lambda_j)\omega_{ik}^j. \tag{12}$$

Since by (11) we have $e_1(\lambda_1) \neq 0$, we claim $\lambda_j \neq \lambda_1$ for $j = 2, 3, 4$. Because, assuming $\lambda_j = \lambda_1$ for some integer $j \neq 1$, we have $e_1(\lambda_j) = e_1(\lambda_1) \neq 0$. On the other hand, from (12) we obtain $0 = (\lambda_1 - \lambda_j)\omega_{j1}^j = e_1(\lambda_j) = e_1(\lambda_1)$. So, we got a contradiction. Therefore, the main claim is affirmed in case $k = 1$.

Case 2: $k = 2$. This case is similar to case $k = 1$, but, the equality (10) will be changed to

$$H_3 = \frac{-1}{2}\lambda_1(\lambda_1^2 - 4H\lambda_1 + 6H_2), \tag{13}$$

which, by assuming H_2 and H to be constant, gives the same result as (11). The rest part of proof is straightforward as Case 1 and give that λ_1 is a non-constant principal curvature of multiplicity one. $\square$

The last proposition can be stated in the case $k = 0$, which may be found in [7] and [17].

Proposition 3.3. *If $\boldsymbol{x} : M_1^4 \to \mathcal{M}^5$ is a L_k-biharmonic Lorentzian hypersurface with diagonal shape operator, exactly three distinct principal curvatures, constant $(k-1)$th and kth mean curvatures and non-constant $(k+1)$th mean curvature, then there exists a locally moving orthonormal tangent frame $\{e_1, e_2, e_3, e_4\}$ of principal vectors of M_1^4 with associated* *principal curvatures $\lambda_1, \lambda_2 = \lambda_3, \lambda_4$, which satisfy the following equalities:*

$$\begin{aligned}
&(i)\nabla_{e_1}e_1 = 0,\ \nabla_{e_2}e_1 = \alpha e_2,\ \nabla_{e_3}e_1 = \alpha e_3,\ \nabla_{e_4}e_1 = -\beta e_4,\\
&(ii)\nabla_{e_2}e_2 = -\alpha e_1 + \omega_{22}^3 e_3 + \gamma e_4,\ \nabla_{e_i}e_2 = \omega_{i2}^3 e_3 \quad for\ i = 1,3,4\ ;\\
&(iii)\nabla_{e_3}e_3 = -\alpha e_1 - \omega_{32}^3 e_3 + \gamma e_4,\quad \nabla_{e_i}e_3 = -\omega_{i2}^3 e_2 \quad for\ i = 1,2,4\ ,\\
&(iv)\nabla_{e_1}e_4 = 0,\ \nabla_{e_2}e_4 = -\gamma e_2,\ \nabla_{e_3}e_4 = -\gamma e_3,\ \nabla_{e_4}e_4 = \beta e_1,
\end{aligned} \tag{14}$$

where $\alpha := \frac{e_1(\lambda_2)}{\lambda_1-\lambda_2}$, $\beta := \frac{e_1(\lambda_1+2\lambda_2)}{\lambda_1-\lambda_4}$, $\gamma := \frac{e_4(\lambda_2)}{\lambda_2-\lambda_4}$.

Proof. Similar to the proof of Proposition 3.2, taking a suitable local basis $\{e_1, e_2, e_3, e_4\}$ for TM, one can see that the equalities (8)$-$(12) occur and λ_1 is of multiplicity one. Also, direct calculations give $[e_2, e_3](\lambda_1) = [e_3, e_4](\lambda_1) = [e_2, e_4](\lambda_1) = 0$, which yields

$$\omega_{23}^1 = \omega_{32}^1, \quad \omega_{34}^1 = \omega_{43}^1, \quad \omega_{24}^1 = \omega_{42}^1. \tag{15}$$

Now, having assumed M_1^4 to has three distinct principal curvatures, (without loss of generality) we can take $\lambda_2 = \lambda_3$, and then $\lambda_4 = 4H_1 - \lambda_1 - 2\lambda_2$. Hence, applying equalities (12) for distinct positive integers i, j and k less than 5, we get $e_2(\lambda_2) = e_3(\lambda_2) = 0$ and then,

$$\begin{aligned}
&(i)\ \omega_{11}^1 = \omega_{12}^1 = \omega_{13}^1 = \omega_{14}^1 = \omega_{31}^2 = \omega_{21}^3 = \omega_{34}^2 = \omega_{24}^3 = \omega_{42}^4 = \omega_{43}^4 = 0,\\
&(ii)\ \omega_{21}^2 = \omega_{31}^3 = \frac{e_1(\lambda_2)}{\lambda_1-\lambda_2},\ \omega_{41}^4 = \frac{-e_1(\lambda_1+2\lambda_2)}{\lambda_1-\lambda_4},\ \omega_{24}^2 = \omega_{34}^3 = \frac{-e_4(\lambda_2)}{\lambda_2-\lambda_4},\\
&(iii)\ (\lambda_1-\lambda_4)\omega_{24}^1 = (\lambda_1-\lambda_2)\omega_{42}^1,\ (\lambda_1-\lambda_4)\omega_{34}^1 = (\lambda_1-\lambda_2)\omega_{43}^1.
\end{aligned} \tag{16}$$

From (15) and (16) we get $\omega_{24}^1 = \omega_{42}^1, = \omega_{34}^1 = \omega_{43}^1 = \omega_{12}^4 = \omega_{13}^4 = 0$. Therefore, all items of the proposition obtain from the above results. □

Proposition 3.4. *If $\boldsymbol{x} : M_1^4 \to \mathcal{M}^5$ is a L_k-biharmonic Lorentzian hypersurface with diagonal shape operator, exactly three distinct principal curvatures, constant $(k-1)$th and kth mean curvatures and non-constant*

$(k+1)$th mean curvature, then there exists an orthonormal (local) tangent frame $\{e_1, e_2, e_3, e_4\}$ of principal vectors of M_1^4 with associated principal curvatures $\lambda_1, \lambda_2 = \lambda_3, \lambda_4$, satisfying $e_4(\lambda_2) = 0$ and

$$e_1(\lambda_2)e_1(\lambda_1 + 2\lambda_2) = \frac{1}{2}\lambda_2(\lambda_1 - \lambda_2)(\lambda_4 - \lambda_1)(2\lambda_1 + 4\lambda_2 + \lambda_4). \quad (17)$$

Proof. From Gauss curvature tensor $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$, by substituting X, Y and Z by different choices from e_1, e_2, e_3 and e_4, using the results of Proposition 3.3, we get the following equalities:

$$\begin{aligned}
&(i)\ e_1(\alpha) + \alpha^2 = -\lambda_1\lambda_2, \quad \beta^2 - e_1(\beta) = -\lambda_1\lambda_4;\\
&(ii)\ e_1\left(\frac{e_4(\lambda_2)}{\lambda_2 - \lambda_4}\right) + \alpha\frac{e_4(\lambda_2)}{\lambda_2 - \lambda_4} = 0;\\
&(iii)\ e_4(\alpha) - (\alpha + \beta)\frac{e_4(\lambda_2)}{\lambda_2 - \lambda_4} = 0;\\
&(iv)\ e_4\left(\frac{e_4(\lambda_2)}{\lambda_2 - \lambda_4}\right) + \alpha\beta - \left(\frac{e_4(\lambda_2)}{\lambda_2 - \lambda_4}\right)^2 = \lambda_2\lambda_4.
\end{aligned} \quad (18)$$

Now, from (2), (5), (6), (7) applying Proposition (3.3) for the case $k = 1$, we obtain

$$\begin{aligned}
&(\lambda_1 - 4H_1)e_1e_1(H_2) - (2(\lambda_2 - 4H_1)\alpha + (\lambda_1 + 2\lambda_2)\beta)e_1(H_2)\\
&= 12H_2(2H_1H_2 - H_3),
\end{aligned} \quad (19)$$

where $\alpha := \frac{e_1(\lambda_2)}{\lambda_1 - \lambda_2}$ and $\beta := \frac{e_1(\lambda_1 + 2\lambda_2)}{\lambda_1 - \lambda_4}$. Similarly, for $k = 2$ we get

$$\begin{aligned}
&-\lambda_2(\lambda_2 + 2\lambda_4)e_1e_1(H_3) + (2(\lambda_1\lambda_2 + \lambda_1\lambda_4 + \lambda_2\lambda_4)\alpha - \lambda_1(\lambda_1 + 2\lambda_2)\beta)e_1(H_3)\\
&= 4H_3(4H_1H_3 - H_4),
\end{aligned} \quad (20)$$

and for $k = 3$ we get

$$(-\lambda_2^2\lambda_4)e_1e_1(H_4) + \lambda_1\lambda_2(2\lambda_4\alpha - \lambda_2\beta)e_1(H_2) = 4H_1H_4^2. \quad (21)$$

On the other hand, from (9) and (14), we obtain

$$e_ie_1(H_{k+1}) = 0, \quad (22)$$

for $i = 2, 3, 4$. Also, by differentiating α and β along e_4, we get

$$(\lambda_1 - \lambda_2)e_4(\alpha) - \alpha e_4(\lambda_2) = e_4 e_1(\lambda_2) = \frac{1}{2}(\lambda_1 - \lambda_4)e_4(\beta) + \beta e_4(\lambda_2),$$

then

$$\frac{1}{2}(\lambda_1 - \lambda_4)e_4(\beta) = (\lambda_1 - \lambda_2)e_4(\alpha) - (\alpha + \beta)e_4(\lambda_2),$$

which, by substituting the value of $e_4(\alpha)$ from (18), gives

$$e_4(\beta) = \frac{-8e_4(\lambda_2)(\alpha + \beta)(\lambda_2 - H_1)}{(\lambda_1 - \lambda_4)(\lambda_2 - \lambda_4)}.$$

Again, differentiating (19), (20) and (21) along e_4 and using (22), (18) and the last value of $e_4(\beta)$, we get $e_4(\lambda_2) = 0$ or

$$\frac{4(\alpha + \beta)[-H_1(8\lambda_1 + 12\lambda_2) + {\lambda_1}^2 + 3\lambda_1\lambda_2 + 16H_1^2]e_1(H_2)}{\lambda_4 - \lambda_1} + 6H_2(\lambda_2 - \lambda_4)^2 = 0.$$

Finally, we claim that $e_4(\lambda_2) = 0$.

Indeed, if the claim be false for example in the case $k = 1$ (the other cases can be followed in similar manners), then we have

$$\frac{4(\alpha + \beta)\gamma e_1(H_2)}{\lambda_1 - \lambda_4} = 6H_2(\lambda_2 - \lambda_4)^2, \tag{23}$$

where $\gamma = -8H_1\lambda_1 + {\lambda_1}^2 + 3\lambda_1\lambda_2 - 12H_1\lambda_2 + 16H_1^2$. Differentiating (23) along e_4, we get

$$\frac{2(\alpha + \beta)\left[6\gamma(\lambda_2 - H_1) + (3\lambda_1 - 12H_1)(\lambda_1 + \lambda_2 - 2H_1)(\lambda_1 + 3\lambda_2 - 4H_1)\right]e_1(H_2)}{(\lambda_1 + \lambda_2 - 2H_1)^2}$$
$$= 36H_2(4H_1 + \lambda_1 + 3\lambda_2)^2. \tag{24}$$

Eliminating $e_1(H_2)$ from (23) and (24), we obtain

$$\gamma(2\lambda_1 - 2H_1) = (\lambda_1 - 4H_1)(\lambda_1 + \lambda_2 - 2H_1)(-4H_1 + \lambda_1 + 3\lambda_2). \tag{25}$$

By differentiating (25) along e_4, we get $4H_1 = \lambda_1$, which is not possible since λ_1 is not constant. Consequently, $e_4(\lambda_2) = 0$. Therefore, the latest equality in (18) gives the main result. $\square$

Theorem 3.5. *If $\boldsymbol{x} : M_1^4 \to \mathcal{M}^5$ is a L_k-biharmonic Lorentzian hypersurface with diagonal shape operator, exactly three distinct principal curvatures, constant $(k-1)$th and kth mean curvatures and non-constant $(k+1)$th mean curvature, then it is k-minimal.*

Proof. First, we assume H_{k+1} is non-constant on M and try to get a contradiction. We continue the proof separately in three cases $k = 1$, $k = 2$ and $k = 3$.

Case 1: $k = 1$. By differentiating (10) in direction of e_1 and using the definition of β, we get

$$e_1(H_2) = \frac{4}{3}(2H_1 - \lambda_1)e_1(\lambda_2) + \frac{4}{3}(\lambda_1 + \lambda_2 - 2H_1)(\lambda_1 - 2H_1)\beta. \tag{26}$$

By Proposition 3.4 and equalities (18), from (26) we obtain

$$\begin{aligned} e_1e_1(H_2) = {} & \frac{4}{3}\lambda_1\lambda_2(\lambda_1 - \lambda_2)(\lambda_1 + 2H_1) \\ & + \frac{4}{3}(4H_1 - \lambda_1 - 2\lambda_2)(\lambda_1 - 2H_1)(4\lambda_1\lambda_2 + {\lambda_1}^2 - 4H_1\lambda_2 - 2H_1\lambda_1) \\ & + \left[3\beta - 4\alpha + 2\frac{(\lambda_1 + \lambda_2 - 2H_1)\beta - (\lambda_1 - \lambda_2)\alpha}{\lambda_1 - 2H_1}\right] e_1(H_2). \end{aligned} \tag{27}$$

Combining (19) and (27), we get

$$(P_{1,2}\alpha + P_{2,2}\beta)e_1(H_2) = P_{3,6}, \tag{28}$$

where $P_{1,2}$, $P_{2,2}$ and $P_{3,6}$ are polynomials in terms of λ_1 and λ_2 of degrees 2, 2 and 6, respectively.

Differentiating (28) along e_1 and using equalities (17), (18)-(i) and (28), we get the following equality

$$P_{4,8}\alpha + P_{5,8}\beta = P_{6,5}e_1(H_2), \tag{29}$$

where $P_{4,8},\ P_{5,8}$ and $P_{6,5}$ are polynomials in terms of λ_1 and λ_2 of degrees 8, 8 and 5, respectively.

Combining (26) and (29), we obtain

$$\begin{aligned} & \left(P_{4,8} + \frac{4}{3}P_{6,5}(\lambda_1 - \lambda_2)(\lambda_1 - 2H_1)\right)\alpha \\ & + \left(P_{5,8} - \frac{4}{3}P_{6,5}(\lambda_1 + \lambda_2 - 2H_1)(\lambda_1 - 2H_1)\right)\beta = 0. \end{aligned} \tag{30}$$

On the other hand, combining (26) with (28) and using Proposition 3.4, we get

$$P_{2,2}(\lambda_1+\lambda_2-2H_1)(\lambda_1-2H_1)\beta^2-P_{1,2}(\lambda_1-\lambda_2)(\lambda_1-2H_1)\alpha^2=\zeta, \quad (31)$$

where ζ stands for

$$\lambda_2(4H_1-\lambda_1-2\lambda_2)(\lambda_1-2H_1)\left(P_{2,2}(\lambda_1-\lambda_2)-P_{1,2}(\lambda_1+\lambda_2-2H_1)\right)+\frac{3}{4}P_{3,6}.$$

Using Proposition 3.4 and equality (30), we get

$$\alpha^2=\frac{\frac{2}{3}P_{6,5}(\lambda_1-\lambda_4)(\lambda_1-2H_1)+P_{5,8}}{P_{4,8}+\frac{4}{3}P_{6,5}(\lambda_1-\lambda_2)(\lambda_1-2H_1)}\lambda_2\lambda_4,$$
$$\beta^2=\frac{\frac{4}{3}P_{6,5}(\lambda_1-\lambda_2)(\lambda_1-2H_1)-P_{4,8}}{P_{5,8}-\frac{2}{3}P_{6,5}(\lambda_1-\lambda_4)(\lambda_1-2H_1)}\lambda_2\lambda_4.$$

Eliminating α^2 and β^2 from (31), we obtain

$$\begin{aligned}&-\lambda_2\lambda_4(\lambda_1+2H_1)(\lambda_2-\lambda_1)P_{1,2}\Big(P_{5,8}-\frac{2}{3}P_{6,5}(\lambda_1-\lambda_4)(\lambda_1-2H_1)\Big)^2\\&-\frac{1}{2}\lambda_2\lambda_4(\lambda_1+2H_1)(\lambda_1-\lambda_4)P_{2,2}\Big(P_{4,8}+\frac{4}{3}P_{6,5}(\lambda_1-\lambda_2)(\lambda_1-2H_1)\Big)^2\\&=\zeta\left(P_{5,8}-\frac{2}{3}P_{6,5}(\lambda_1-\lambda_4)(\lambda_1-2H_1)\right)\left(P_{4,8}+\frac{4}{3}P_{6,5}(\lambda_1-\lambda_2)(\lambda_1-2H_1)\right),\end{aligned} \quad (32)$$

which is a polynomial equation of degree 22 in terms of λ_2 and λ_1.

Now consider an integral curve of e_1 passing through $p=\gamma(t_0)$ as $\gamma(t),\ t\in I$. Since $e_i(\lambda_1)=e_i(\lambda_2)=0$ for $i=2,3,4$ and $e_1(\lambda_1),\ e_1(\lambda_2)\neq 0$, we can assume $\lambda_2=\lambda_2(t)$ and $\lambda_1=\lambda_1(\lambda_2)$ in some neighborhood of $\lambda_0=\lambda_2(t_0)$. Using (30), we have

$$\begin{aligned}\frac{d\lambda_1}{d\lambda_2}&=\frac{d\lambda_1}{dt}\frac{dt}{d\lambda_2}=\frac{e_1(\lambda_1)}{e_1(\lambda_2)}\\&=2\frac{(\lambda_1+\lambda_2-2H_1)\beta-(\lambda_1-\lambda_2)\alpha}{(\lambda_1-\lambda_2)\alpha}\\&=\frac{2\left(P_{4,8}+\frac{4}{3}P_{6,5}(\lambda_1-\lambda_2)(\lambda_1-2H_1)\right)(\lambda_1+\lambda_2-2H_1)}{\left(\frac{4}{3}P_{6,5}(\lambda_1+\lambda_2-2H_1)(\lambda_1-2H_1)-P_{5,8}\right)(\lambda_1-\lambda_2)}-2\end{aligned} \quad (33)$$

Differentiating (32) with respect to λ_2 and substituting $\frac{d\lambda_1}{d\lambda_2}$ from (33), we get

$$f(\lambda_1, \lambda_2) = 0, \tag{34}$$

another algebraic equation of degree 30 in terms of λ_1 and λ_2.

We rewrite (32) and (34) respectively in the following forms

$$\sum_{i=0}^{22} f_i(\lambda_1)\lambda_2^i = 0, \qquad \sum_{i=0}^{30} g_i(\lambda_1)\lambda_2^i = 0, \tag{35}$$

where $f_i(\lambda_1)$ and $g_j(\lambda_1)$ are polynomial functions of λ_1. We eliminate λ_2^{30} between these two polynomials of (35) by multiplying $g_{30}\lambda_2^8$ and f_{22} respectively on the first and second equations of (35), we obtain a new polynomial equation in λ_2 of degree 29. Combining this equation with the first equation of (35), we successively obtain a polynomial equation in λ_2 of degree 28. In a similar way, by using the first equation of (35) and its consequences we are able to gradually eliminate λ_2. At last, we obtain a non-trivial algebraic polynomial equation in λ_1 with constant coefficients. Therefore, we get that the real function λ_1 is constant and then by (10), H_2 is constant, which contradicts with the first assumption. Hence, H_2 is constant on M^4.

Now, we claim that $H_2 = 0$.

Having assumed that $H_2 \neq 0$, by (5)-(i), we obtain that H_3 is constant. Therefore all the mean curvatures H_i are constant functions, this is equivalent to M^4 is an isoparametric hypersurface of E_1^5. By Corollary 2.7 in [10], an isoparametric Lorentzian hypersurface of type I has at most one nonzero principal curvature, which contradicts with the assumption that, three principal curvatures of M are assumed to be mutually distinct. So $H_2 \equiv 0$.

Case 2: $k = 2$. By differentiating (13) in direction of e_1 and using the definition of β, we get

$$e_1(H_2) = (6H_2 - 8H_1\lambda_1 + 3\lambda_1^2)[e_1(\lambda_2) - (\lambda_1 + \lambda_2 - 2H_1)\beta]. \tag{36}$$

By Proposition 3.4 and equalities (18), from (36) we obtain

$$
\begin{aligned}
e_1e_1(H_2) &= \frac{4}{3}\lambda_1\lambda_2(\lambda_1-\lambda_2)(\lambda_1+2H_1)\\
&+\frac{4}{3}(4H_1-\lambda_1-2\lambda_2)(\lambda_1-2H_1)(4\lambda_1\lambda_2+{\lambda_1}^2-4H_1\lambda_2-2H_1\lambda_1)\\
&+\left[3\beta-4\alpha+2\frac{(\lambda_1+\lambda_2-2H_1)\beta-(\lambda_1-\lambda_2)\alpha}{\lambda_1-2H_1}\right]e_1(H_2).
\end{aligned}
\tag{37}
$$

Combining (20) and (37), we get

$$
(P_{1,2}\alpha+P_{2,2}\beta)e_1(H_2)=P_{3,6},
$$

□

4 Nondiagonal shape operator

In this section, we show that some L_k-biharmonic connected orientable Lorentzian hypersurface with nondiagonal shape operator has to be k-minimal.

Theorem 4.1. *Let $x : M_1^4 \to \mathcal{M}^5$ be an L_k-biharmonic connected orientable Lorentzian hypersurface with shape operator of type $\mathcal{F}_2$. If M_1^4 has constant ordinary mean curvature and a constant real principal curvature, then it is k-minimal.*

Proof. The first stage is to show that H_{k+1} is constant. We suppose that H_{k+1} is nonconstant. Using the open subset $\mathcal{U} = \{p \in M : \nabla H_{k+1}^2(p) \neq 0\}$ we get a contradiction. With respect to a suitable (local) orthonormal tangent frame $\{e_1, \cdots, e_4\}$ on M_1^4, we have $Se_1 = \kappa e_1 - \lambda e_2$, $Se_2 = \lambda e_1 + \kappa e_2$, $Se_3 = \eta_1 e_3$, $Se_4 = \eta_2 e_4$ and then, we have $P_2 e_1 = [\kappa(\eta_1+\eta_2)+\eta_1\eta_2]e_1 + \lambda(\eta_1+\eta_2)e_2$, $P_2e_2 = -\lambda(\eta_1+\eta_2)e_1 + [\kappa(\eta_1+\eta_2)+\eta_1\eta_2]e_2$, $P_2e_3 = (\kappa^2+\lambda^2+2\kappa\eta_2)e_3$ and $P_2e_4 = (\kappa^2+\lambda^2+2\kappa\eta_1)e_4$.

Using the polar decomposition $\nabla H_{k+1} = \sum_{i=1}^{4}\epsilon_i e_i(H_{k+1})e_i$, from condition (4)(ii) we get

$$
\begin{aligned}
&\text{(i) } (\kappa\mu_{1,2;k}+\mu_{1,2;k+1}-3(4-k)H_{k+1})\epsilon_1 e_1(H_2)-\lambda\mu_{1,2;k}\epsilon_2 e_2(H_{k+1})=0\\
&\text{(ii) } \lambda\mu_{1,2;k}\epsilon_1 e_1(H_{k+1})+(\kappa\mu_{1,2;k}+\mu_{1,2;k+1}-3(4-k)H_{k+1})\epsilon_2 e_2(H_{k+1})=0\\
&\text{(iii) } (\mu_{3;k+1}-3(4-k)H_{k+1})\epsilon_3 e_3(H_{k+1})=0,\\
&\text{(iv) } (\mu_{4;k+1}-3(4-k)H_{k+1})\epsilon_4 e_4(H_{k+1})=0.
\end{aligned}
\tag{38}
$$

It is enough to prove four simple claims as

$$e_1(H_{k+1})=e_2(H_{k+1})=e_3(H_{k+1})=e_4(H_{k+1})=0.$$

Claim 1: $e_1(H_{k+1})=0$. If $e_1(H_{k+1})\neq 0$, then dividing both sides of equalities (38)(i, ii) by $\epsilon_1 e_1(H_{k+1})$ and taking $u:=\frac{\epsilon_2 e_2(H_{k+1})}{\epsilon_1 e_1(H_{k+1})}$ we get

$$
\begin{aligned}
&(i)\quad \kappa\mu_{1,2;k}+\mu_{1,2;k+1}-3(4-k)H_{k+1}=\lambda\mu_{1,2;k}u,\\
&(ii)\quad (\kappa\mu_{1,2;k}+\mu_{1,2;k+1}-3(4-k)H_{k+1})u=-\lambda\mu_{1,2;k},
\end{aligned}
\tag{39}
$$

which, by substituting (i) in (ii), gives $\lambda\mu_{1,2;k}(1+u^2)=0$, then $\lambda\mu_{1,2;k}=0$. Since $\lambda\neq 0$ (by definition), we get $\mu_{1,2;k}=0$. So, by (39)(i), we obtain

$$\mu_{1,2;k+1}=3(4-k)H_{k+1}. \tag{40}$$

In the case $k=1$, from $\mu_{1,2;1}=0$ we have $\eta_1+\eta_2=0$. Since η_1 is assumed to be constant, from (40) we get that $9H_2=-\eta_1^2=-\eta_1^2$ is constant which contradicts with assumption $e_1(H_2)\neq 0$. So, we have $e_1(H_2)=0$.

In the case $k=2$, from $\mu_{1,2;2}=0$ we have $\eta_1\eta_2=0$ and $\mu_{1,2;3}=0$. So, by (40) we get $H_3=0$ which contradicts with assumption $e_1(H_3)\neq 0$. Hence, we have $e_1(H_3)=0$.

In the case $k=3$, condition (7)(ii) gives that H_4 is constant and then $e_1(H_3)=0$.

Therefore, Claim 1 is affirmed.

Claim 2: $e_2(H_{k+1})=0$. if $e_2(H_{k+1})\neq 0$, then dividing both sides of (38)(i,ii) by $\epsilon_2 e_2(H_{k+1})$ and taking $v:=\frac{\epsilon_1 e_1(H_{k+1})}{\epsilon_2 e_2(H_{k+1})}$, we get $\lambda\mu_{1,2;k}(1+v^2)=0$, which gives $\lambda\mu_{1,2;k}=0$. in a similar way, one can get the same results. Hence, the second claim (i.e. $e_2(H_{k+1}) = 0$) is satisfied.

Claim 3: $e_3(H_{k+1}) = 0$. In order to prove the third one, we assume $e_3(H_{k+1}) \neq 0$ and get a contradiction. From equality (38)(iii) we have

$$\mu_{3;k+1} = 3(4-k)H_{k+1}. \tag{41}$$

In the case $k = 1$, from (41) we get $-3\kappa^2 + (2\kappa + 3\eta_1)(4H_1 - \eta_1) = -\lambda^2 < 0$, then, $-2[2\kappa^2 + (\eta_1 - 4H_1)\kappa + 2\eta_1(\eta_1 - 3H_1)] = -(\lambda^2 + \kappa^2 + \eta_1^2) < 0$.

Remember that the last inequality occurs if and only if we have $\delta < 0$ where

$$\delta = (\eta_1 - 4H_1)^2 - 16\eta_1(\eta_1 - 3H_1) = -15\eta_1^2 + 40\eta_1 H_1 + 16H_1^2.$$

The condition $\delta < 0$ is equivalent to a new inequality $\bar{\delta} < 0$ where

$$\bar{\delta} = (40H_1)^2 + (4 \times 15 \times 16)H_1^2 = 2560H_1^2,$$

which is a contradiction.

In the case $k = 2$, we have $\mu_{3;3} = 6H_3$.

In the case $k = 3$, condition (7)(ii) gives that H_4 is constant and then $e_3(H_4) = 0$.

So, the third claim is proved.

Claim 4: $e_4(H_{k+1}) = 0$. If $e_4(H_{k+1}) \neq 0$, then from equality (38)(iv) we have $\mu_{4;k+1} = 3(4-k)H_{k+1}$.

In case $k = 1$, we get

$$-11\kappa^2 + (24H_1 - 10\eta_1)\kappa + 12\eta_1 H_1 - 3\eta_1^2 = -\lambda^2 < 0,$$

then,

$$-2[6\kappa^2 + (5\eta_1 - 12H_1)\kappa + 2\eta_1(\eta_1 - 3H_1)] = -(\lambda^2 + \kappa^2 + \eta_1^2) < 0.$$

Remember that the last inequality occurs if and only if we have $\delta < 0$ where

$$\delta = (5\eta_1 - 12H_1)^2 - 48\eta_1(\eta_1 - 3H_1) = -23\eta_1^2 + 24\eta_1 H_1 + 144H_1^2.$$

The condition $\delta < 0$ is equivalent to a new inequality $\bar{\delta} < 0$ where

$$\bar{\delta} = (24H_1)^2 + (4 \times 23 \times 144)H_1^2 = 13824H_1^2,$$

which is a contradiction. So, $e_4(H_2) = 0$.

In the case $k = 2$, we have $\mu_{4;3} = 6H_3$.

In the second stage, we prove that $H_{k+1} = 0$. since H_{k+1} is constant, we have $L_k H_{k+1} = 0$. Then, by (4)(i), we have $H_{k+1}(4H_1H_{k+1} - (4-k-1)H_{k+2}) = 0$. Assuming $H_{k+1} \neq 0$ we get $4H_1H_{k+1} = (4-k-1)H_{k+2}$, which implies that H_{k+2} is constant. Therefore, M_1^4 is a Lorentzian isoparametric hypersurface of E_1^5 which, by Corollary 2.9 in [10], has at most one non-zero real principal curvature. Hence, we have $\eta_1\eta_2 = 0$ which gives $H_4 = (\kappa^2 + \lambda^2)\eta_1\eta_2 = 0$. Therefore, M_1^4 is 3-minimal. □

Proposition 4.2. *Let k be a positive integer number less than 4, $x : M_1^4 \to E_1^5$ be an L_k-biharmonic connected orientable lorentzian hypersurface with shape operator of type $\tilde{\mathcal{F}}_3$ in E_1^5. If M_1^4 has three distinct principal curvatures and constant kth mean curvature, then its $(k+1)$th mean curvature has to be constant.*

Proof. Suppose that, H_{k+1} be non-constant. Considering the open subset $\mathcal{U} = \{p \in M : \nabla H_{k+1}^2(p) \neq 0\}$, we try to show $\mathcal{U} = \emptyset$. By the assumption, with respect to a suitable (local) orthonormal tangent frame $\{e_1, \cdots, e_4\}$ on M, the shape operator A has the matrix form $\tilde{B}_2$, such that $Ae_1 = (\kappa + \frac{1}{2})e_1 - \frac{1}{2}e_2$, $Ae_2 = \frac{1}{2}e_1 + (\kappa - \frac{1}{2})e_2$, $Ae_3 = \lambda_1 e_3$ and $Ae_4 = \lambda_2 e_4$, and then, for $j = 1, 2, 3$ we have $P_j e_1 = [\mu_{1,2;j} + (\kappa - \frac{1}{2})\mu_{1,2;j-1}]e_1 + \frac{1}{2}\mu_{1,2;j-1}e_2$, $P_2 e_2 = -\frac{1}{2}\mu_{1,2;j-1}e_1 + [\mu_{1,2;j} + (\kappa - \frac{1}{2})\mu_{1,2;j-1}]e_2$, and $P_2 e_3 = \mu_{3;j}e_3$ and $P_2 e_4 = \mu_{4;j}e_4$.

We continue the proof separately in three cases $k = 1$, $k = 2$ and $k = 3$.

Case 1: $k = 1$. Using the polar decomposition $\nabla H_2 = \sum_{i=1}^{4} \epsilon_i e_i(H_2)e_i$, from conditions (5)(ii), we get

$$
\begin{aligned}
&(i)\ [\lambda_1\lambda_2 + (\kappa - \frac{1}{2})(\lambda_1 + \lambda_2) - 9H_2]\epsilon_1 e_1(H_2) = \frac{1}{2}(\lambda_1 + \lambda_2)\epsilon_2 e_2(H_2),\\
&(ii)\ [\lambda_1\lambda_2 + (\kappa + \frac{1}{2})(\lambda_1 + \lambda_2) - 9H_2]\epsilon_2 e_2(H_2) = -\frac{1}{2}(\lambda_1 + \lambda_2)\epsilon_1 e_1(H_2),\\
&(iii)\ (\kappa^2 + 2\kappa\lambda_2 - 9H_2)\epsilon_3 e_3(H_2) = 0,\\
&(iv)\ (\kappa^2 + 2\kappa\lambda_1 - 9H_2)\epsilon_3 e_4(H_2) = 0.
\end{aligned}
\tag{42}
$$

Now, we prove the following claim.

Claim: $e_i(H_2) = 0$ for $i = 1, 2, 3, 4$.
If $e_1(H_2) \neq 0$, then by dividing both sides of equalities (42)(i, ii) by $\epsilon_1 e_1(H_2)$ we get

$$\begin{aligned} &(i)\ \lambda_1\lambda_2 + (\kappa - \frac{1}{2})(\lambda_1 + \lambda_2) - 9H_2 = \frac{1}{2}(\lambda_1 + \lambda_2)u, \\ &(ii)\ [\lambda_1\lambda_2 + (\kappa + \frac{1}{2})(\lambda_1 + \lambda_2) - 9H_2]u = -\frac{1}{2}(\lambda_1 + \lambda_2), \end{aligned} \tag{43}$$

where $u := \frac{\epsilon_2 e_2(H_2)}{\epsilon_1 e_1(H_2)}$. By substituting (i) in (ii), we obtain $(\lambda_1 + \lambda_2)(1 + u)^2 = 0$, then $\lambda_1 + \lambda_2 = 0$ or $u = -1$.
If $\lambda_1 + \lambda_2 = 0$, then, from (43)$(i)$ we obtain $9H_2 = -\lambda_1^2$, which gives $3\kappa^2 = -\lambda_1^2$. Since H_1 is assumed to be constant on M, then $\kappa = 2H_1$ is constant on M. Hence, λ_1 and λ_2 are also constant on M. Therefore, M_1^4 is an isoparametric Lorentzian hypersurface of real principal curvatures in E_1^5, which by Corollary 2.7 in [10], cannot has more than one nonzero principal curvature contradicting with the assumptions. So, $\lambda_1 + \lambda_2 \neq 0$ and then $u = -1$.

From $u = -1$, we get $\lambda_1\lambda_2 + \kappa(\lambda_1 + \lambda_2) = 9H_2$, then

$$3\kappa^2 + 4\kappa(\lambda_1 + \lambda_2) + \lambda_1\lambda_2 = 0.$$

Since $4H_1 = 2\kappa + \lambda_1 + \lambda_2$ is assumed to be constant on M, by substituting which in the last equality, we get $\lambda^2 - H_1\lambda - 3H_1^2 = 0$, which means λ, κ and the kth mean curvatures (for $k = 2, 3, 4$) are constant on M. So, we got a contradiction and therefore, the first part of the claim is proved.

By a similar manner, each of assumptions $e_i(H_2) \neq 0$ for $i = 2, 3, 4$, gives the equality $\lambda^2 + 2\kappa\lambda = 9H_2$, which implies the contradiction that H_2 is constant on M. So, the claim is affirmed. □

Theorem 4.3. *Let $x : M_1^3 \to E_1^4$ be a L_1-biharmonic timlike hypersurface with shape operator of type $\tilde{\mathcal{F}}_3$ in E_1^4. If M_1^3 has at most two distinct principal curvature and constant ordinary mean curvature, then it is* 1*-minimal.*

Proof. By assumption H_1 is assumed to be constant and then, by Proposition 4.2 it is proved that H_2 has to be constant. By (4)(i) we obtain that H_3 is constant. Therefore, M_1^4 is isoparametric. On the other hand, by Corollary 2.7 in [10], an isoparametric Lorentzian hypersurface of Case II in the E_1^5 has at most one nonzero principal curvature, so we get $\lambda = 0$ (for example). Then $H_1 = \frac{1}{2}\kappa$, $H_2 = \frac{1}{6}\kappa^2$ and $H_3 = 0$, hence, by (4)(i), we get $\kappa = 0$. Therefore $H_2 = 0$. $\square$

Proposition 4.4. *Let $x : M_1^4 \to E_1^5$ be an L_1-biharmonic connected orientable lorentzian hypersurface with shape operator of type $\tilde{\mathcal{F}}_3$ in E_1^5. Assume that M_1^4 has one constant principal curvature and constant ordinary mean curvature. Then its 2th mean curvature has to be constant. Furthermore, all of principal curvatures of M_1^4 are constant and M_1^4 is isoparametric.*

Proof. Suppose that, H_2 be non-constant. Considering the open subset $\mathcal{U} = \{p \in M : \nabla H_2^2(p) \neq 0\}$, we try to show $\mathcal{U} = \emptyset$. By the assumption, with respect to a suitable (local) orthonormal tangent frame $\{e_1, \cdots, e_4\}$ on M, the shape operator A has the matrix form $\tilde{B}_2$, such that $Ae_1 = (\kappa + \frac{1}{2})e_1 - \frac{1}{2}e_2$, $Ae_2 = \frac{1}{2}e_1 + (\kappa - \frac{1}{2})e_2$, $Ae_3 = \lambda_1 e_3$ and $Ae_4 = \lambda_2 e_4$, and then, we have $P_2 e_1 = [\lambda_1\lambda_2 + (\kappa - \frac{1}{2})(\lambda_1 + \lambda_2)]e_1 + \frac{1}{2}(\lambda_1 + \lambda_2)e_2$, $P_2 e_2 = -\frac{1}{2}(\lambda_1 + \lambda_2)e_1 + [\lambda_1\lambda_2 + (\kappa + \frac{1}{2})(\lambda_1 + \lambda_2)]e_2$, and $P_2 e_3 = (\kappa^2 + 2\kappa\lambda_2)e_3$ and $P_2 e_4 = (\kappa^2 + 2\kappa\lambda_1)e_4$.

Using the polar decomposition $\nabla H_2 = \sum_{i=1}^{4} \epsilon_i e_i(H_2) e_i$, from condition (4)(ii) we get

$$
\begin{aligned}
&(i) \quad [\lambda_1\lambda_2 + (\kappa - \frac{1}{2})(\lambda_1 + \lambda_2) - 9H_2]\epsilon_1 e_1(H_2) = \frac{1}{2}(\lambda_1 + \lambda_2)\epsilon_2 e_2(H_2),\\
&(ii) \quad [\lambda_1\lambda_2 + (\kappa + \frac{1}{2})(\lambda_1 + \lambda_2) - 9H_2]\epsilon_2 e_2(H_2) = \frac{1}{2}(\lambda_1 + \lambda_2)\epsilon_1 e_1(H_2),\\
&(iii) \quad (\kappa^2 + 2\kappa\lambda_2 - 9H_2)\epsilon_3 e_3(H_2) = 0,\\
&(iv) \quad (\kappa^2 + 2\kappa\lambda_1 - 9H_2)\epsilon_3 e_4(H_2) = 0.
\end{aligned}
\tag{44}
$$

Now, we prove some simple claims.

Claim: $e_1(H_2) = e_2(H_2) = e_3(H_2) = e_4(H_2) = 0$.

If $e_1(H_2) \neq 0$, then by dividing both sides of equalities (44)(i, ii) by

$\epsilon_1 e_1(H_2)$ we get

$$\begin{aligned} &(i) \quad \lambda_1\lambda_2 + (\kappa - \frac{1}{2})(\lambda_1+\lambda_2) - 9H_2 = \frac{1}{2}(\lambda_1+\lambda_2)u, \\ &(ii) \quad [\lambda_1\lambda_2 + (\kappa + \frac{1}{2})(\lambda_1+\lambda_2) - 9H_2]u = -\frac{1}{2}(\lambda_1+\lambda_2), \end{aligned} \tag{45}$$

where $u := \frac{\epsilon_2 e_2(H_2)}{\epsilon_1 e_1(H_2)}$. By substituting (i) in (ii), we obtain $\frac{1}{2}(\lambda_1+\lambda_2)(1+u)^2 = 0$, Then $\lambda_1+\lambda_2 = 0$ or $u = -1$. If $\lambda_1+\lambda_2 = 0$, then, by assumption we get that $\kappa = 2H_1$ is constant, and also, from (43)(i) we obtain $H_2 = \frac{-1}{9}\lambda_1^2$ which gives $\frac{1}{6}(\kappa^2-\lambda_1^2) = \frac{-1}{9}\lambda_1^2$ and then $\lambda_1^2 = 3\kappa^2$. Hence, we get $H_2 = \frac{-1}{3}\kappa^2$, which means H_2 is constant.

Also, by assumption $\lambda_1+\lambda_2 \neq 0$ we get $u=-1$, which, using (45)(i) and $4H_1 = 2\kappa+\lambda_1+\lambda_2$, gives $5\kappa^2 - 16\kappa H_1 - \lambda_1(4H_1 - 2\kappa - \lambda_1) = 0$. Without loss of generality, we assume that λ_1 is constant on M. So, from the last equation we get that κ, λ_2 and H_2 are constant on $\mathcal{U}$, which is a contradiction. Therefore, the first claim is proved. The second claim (i.e. $e_2(H_2) = 0$) can be proven by a similar manner.

Now, if $e_3(H_2) \neq 0$, then using (44(iii)) and $4H_1 = 2\kappa+\lambda_1+\lambda_2$ and by assuming λ_1 to be constant on M, we get

$$\kappa^2 - (\frac{16}{3}H_1 - \frac{2}{3}\lambda_1)\kappa - 4\lambda_1 H_1 + \lambda_1^2 = 0,$$

which gives that κ, λ_2 and H_2 are constant on $\mathcal{U}$, which is a contradiction. Therefore, the third claim is proved.

The forth claim (i.e. $e_4(H_2) = 0$) can be proven by a manner exactly similar to third one. $\square$

Theorem 4.5. *Let $x : M_1^4 \to E_1^5$ be a L_1-biharmonic timlike hypersurface with shape operator of type $\tilde{\mathcal{F}}_3$ in E_1^5. Assume that M_1^4 has one constant principal curvature and constant ordinary mean curvature. Then, it is* 1*-minimal.*

Proof. By Proposition 4.4, all of principal curvatures of M_1^3 are constant and M_1^3 is isoparametric. We claim that H_2 is null. Since, by Corollary 2.7 in [10], an isoparametric Lorentzian hypersurface of real principal curvatures in E_1^5 has at most one nonzero principal curvature, we get $H_2 = 0$. $\square$

Proposition 4.6. *Let $x : M_1^4 \to E_1^5$ be an L_1-biharmonic connected orientable Lorentzian hypersurface with shape operator of type $\tilde{\mathcal{F}}_4$ in E_1^5. If M_1^4 has constant ordinary mean curvature, then its 2th mean curvature is constant.*

Proof. Suppose that, H_2 be non-constant. Considering the open subset $\mathcal{U} = \{p \in M : \nabla H_2^2(p) \neq 0\}$, we try to show $\mathcal{U} = \emptyset$. By the assumption, with respect to a suitable (local) orthonormal tangent frame $\{e_1, \cdots, e_4\}$ on M, the shape operator A has the matrix form $\tilde{B}_3$, such that $Ae_1 = \kappa e_1 - \frac{\sqrt{2}}{2}e_3$, $Ae_2 = \kappa e_2 - \frac{\sqrt{2}}{2}e_3$, $Ae_3 = \frac{\sqrt{2}}{2}e_1 - \frac{\sqrt{2}}{2}e_2 + \kappa e_3$ and $Ae_4 = \lambda e_4$ and then, we have $P_2e_1 = (\kappa^2 + 2\kappa\lambda - \frac{1}{2})e_1 + \frac{1}{2}e_2 + \frac{\sqrt{2}}{2}(\kappa+\lambda)e_3$, $P_2e_2 = \frac{-1}{2}e_1 + (\kappa^2 + 2\kappa\lambda + \frac{1}{2})e_2 + \frac{\sqrt{2}}{2}(\kappa+\lambda)e_3$, $P_2e_3 = \frac{-\sqrt{2}}{2}(\kappa+\lambda)e_1 + \frac{\sqrt{2}}{2}(\kappa+\lambda)e_2 + (\kappa^2+2\kappa\lambda)e_3$ and $P_2e_4 = 3\kappa^2 e_4$.

Using the polar decomposition $\nabla H_2 = \sum_{i=1}^{4} \epsilon_i e_i(H_2)e_i$, from condition (4)(ii) we get

$$\begin{aligned}
&(i) \quad (\kappa^2 + 2\kappa\lambda - \frac{1}{2} - 9H_2)\epsilon_1 e_1(H_2) - \frac{1}{2}\epsilon_2 e_2(H_2) - \frac{\sqrt{2}}{2}(\kappa+\lambda)\epsilon_3 e_3(H_2) = 0\\
&(ii) \quad \frac{1}{2}\epsilon_1 e_1(H_2) + (\kappa^2 + 2\kappa\lambda + \frac{1}{2} - 9H_2)\epsilon_2 e_2(H_2) + \frac{\sqrt{2}}{2}(\kappa+\lambda)\epsilon_3 e_3(H_2) = 0\\
&(iii) \quad \frac{\sqrt{2}}{2}(\kappa+\lambda)(\epsilon_1 e_1(H_2) + \epsilon_2 e_2(H_2)) + (\kappa^2 + 2\kappa\lambda - 9H_2)\epsilon_3 e_3(H_2) = 0,\\
&(iv) \quad (3\kappa^2 - 9H_2)\epsilon_4 e_4(H_2) = 0.
\end{aligned} \tag{46}$$

Now, we prove some simple claims.

Claim: $e_1(H_2) = e_2(H_2) = e_3(H_2) = e_4(H_2) = 0$.

If $e_1(H_2) \neq 0$, then by dividing both sides of equalities (46)(i, ii, iii) by $\epsilon_1 e_1(H_2)$, and using the identity $2H_2 = \kappa^2 + \kappa\lambda$ in Case III, putting $u_1 := \frac{\epsilon_2 e_2(H_2)}{\epsilon_1 e_1(H_2)}$ and $u_2 := \frac{\epsilon_3 e_3(H_2)}{\epsilon_1 e_1(H_2)}$, we get

$$\begin{aligned}
&(i) \quad -\frac{1}{2} - \frac{7}{2}\kappa^2 - \frac{5}{2}\kappa\lambda - \frac{1}{2}u_1 - \frac{\sqrt{2}}{2}(\kappa+\lambda)u_2 = 0,\\
&(ii) \ \frac{1}{2} + (\frac{1}{2} - \frac{7}{2}\kappa^2 - \frac{5}{2}\kappa\lambda)u_1 + \frac{\sqrt{2}}{2}(\kappa+\lambda)u_2 = 0,\\
&(iii) \ \frac{-\sqrt{2}}{2}(\kappa+\lambda)(1+u_1) - (\frac{7}{2}\kappa^2 + \frac{5}{2}\kappa\lambda)u_2 = 0,
\end{aligned} \tag{47}$$

which, by comparing (i) and (ii), gives $\frac{-1}{2}\kappa(7\kappa+5\lambda)(1+u_1)=0$. If $\kappa=0$, then $H_2=0$. Assuming $\kappa\neq 0$, we get $u_1=-1$ or $\lambda=-\frac{7}{5}\kappa$. If $u_1\neq -1$ then $\lambda=-\frac{7}{5}\kappa$, then by (47)$(iii)$ we obtain $u_1=-1$, which is a contradiction. Hence we have $u_1=-1$, which by (47)(i,iii) implies $u_2=0$.

Now we discuss on two cases $\lambda=-\frac{7}{5}\kappa$ or $\lambda\neq -\frac{7}{5}\kappa$. If $\lambda=-\frac{7}{5}\kappa$, then, $\kappa=\frac{5}{2}H_1$, $H_2=\frac{-1}{5}\kappa^2$, $H_3=\frac{-4}{5}\kappa^3$ and $H_4=\frac{-7}{5}\kappa^4$ are all constants on $\mathcal{U}$. Also, the case $\lambda\neq -\frac{7}{5}\kappa$ is in contradiction with (47)(ii).

Hence, the first claim $e_1(H_2)\equiv 0$ is affirmed. Similarly, the second claim (i.e. $e_2(H_2)=0$) can be proved.

Now, applying the results $e_1(H_2)=e_2(H_2)=0$, from (47)(ii,iii) we get $e_3(H_2)=0$.

The final claim (i.e. $e_2(H_2)=0$), can be proved using (47)(iv), in a straightforward manner. $\square$

Theorem 4.7. *Let $x: M_1^4\to E_1^5$ be an L_1-biharmonic connected orientable Lorentzian hypersurface with shape operator of type $\tilde{\mathcal{F}}_4$ in E_1^5. If M_1^4 has constant ordinary mean curvature, then, it is 1-minimal.*

Proof. By Proposition 4.6, the 2th mean curvature of M_1^3 is constant, which, by (4(i)), gives $L_1H_2=9H_1H_2^2-3H_2H_3=0$. If $H_2=0$, it remains nothing to prove. By assumption $H_2\neq 0$, we get $3H_1H_2=H_3$, which gives $\kappa(\kappa^2-3H_1\kappa+3H_1^2)=0$, where $\kappa^2-3H_1\kappa+3H_1^2>0$, Hence, $\kappa=0$. Therefore, $H_2=H_3=H_4=0$. $\square$

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