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# A Remark on Orbit Spaces of Lorentzian Spaces

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**Abstract.** We study the orbit space of Lorentzian spaces under the action of a connected and closed subgroup of the linear isometries. If dimension of the orbit space is bigger than two then topological characterization of the orbit space reduces to a problem in Riemannian geometry.

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## 1 Introduction

Let $X$ be a topological space and $G$ be a group acting continuously on $X$. If $x \in X$ then the $G$-orbit passing from $x$ is $G(x) = \{gx : g \in G\}$. We will denote by $\frac{X}{G}$ the set of all orbits, that is

$$\frac{X}{G} = \{G(x) : x \in X\}.$$

$\frac{X}{G}$ is a topological space with the quotient topology (a topology with the property that the map $\kappa : X \to \frac{X}{G}$ defined by $\kappa(x) = G(x)$ is open

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and continuous). Any $G$-invariant problem on $X$ can be reduced to a problem on $\frac{X}{G}$, which usually is an easier problem. This is an important motivation to study orbit spaces as one of the main problems in topology and geometry. Topology of the orbit spaces are very complicated in general. So, it is necessary to put some conditions on $X$ and $G$ to characterize the orbit space. One of the main category of the orbit spaces is the orbit space of a differentiable manifold $X$ under the action of a subgroup $G$ of diffeomorphisms and a more interesting situation is the case where $X$ is a Riemannian manifold and $G$ is a subgroup of the isometries. An orbit space in this case has the structure of a stratified manifold with the smooth strata. In fact, the orbit space is a metric space and one can relate the geodesics normal to the orbits in $X$ to length minimizing curves in the orbit space (see [3]).
When dimension of the orbit space (which is also called cohomogeneity) is small, then one can find many interesting results about the structure of the orbit space. For example, when $G$ is a closed and connected subgroup of the isometries of a Riemannian manifold $X$ and $dim\frac{X}{G} = 1$, then $\frac{X}{G}$ is homeomorphic to one of the spaces $R, [0,1], [0,\infty)$ and $S^1$ (see [13], [11]). For results about the orbit space of cohomogeneity two Riemannian manifolds, see [6], [8], [12].
The situation for Lorentzian manifolds is more complicated in compare with the Riemannian manifolds. Because, usually the orbit space is not Hausdorff in this case. There is no general topological classification results, even in the small cohomogeneities. There are some partial results under extra conditions on the manifold. For example, when the manifold is globally hyperbolic and $G$ is compact with cohomogeneity one action, then the orbit space is homeomorphic to $R$ (see [15]). If there is no null geodesic orbit then the orbit space is characterized in [11] for flat Lorentzian manifolds of cohomogeneity smaller than 4. Also, we refer to [1], [2] and [9] for more relevant results.
In the present article, we consider the orbit space of a Lorentzian space (an special case of Lorentzian manifold) under the action of a connected subgroup $G$ of the linear isometries. First, we consider the case where $G$ is compact. We show that the orbits, under the induced scalar product, are Riemannian and we use it to characterize the orbit space. Then, we consider the more general case where $G$ can be noncompact. Sur-

prisingly, when dimension of the orbit space is bigger than two, characterization of the orbit space seems to be more normal than the case of dimension one or two. We show that when dimension of the orbit space is bigger than two ($G$ compact or non-compact), then the problem reduces to characterization of the orbit space of isometric actions on the usual spheres (a Riemannian problem). Also, we give a topological explanation of the orbit space of spacelike and timelike part of the Lorentzian space, when dimension of the orbit space is one or two.

## 2 Preliminaries

Let $L^{n+1}(=R_1^{n+1})$ be the usual Lorentzian space with the scalar product $< v, w >= -v_1w_1 + v_2w_2 + ... + v_{n+1}w_{n+1}$. Let $SO_0(n,1)$ be the connected component of the identity in the Lie group of all linear isometries of $L^{n+1}$ and $G$ be a Lie subgroup of $SO_0(n,1)$. The action of $G$ is said irreducible if $G$ does not leave invariant any proper subspace of $L^{n+1}$ and weakly irreducible if any $G$-invariant subspace has a degenerate induced metric.

We will use the following symbols:
$C^+ = \{v = (v_1, ..., v_{n+1}) \in R_1^{n+1} : \ < v, v >= 0, v_1 > 0\}$, the upper cone.
$C^- = -C^+$ the lower cone.
$S_1^n(r) = \{x \in R_1^{n+1} :< x, x >= r^2, r > 0\}$, the pseudo-sphere of radius $r$.
$H^n(r) = \{x \in R_1^{n+1} :< x, x >= -r^2, x_1 > 0\}$, the hyperbolic space.
$H_-^n(r) = \{x \in R_1^{n+1} :< x, x >= -r^2, x_1 < 0\}$.
$S =$ union of all spacelike vectors.
$T^+(T^-)$ union of all future (past) directed timelike vectors. $T = T^+ \cup T^-$.
We will write $X = Y$ if $X$ and $Y$ are homeomorphic topological spaces.

We recall that the infinity $H^n(\infty)$ of the hyperbolic space $H^n(r)$, $r > 0$, is defined as the classes of equivalence of asymptotic geodesics (see [10]). We refer to [10] for definition of the horosphere centered at a point $z \in H^n(\infty)$. Also note that in Poincare ball model of $H^n(r)$, $H^n(\infty)$

can be interpreted as the boundary sphere $S^{n-1}(r)$. In the Lorentz model of $H^n(r)$, each null line corresponds to a point of $H^n(\infty)$.

In the context of the classification list of M. Berger for Lorentzian holonomy and irreducible Lorentzian locally symmetric spaces ([4], [5]), it is proved by A. Scala and C. Olmos that there are no connected irreducible proper subgroups of $SO_0(n, 1)$.

**Theorem 2.1** ([14]). *Let $G$ be a connected (non nec. closed) Lie subgroup of $SO_0(n, 1)$ and assume that the action of $G$ on the Lorentzian space $L^{n+1}$ is irreducible. Then $G = SO_0(n, 1)$.*

They also proved the following theorem.

**Theorem 2.2** ( see[14]). *Let $G$ be a connected Lie subgroup of $SO_0(n, 1)$ and assume that the action of $G$ on the Lorentzian space $L^{n+1}$ is weakly irreducible. Then, either $G$ acts transitively on $H^n(r)$, $r > 0$, or there is a point $z$ at infinity such that $G$ acts transitively on each horosphere of $H^n(r)$ centered at $z$.*

We should also recall the following theorem.

**Theorem 2.3** ([11]). *If $G$ is a closed and connected subgroup of $SO(n, 1)$ without null eigenvector, then either $G = SO_0(n, 1)$ or there is a non-negative integer $m < n$ and a closed and connected subgroup $H$ of the isometries of $\mathbb{R}^{n-m}$ such that $G = SO_0(m, 1) \times H$.*

The above theorems will play an important role in our proofs.

## 3 Results

According to the following corollary, when $G$ is compact, study of the orbits essentially reduces to a problem in Riemannian geometry. In fact, the corollary is proved in a more general case, for globally hyperbolic Lorentzian manifolds (see [15]). For facility, we give a different proof for $L^{n+1}$ which seems to be useful.

**Proposition 3.1.** *If $G$ is a compact and connected subgroup of $Iso(L^{n+1})$, then each $G$-orbit is a Riemannian submanifold of $L^{n+1}$ (i.e, the induced scaler product on tangent spaces of each orbit is positive definite).*

**Proof.** Since $G(H^n(r)) = H^n(r)$, $r > 0$, then $G$ can be considered as a compact subgroup of the isometries of $H^n(r)$. Consider $H^n = H^n(1)$. $G$ has a fixed point $p$ in $H^n$( it is well known that a compact subgroup of the isometries of a simply connected Riemannian manifold of non-positive curvature has a fixed point). We can translate the origin to $p$. Then, $G$ can be considered as a subgoup of $SO_0(n, 1)$. Thus in the remaining part of the proof, we suppose without lose of generality, that $G$ is a subgroup of $SO_0(n, 1)$. If the action of $G$ on $L^{n+1}$ is irreducible then by Theorem 2.1, $G = SO_0(n, 1)$ which is in contrast to the compactness of $G$ . Suppose that the action of $G$ is not irreducible. Since $G$ fixes a point $v$ in $H^n$ and $G \subset SO_0(n, 1)$, then all points of the timelike line $L_V = \{tv : t \in R\}$ are fixed by $G$. Now, consider two cases bellow:
Case 1: $G$ fixes invariant a null direction.
Case 2: $G$ fixes invariant no null direction.

Case 1: Let $w$ be a null vector such that for all $g \in G$, $g(w) = a_g w$, $a_g \in R$. If for some $g \in G$, $|a_g| > 1$ then $\{w, a_g w, a_g^2 w, ...\}$ is a non-compact and closed subset of $G(w)$. Since $G$ is compact, $G(w)$ must be compact and we have a contradiction. If $|a_g| < 1$, put $b = \frac{1}{a_g}$ and $w_1 = a_g w$. Then $g^{-1}(w_1) = bw_1$, $|b| > 1$. So, we can find a noncompact closed subset in $G(w_1)$ which is contradiction.
Thus, for all $g \in G$, $a_g = 1$ and $L_w$ is fixed invariant by $G$ point wisely. Now, put $S = \{r_1 w + r_2 v : r_1, r_2 \in R\}$. S is a two dimensional Minkowsky space. Then, after a suitable change of coordinates, we can assume that $S = R_1^2$ and $L^{n+1} = S \times R^{n-1} = R_1^2 \times R^{n-1}$. Since $L_w$ and $L_v$ are fixed by $G$ point wisely, then $S(= R_1^2)$ is fixed by $G$ point wisely, so $G = \{I\} \times K$, where $I$ is the identity map on $S(= R_1^2)$ and $K$ is a connected and compact subgroup of $O(n - 1)$. Then, for all $x = (x_1, x_2) \in R_1^2 \times R^{n-1} = L^{n+1}$ we have $G(x) = \{x_1\} \times K(x_2) \simeq K(x_2)$ and $G(x)$ is Riemannian.

Case 2: Since $G$ fixes no null direction, then by Theorem 2.3, there is a nonnegative integer $m$ such that after suitable change of coordinates, $G = SO_0(m,1) \times K, K \subset O(n-m)$. But, $G$ is compact and this is impossible, unless $m = 0$ and $G = \{I\} \times K$, $K \subset O(n)$. Thus, all $G$-orbits are Riemannian. □

**Remark 3.2.** By proof of the above corollary, if $G$ is a compact and connected subgroup of the isometries of $L^{n+1}$, then $\frac{L^{n+1}}{G}$ is homeomorphic to one of the spaces $R^2 \times \frac{R^{n-1}}{K}$ or $R \times \frac{R^n}{K}$, where $K$ is a connected and compact subgroup of $O(n-1)$ or $O(n)$, respectively.

In the following theorems 3.3, 3.4, we characterize the orbit spaces arising from isometric actions of connected Lie groups on $L^{n+1}$, $n \geq 2$.

**Theorem 3.3.** *Let $G$ be a connected and compact subgroup of the isometries of $L^{n+1}$, $n \geq 3$. Then, $c = coh(L^{n+1}, G) \geq 2$.*
*(1) If $c = 2$ then $\frac{L^{n+1}}{G}$ is homeomorphic to $R \times [0, \infty)$.*
*(2)If $c = 3$ then $\frac{L^{n+1}}{G}$ is homeomorphic to $R^2 \times [0, \infty)$.*

**Proof.** First, note that if $K$ is a closed and connected subgroup of $O(m)$ and $dim\frac{R^m}{K} = 1$, then for each point $x \in R^m$, with $|x| \neq 0$, we have $G(x) = S^{m-1}(|x|)$. Then, the following map is a homeomorphism

$$\phi : \frac{R^m}{K} \to [0, \infty), \quad \phi(G(x)) = |x|.$$

Now, by Remark 3.2, we get the result. □

In the following theorem we show that when dimension of the orbit space is big, then characterization of the orbit space reduces to characterization of the orbit space of isometric actions on the usual spheres.

**Theorem 3.4.** *Let $G$ be a closed and connected subgroup of $SO(n,1)$. If $dim\frac{L^{n+1}}{G} > 2$ then there is a positive integer $m$ and a compact connected subgroup $K$ of $O(n-m)$ such that $\frac{L^{n+1}}{G}$ is homeomorphic to $\frac{L^{m+1}}{SO_0(m,1)} \times \frac{S^{n-m-1}}{K} \times [0, +\infty)$, where $S^{n-m-1}$ is the usual sphere of radius one in $R^{n-m}$.*

**Proof.** If the action of $G$ is weakly irreducible, then by Theorem 2.2, either $G$ acts transitively on $H^n$ or it acts transitively on a horosphere. In the first case the cohomogeneity of the action is one and in the second case is two. Thus, $dim\frac{L^{n+1}}{G} \leq 2$ which is contradiction. Therefore, the action of $G$ on $L^{n+1}$ is not weakly irreducible, so there is no null eigenvector . Then, by Theorem 2.3, either $G = SO_0(n,1)$ (which is not possible, because $dim\frac{L^{n+1}}{SO(n,1)} = 1$) or for a positive integer $m$, $G$ splits as $G = SO_0(m,1) \times K$, $K \subset O(n-m)$ . Thus, $\frac{L^{n+1}}{G} = \frac{L^{m+1}}{SO_0(m,1)} \times \frac{R^{n-m}}{K}$. In other way, for each number $r > 0$, $K(S^{n-m-1}(r)) = S^{n-m-1}(r)$. Consider the following map:

$$\varphi : \frac{S^{n-m-1}(1)}{K} \times [0,\infty) \to \frac{R^{n-m}}{K}, \;\; \varphi(K(x),t) = K(tx), \;\; x \in S^{n-m-1}(1).$$

It is easy to show that $\varphi$ is well defined ( independent to the choice of $x \in S^{n-m-1}(1)$) and it is a homeomorphism. This proves the theorem. □

As we can see in proof of the previous theorem, if $dim\frac{L^{n+1}}{G} > 2$ then the action of $G$ is not weakly irreducible. If $dim\frac{L^{n+1}}{G} \leq 2$ then possibly $G$ acts weakly irreducible and it has null eigenvector ( a non zero vector $v$ with the property $g(v) = v$ for all $g \in G$). In the remaining part of the article we consider the orbit space when $dim\frac{L^{n+1}}{G} \leq 2$. We consider the orbit space in two cases, $G$ with null eigenvector (Corollary 3.5) and $G$ without null eigenvector (Remark 3.6).

**Corollary 3.5** ( see [11] Theorem 3). *Let $G$ be a closed and connected subgroup of $SO_0(n,1)$ such that there is no null eigenvector.*
*(a) If $dim\frac{L^{n+1}}{G} = 1$ then $\frac{L^{n+1}}{G}$ is homeomorphic to $\frac{L^{n+1}}{SO_o(n,1)}$.*
*(b) If $dim\frac{L^{n+1}}{G} = 2$ then there is a nonnegative integer $m$ such that $\frac{L^{n+1}}{G}$ is homeomorphic to $\frac{L^{m+1}}{SO_0(m,1)} \times [0,\infty)$.*

**Remark 3.6.** When there is a null $G$-eigenvector and $dim\frac{L^{n+1}}{G} = 1$ or 2, the situation is a little more complicated. In this case, if we consider $G$ as a subgroup of the isometries of $H^n(r)$, $r > 0$, then it has fixed point at $H^n(r)(\infty)$ ( each null eigenvector corresponds to a fixed point of $G$ at infinity). Note that the existence of null eigenvector for $G$

is equivalent to weak irreducibility. Thus, by Theorem 2.2, one of the following is true:
(1) $G$ acts transitively on $H^n(r)$.
(2) For a fixed point $z$ at infinity, $G$ acts transitively on each horosphere of $H^n(r)$ centered at $z$.
Since $G \subset SO_0(n,1)$ then either (1) or (2) is true for all $r > 0$ simultaneously.

Case (1), $\frac{S}{G}, \frac{T}{G}$:
We can show that $G$ acts transitively on $S_1^n(r)$ for all $r > 0$. Then, $S_1^n(r)$, $r > 0$, is a $G$-orbit. The map which sends $S_1^n(r)$ to the positive number $r$, makes the following homeomorphism:

$$\frac{S}{G} = (0, \infty).$$

Similarly, the map which sends the orbit $H^n(r)$ to $r > 0$ and sends $H_-^n(r)$ to $-r$, $r > 0$, makes the following homeomorphism:

$$\frac{T}{G} = (-\infty, 0) \cup (0, \infty).$$

Case 2, $\frac{S}{G}, \frac{T}{G}$:
Consider a unit speed geodesic $\gamma : R \to H^n(r)$ such that $[\gamma] = z$. For each $t \in R$, $\gamma(t)$ belongs to a unique horosphere centered at $z$, which we denote it by $D_t$. Also, each horosphere centered at $z$ intersects $\gamma$. Since $G$ acts transitively on $D_t$ for all $t \in R$, then corresponding each orbit $D_t$ to the point $t$ makes the following homeomorphism:

$$\frac{H^n(r)}{G} = R.$$

Then,

$$\frac{T^+}{G} = \bigcup_{r>0} \frac{H^n(r)}{G} = \bigcup_{r>0} R = R \times (0, \infty)$$

Similarly:

$$\frac{T^-}{G} = R \times (-\infty, 0)$$

It is easy to see that in this case $G$ acts by cohomogeneity one on $S_1^n(r)$, $r>0$. The map which sends each orbit $G(x)$ to $(G(\frac{x}{|x|}),|x|)$ makes the following homeomorphism:

$$\frac{S}{G}=\frac{S_1^n(1)}{G}\times(0,\infty).$$

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