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Original Research Paper

Probabilistic Measurement, a New Tool to Computing the Spectrals-base Equilibrium Points

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Abstract. In a quantum system, equilibrium points are usually defined by the equation of evolution. The analysis of this process is often done by Schrödinger equation, and by linear operators on a Hilbert space. Regardless of the fact that calculations are based on static–point metrics, due to the chaotic behavior, the realization of practical conditions on space and mappings will be relatively difficult. In addition, access to constructive arguments will enable us to provide a computational method. In this paper, at first we define a probabilistic measurement and combine it with probabilistic domains to obtain a probabilistic model suitable for quantum systems. By extending the mappings to nonlinear operators, we examine the conditions under which stable equilibrium points can be reached. In the application section, using the evolution equation, we will determine the stationary points based on their spectral properties. Also there will be possible to generalize this method to simultaneous measurements.

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1 Introduction

Dynamical systems in physics evolve at time. The theoretical framework to examine them in classical mechanical systems, are Newton's laws of motion and in classical electromagnetic are determined by Maxwell's equations. There is some unification provided in terms of Hamilton's principle, a universal way of stating the laws of classical dynamics. Here the dynamical properties of quantum systems can be described by a quantum law of evolution. This rule can be expressed simply as follows: Suppose that a state vector is a repository of information known about a system. This law tells us how the information evolves in time, in response to the particular physical circumstance that the system of interest finds itself in.

In a quantum system, any stationary state refers to a state in which there is no change in time but this does not mean that nothing happens. If the given space is equipped with a suitable measurement, any stationary point turns out to be one for which the estimated probability outcomes are the same no matter at what time the measurement is made ([17], [19] and [20]).

A key point is that if a suitable measurement is used to describe each of the basic properties, then the simultaneous application of all measurements will lead to a more comprehensive description of the information within the system. This will be the final stage of our work.

Quantum mechanics is a good tool to predict properties of atoms and nuclei. Also study the equilibrium states of nuclei and their energies requires us to look at a time-dependent description of quantum mechanical systems. At the same time, to describe dynamical processes, such as radiation decays, scattering and nuclear reaction, we need to study how quantum mechanical systems evolve in time ([16], [24]). One common model is to use the Schrödinger equation to study such a point. From this point of view, the evolution of closed systems can be controlled by a differential equation [30]. The basis of the view is established on two facts. First by developing such an equation, we finally obtain a linear operator on a Hilbert space, and we must note the derivability may in most cases be unattainable. Also, if such conditions are met, access to a constructive argument that relies on a fast and efficient computational method is of particular importance. The second fact is that the mathematical framework used in this type of study is based on the structure of Euclidean spaces. In Euclidean model, the system analysis is based on static–point position, which does not fit well with quantum structures. To compensate for such a defect, it is usually possible to reduce the measurement error somewhat by using the wave function to justify particle motion, but if a metric design based on distribution functions can be applied, the approximation method will be considerably more accurate. Achieving such a model makes the use of probabilistic spaces fully justified.

In 1942, K. Menger replaced the number $d(p, q)$, the distance between to points p and q, by a probabilistic distribution function $F_{p,q}$, and defined probabilistic metric spaces [21]. That idea was a starting point for establishing a remarkable field of mathematics called probabilistic analysis ([5], [13] and [15]). In 1962, A. N. Serstnrv used the Menger's idea and endowed a set having an algebraic structure of linear space with a probabilistic norm [28]. For extensive view of this subject, please see [5], [21] and [29].

The last point that can be raised in the discussion of measurement is how to approximate. Because the nature of approximation depends on the recovery process compared to the previous cases, it seems that equipping the structure in one order will help this process completely. The design of such a discussion leads us to take the advantage of domain theory. The nature of equilibrium points in a dynamical system is such that studying them in a spectrum seems more reasonable than point state. The realization of such conditions can be fully accomplished in evolution function. For more details about the mathematical structures of domains, we suggest the readers to review [1], [2] and [13] and for applications of this theory in other disciplines, we recommend them to study [31], [32],[14] .

The needs raised lead us to the conclusion that in this paper, first, by developing probabilistic spaces, we achieve a more constructive structure called probabilistic domains. The main advantage of this work is that the concept of approximation and improvement of approximation can be done more effectively according to the order obtained on the new space.

In the second step, it is necessary to define a probabilistic measurement that can obtain a clear picture of the information within the system. This will lead us to the concept of p–measurable domains. Given that the Scott topology is one of the most well–known topologies on the domains, comparing the topology obtained from this measurement with the Scott one will have many results. Equilibrium results and stable equilibrium points on the obtained space are of special importance, because that will be the basis of our study in the second part of the article. In such cases, reducing conditions for special mappings is of particular importance, since in turbulent models even continuity can be a limitation.

Systematic analysis for discrete topologies and non–linear mappings are the most important results of the first part. The second part is devoted to the application of introduced ideas in quantum systems. A new look at Schrödinger equation is one of the main points in this part. Using a new tool to study wave functions, developing results for nonlinear mappings and not necessarily considered on a Hilbert space is among the most important points in this section. Another notable result is the study of equilibrium in spectral form (in degrees relative to the origin). This is important because in chaotic dynamical systems it makes much more sense to talk about a bounded limit of equilibrium. Finally, a suggestion is made for the next study. It is natural that increasing the monitoring criteria for the measurement will lead to a more accurate framework to predict the behavior of system.

With the explanation provided, the layout of the different parts in this article will be as follows: Section 2 is dedicated to providing the necessary elementary concepts. Section 3 will present basic results on probabilistic spaces that provide the necessary tools for the application section. Section 4, which is the practical part of the article, is dedicated to the application of previous results. The most significant strength of this part is a completely new look at the notion equilibrium in quantum theory. Finally, section 5 states that how to achieve a more comprehensive understanding of the systems by applying different measurements. Also [8], [22], [23], [26] and [27] can be helpful for more related extensions or generalizations of the results in this paper in the future research works.

2 Preliminaries

This section will be devoted to introducing perquisites from domains and probabilistic spaces.

Definition 2.1. *Let $(D, \leq)$ be a partially ordered set (poset). An infinite sequence*

$$d_1 \leq d_2 \leq \ldots \leq d_n \leq \ldots,$$

of elements in D, is called a chain. Such a chain is called stationary if there is some positive integer n such that $d_m = d_{m+1}, \forall m \geq n$. For $S \subseteq D$, an element $p \in D$ is called an upper bound (lower bound) of S if and only if $x \leq p$ $(p \leq x)$, for each $x \in S$. Furthermore, $p \in D$ is called the least upper bound (the greatest lower bound) of S and denoted by lub or sup *(glb or* inf*) if and only if p is an upper (lower) bound and for any other upper (lower) bound q of S, it is the case that $p \leq q$ $(q \leq p)$. In this context, we denote by $\vee S$ and $\wedge S$, the lub and glb of S, respectively. If $S = D$, we denote them by $\top$ and $\bot$, respectively. A poset $(D, \leq)$ is called a complete partial order (cpo), if and only if any of its chains has a lub.*

Definition 2.2. [17] *Let $(D, \leq)$ be a poset. A nonempty subset $S \subseteq D$ is called directed, if $\forall x, y \in S$, $\exists z \in S$ such that $x, y \leq z$. A directed complete partial order (dcpo) is a poset in which every directed subset has a supremum.*

Lattices are efficient structure mathematical modeling, related to patterns that do not have the usual algebraic and analytical structures. At the same time, their inherent freedom helps us to more closer to modeling the behavior of turbulent systems.

Definition 2.3. [17] *Suppose that $(D, \leq)$ be a nonempty poset. If for any $x, y \in D$, $x \wedge y$ and $x \vee y$ exist then the system $(D, \leq)$ is called a lattice. If for any arbitrary subset A of D, $\wedge A$ and $\vee A$ exist then we say that $(D, \leq)$ is called a complete lattice.*

Continuity in the mapping defined on a topological structure is a fundamental property that is absolutely necessary to preserve the basic properties. In discrete structures, it can be changed into monotonicity with a lattice extra conditions.

Definition 2.4. [17] *Let* $(D_1, \leq_1)$ *and* $(D_2, \leq_2)$ *be two posets. A function* $f; D_1 \longrightarrow D_2$ *is called increasing (decreasing) if for each* $x, y \in D_1$, $x \leq_1 y$ *implies that* $f(x) \leq_2 f(y)$ $(f(y) \leq_2 f(x))$. f *is called monotone if* f *is increasing or decreasing. For* $f : (D, \leq) \longrightarrow (D, \leq)$, *we say that* f *is extensive if* $x \leq f(x), \forall x \in D$. *Also* $d \in D$ *is called a fixed point of* f *if* $f(d) = d$, *the set of fixed points of* f *is denoted by* $Fix(f)$. *A fixed point* $d \in D$ *of* f *is called minimal (maximal) if for other fixed points of* f *such* v, *we have not* $v \leq d$ $(d \leq v)$. *If* f *has exactly one minimal (maximal) fixed point, then this fixed point is called the least (greatest) fixed point.*

One of important topologies on a poset is the Scott topology, which will be mentioned here.

Definition 2.5. [17] *A subset* U *of a dcpo* D *is Scott open, if*
(i) U *is an upper set:* $x \in U$ *and* $x \leq y$ *then* $y \in U$,
(ii) U *is inaccessible by directed suprema: For every directed* $S \subseteq D$,

$$\vee S \in U \Rightarrow S \cap U \neq \varnothing.$$

The collection σ_D *of all Scott open sets on* D *is called the Scott topology.*

Theorem 2.6. *[17] A map* $f : D \to E$ *between dcpo's is Scott continuous, iff*
(i) f *is monotone.*
(ii) f *preserves directed suprema: For every directed* $S \subseteq D$,

$$f(\vee S) = \vee f(S).$$

Definition 2.7. [17] *In a dcpo* $(D, \leq)$, $a \ll x$ *if for all directed subsets* $S \subseteq D$,

$$x \leq \vee S \Rightarrow (\exists s \in S) a \leq s.$$

We set $\twoheaddownarrow x = \{a \in D : a \ll x\}$. *A dcpo* D *is continuous if* $\twoheaddownarrow x$ *is directed with supremum* x *for all* $x \in D$. *The sets* $\twoheaduparrow x = \{y \in D : x \ll y\}$ *for* $x \in D$, *form a basis for the Scott topology on a continuous dcpo* D.

Definition 2.8. [17] *A domain is a continuous dcpo* D *such that for all* $x, y \in D$, *there is* $z \in D$ *with* $z \leq x, y$.

To describe a chaotic system of information more accurately, the best way is to use the concept of measurement, which will be presented in the following definition.

Definition 2.9. [17] *A Scott continuous map $\mu : D \to [0,\infty)^*$ on a continuous dcpo D induces the Scott topology near $X \subseteq D$ if for all $x \in X$ and all sequences $x_n \ll x$,*

$$\lim_{n\to\infty} \mu x_n = \mu x \Rightarrow \vee x_n = x,$$

and this supremum is directed. We write this as $\mu \to \sigma_x$. A measurement is a Scott continuous map $\mu : D \to [0,\infty)^*$ that measures the set

$$ker\mu = \{x \in D : \mu x = 0\}.$$

We remind that, a first-order discrete dynamical system on a normed space is a map $U : X \to X$ by $x_{n+1} = U(x_n)$, where n is a positive integer, X is a normed space and $\{x_n\}$ is called the recursion sequence. Our focus in the study will be mainly on probabilistic product spaces and specially on $\mathbb{R}^n$. Also, in any space X, and for any continuous map $g : X \to X$, a fixed point $p \in X$ is called an attractor if there exists an open set V near p such that for all $y \in V$ and $n \geq 0$, $g^n(y) \to p$.

probabilistic spaces as a generalization of metric spaces can provide a more accurate description of modeling the behavior of particles at the quantum scale. Combining this structure with the concept of domains will lead us to a better model that will be very effective in describing the theory of evolution and determining the equilibrium points. More detailed information about these spaces can be found in [18], [19] and [20]. Here we will confine ourselves to the definitions and results we use in the paper.

A function $F : [-\infty, +\infty] \longrightarrow [0,1]$ is called a distribution function if it is non–decreasing and left–continuous with $F(-\infty) = 0$, $F(+\infty) = 1$ and $F(0) = 0$. The set of all distance distribution functions is denoted by D and $D^+ := \{F \in D : F(0) = 0\}$ will mark as the distance distribution functions. Two special distance distribution function are given by

$$H_\infty(x) = \begin{cases} 0 & x \leq \infty, \\ 1 & x = +\infty, \end{cases}, \quad H_0 = \begin{cases} 0 & x \leq 0, \\ 1 & x > 0. \end{cases}$$

Definition 2.10. [28] *A probabilistic normed space (or PN-space) is a quadruple* (X, ν, τ, τ^*), *where* X *is a vector space,* τ *and* τ^* *are continuous triangle functions such that* $\tau \leq \tau^*$ *and* ν *is a mapping from* X *into* D^+ *, called the probabilistic norm , such that for every choice of* $x, y \in X$ *the following conditions hold:*
(PN–1) $\nu_p = H_0$ *iff* $p = \theta$ *(*θ *is the null vector in* X*),*
(PN–2) $\nu_{-p} = \nu_p$,
(PN–3) $\nu_p \leq \tau^*(\nu_{\lambda p}, \nu_{(1-\lambda)p})$ *for every* $\lambda \in [0, 1]$,
(PN–4) $\nu_{p+q} \geq \tau(\nu_p, \nu_q)$.

Suppose that (X, ν, T) be a *PN* space under a continuous *t–norm* T. There is a natural topology called the strong topology [13] that is defined by neighborhoods

$$N_p(t) = \{q \in X : \nu_{q-p}(t) > 1 - t\}.$$

A normed space $(X, ||.||)$ can always be regarded as a *PN* space. In fact, define $\nu : X \to \Delta^+$ by $\nu_p = H_{||p||}$, for each $p \in X$. For any triangle function τ with

$$\tau(H_a, H_b) = H_{a+b},$$

for all $a, b \geq 0, (X, \nu, \tau)$ is a Menger *PN* space [13]. What we will call as Menger *PN* space (or Menger space) has the following feature.

Definition 2.11. [28] *Let* X *be a Linear space,* τ *a triangle function and* $\nu : L \to D^+$ *be such that:*

1. $\nu_x = H_0$ *iff* $x = 0$;
2. $\nu_{\alpha x} = \nu_x(t/\alpha)$, *for any* $t > 0, \alpha \in \mathbb{R}$ *and* $x \in X$;
3. $\nu_{x+y} \geq \tau(\nu_x, \nu_y)$, *whenever* $x, y \in X$.

If in addition to 1 and 2, the probabilistic triangle inequality 3 is given under a t–norm *T:*

$$\nu_{x+y}(t_1 + t_2) \geq T(\nu_x(t_1), \nu_y(t_2)),$$

for all $t_1, t_2 \in \mathbb{R}^+$ *and* $x, y \in X$, *then* (X, N, T) *is called a Menger space.*

Now take

$$V(\varepsilon, \lambda) = \{x \in X : \nu_x(\varepsilon) > 1 - \lambda\},$$

then $\mathcal{B}_V = \{V(\varepsilon, \lambda) : \varepsilon > 0, 0 < \lambda < 1\}$ is a complete system of neighborhoods of null vector for strong topology on X (see [13]).

3 Main Result

First of all, we need to build a domain structure on any Menger space. The advantage of this reproduction is that the concept of order can be considered on the produced spaces. A formal ball in a Menger space (X, ν, T) is the ordered triple $[x, \lambda, t]$ where $x \in X, \lambda \in [0, 1]$ and $t \in \mathbb{R}^+$ in which for every $y \in X, 0 < \beta < \lambda < 1, t_1 > t_2 > 0$, one can define:

$$[x, \lambda, t_1] \leq [y, \beta, t_2] \ iff \ \nu(x - y, t_1 - t_2) \geq 1 - (\lambda - \beta).$$

If we denote the set of formal balls by $p - BX$, this order turns $p - BX$ into a poset that we write it as $(p - BX, \leq)$.

A direct calculation shows that $(p - BX, \leq)$ is actually a domain. By transitional property in a linear normed space, any complete system of neighborhoods at any point $x \in X$ can be transferred to O_X. So, all systems will study around the O_X, the advantage of this system behind its strong topology, is that it has a computational capability. But by repairing this structure, one can move to the spaces that are not necessarily normable and at most be a dcpo.

In order to better understand, imagine that a reaction (i.e. an information system) starts from a space–time point $x \in X$. Gradually, as we move away from the zero point (x can be transformed to O_X), more information is found than at the starting point. If we consider O_X as the unique point possessing all information, the next moments give us more information, and gradually the chaos of the system is increasing in different directions.

As stated previously, the general diagram of this process in a probabilistic normed linear space can be plotted as Figure 1, by $B_n(O_X, \lambda_n, t)$.

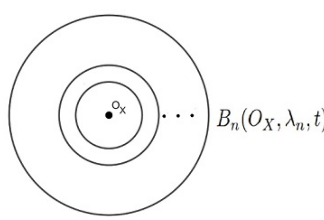

Figure 1: The distribution of information in a Menger space

The complete core of the information is at the zero point, and the circles are relative approximations of this system, respectively. Information is distributed randomly, and over time is a continuous function. This structure has two general advantages. First, a linear structure governs the space that allows algebraic operations. Second, the normability of the space is considered as a basic instrument to reach a scale. What we want to do here as a special job is a norm replacement with a new concept called p–measurement. In fact our goal is to design a generalized probabilistic space extended by a measurement over any probabilistic norm. Furthermore it turns it into a constructive structure. With a little effort it can be proved that $(p - BX, \leq)$, the ordered family of $\{[o_x, \lambda, t] : \lambda \in (0,1), t \in \mathbb{R}\}$, is a continuous dcpo having the Scott topology.

In the next step one can show that D^+ is a domain, the essential thing we need to define the notion of p–measurement.

Lemma 3.1. *D^+ is a domain with H_∞ and H_0 as its bottom and top, respectively.*

Proof. Here $f \leq g$ if $\forall t \in \mathbb{R}, f(t) \leq g(t)$. It is clear that D^+ is a poset, for each $f \in D^+$, $H_\infty \leq f \leq H_0$. For each $f, g \in D^+$, take

$$f \vee g = max\{f, g\},\ f \wedge g = min\{f, g\}.$$

Then D^+ is a directed set. For any $\varnothing \neq B \subseteq D^+$, putt

$$\vee B = sup\{g : g \in B\},\ \wedge B = inf\{g : g \in B\}.$$

This shows that D^+ is a directed set that has a supremum and even an infimum, For any $f \in D^+, f \twoheaddownarrow$ is directed with supremum f. So D^+ is a continuous dcpo. □

One effective way to investigate and predict the results in a dynamical system is to use the concept of a measure related to the system. Previously, we observed that if the space under the study has a linear and normable structure, such a thing is much more practical and relatively simple. The nature of this simplicity stems from the fact that despite the chaotic nature of the system, the topology of space is formed by concentric balls at origin (starting point of evolution). But when the system under study does not have any of the above advantages, how this problem can be overcome? In this regard, by defining the concept of real–valued measurements on the domains, as well as the lattice valued measurements between to domains, the problem has been solved approximately. To see details, we refer the readers to [17].

The main drawback of such an assessment is that the defined measurements are based on static–point model that are more compatible with Euclidean geometry and ultimately with Newtonian mechanics. If we want to talk about evaluating the system in the realm of quantum mechanics, such tools do not have the necessary efficiency at all. Therefore, the basement of evaluation should be emphasized on probabilistic patterns and this justifies our study.

Definition 3.2. *Suppose that D is a domain with bottom $\perp_D$. Each ascending chain C in D with initial point $\perp_D$ will be called a component of D. Two chains C_1 and C_2 are called equivalent if there exists a one-to-one corresponding $f : C_1 \to C_2$, that has order preserving property.*

It is clear that any ascending chain with initial point will be the same (equivalent) as a chain starting from $\perp_D$. Physically, the meaning of this property is that the evolution of both chains takes place in the same way, while one of them is in progress with a phase difference compared to the other. So with a little tolerance, we merge such chains and use a class of equivalent chains instead. A domain is said to be decomposable if it can be written as a product of its non–equivalent chains.

As a concrete example that we will use soon, one can consider the domain D^+. A component of D^+ can be presented as follows and called linearity probabilistic domain (LPD):

$$LPD = \{H_a : a \in \mathbb{R}\}.$$

By considering the triangular function τ such that $\tau(H_a, H_b) = H_{a+b}$, for each $a, b \in \mathbb{R}$, LPD is a complete lattice with the relation:

$$H_a \leq H_b \text{ iff } b \leq a,$$

so that H_0 and H_∞ are its top and bottom respectively. The ultimate goal in the first part of this paper is determining a product p–measurement on the space D^+ by imposing some p–measurements (not necessarily distinct) on its components. Therefore we first define and examine the results on a component like LPD.

If we denote LPD_X as the component of D^+ along the X–axis (or real axis) and LPD_Y as the related one along the Y–axis (or imaginary axis), then our probabilistic measurement can be considered as a product p–measurement on $D^+ \times D^+$. This will help us to study a probabilistic measurement as a combination of p–measurements on LPD_X and LPD_Y. So we are naturally led to the following definition.

Definition 3.3. *For any continuous dcpo D, A Scott continuous map $\mu_X : D \to LPD_X$ induces a Scott topology near $S \subseteq D$ if for all $s \in S$ and all $s_n \ll s$, $\lim\limits_{n\to\infty} \mu_X s_n = \mu_X s$ implies that $\vee_{n=1}^{\infty} x_n = x$. We write this as $\mu_X \to \sigma_S$. A p–measurement is a Scott continuous map $\mu_X : D \to LPD_X$ that measures the set*

$$ker\mu_X = \{s \in D : \mu_X s = H_0\}.$$

The map μ_X randomly measures the content of information for objects in S. So one can say that μ_X practically measures S by what we will call as quantum measurement in the future. The essence of this idea is that the quantum measurement of information within a system can be a good estimate of its reality. We will also see the extrem points, with desired p–measurement occur at probabilistic zero. The following theorem largely makes clear what we infer from Definition 3.3.

Theorem 3.4. *Let $\mu_X : D \to LPD_X$ be a p–measurement with $\mu_X \to \sigma_S$. Then*

i) $\forall x, y \in D$ *with* $x \leq y$ *and* $\mu_X x = \mu_X y$, *we have* $x = y$.

ii) *For each* $x \in D$ *and* $\mu_X x = H_0$, *we implies that* $x \in maxD$ *(extreme points of* D*).*

iii) *Any monotone map* $g : D \to D$ *is Scott continuous iff* $\mu_X g : D \to LPD_X$ *is Scott continuous.*

Proof. *i)* According to Definition 3.3 and for $x \in D$, by taking $S = \{x\}$, one can show that μ_X measures the content of x if for all Scott open sets $U \subseteq D$,

$$x \in U \Longrightarrow \exists \epsilon > 0 \text{ such that } x \in \mu_X^\epsilon(x) \subseteq U,$$

where

$$\mu_X^\epsilon(x) = \{y \in D : y \leq x \text{ and } H_y \gg H_\epsilon\}. \tag{1}$$

Now suppose that $\mu_X x = \mu_X y$ and U is any Scott open set near y. Then there exists $\epsilon > 0$ such that $y \in \mu_X^\epsilon y \subseteq U$. But $x \leq y$ and $\mu_X x \gg H_\epsilon$. By considering $\mu_X x = \mu_X y$, we see that $\mu_X y \gg H_\epsilon$ and hence $x \in \mu_X^\epsilon y \subseteq U$. This means that each Scott open set concluding y also contains x and then $y \leq x$.

ii) By theorem 1 μ_X is a monotone map if we see LPD_X as a dcpo. Now suppose that $x \leq y$ and $\mu_X x = H_0$. Then $H_0 = \mu_X x \leq \mu_X y$. So $H_0 = \mu_X y$ because H_0 is the top element in LPD_X. Hence $\mu_X x = \mu_X y$ and by absorption property from below of μ implied from (i), we have $x = y$.

iii) If g is Scott continuous then by Definition 3.3, $\mu_X g$ is also continuous. Let $X \subseteq D$ be any directed set with $\sup X = \alpha$ and $\mu_X g$ is Scott continuous. Then $\mu_X g$ is monotone. By hypothesis, $g(\alpha)$ is an upper bound for the directed set $g(X)$. on the other hand

$$\mu_X g(\alpha) = \mu_X g(\sup X) = \sup \mu_X(g(X)).$$

Since μ_X preserves suprema, $\sup g(X)$ exists and we have $g(\alpha) = \sup g(X)$. Now the result is followed from *i)*. □

One of the main issues in the study of dynamical systems is to consider stable equilibrium and providing a method to determine such a point. But in the situations where maximum condition at our hand is monotonicity, the work is definitely more difficult than usual. Under such conditions, fixed point techniques seem to be very effective.

A mapping $g : D \to D$ is called a returner with respect to μ_X if there is a constant $0 < c < 1$ such that for all $x \in D$,

$$\mu_X(f(x), t) \geq \mu_X(x, \frac{t}{c}), t > 0. \tag{2}$$

Important feature of this condition is that it is possible to achieve stable equilibrium points with a constructive nature. The following case is an excellent example of this assertion.

Theorem 3.5. *Let (D, μ_X) be a p–measurable domain (a domain with a p–measurement) and g be a returner and monotone mapping on D. Moreover, suppose that at least for a point $x \in D$, we have $x \leq g(x)$. Take*

$$x^* = \vee_{n=0}^{\infty} g^n(x).$$

Then x^* is a fixed point of g such that for all $y \ll x^*$, we have $\vee_{n=0}^{\infty} g^n(y) = x^*$ (This means that x^* is an attractive stable point). Then the following statements are equivalent:

i) x^* is unique as a fixed point of g,

ii) $Fix(g)$ under the induced structure on D is a dcpo.

Proof. Let $x \in D$. Then for any $n \geq 0, t > 0$, and by induction,

$$\begin{aligned}\mu_X(g^n(x), t) &\geq \mu_X(g^{n-1}(x), \frac{t}{c}) \\ &\geq \mu_X(g^{n-2}(x), \frac{t}{c^2}) \\ &\vdots \\ &\geq \mu_X(g(x), \frac{t}{c^n}).\end{aligned}$$

Given a point $x \in D$ such that $x \leq g(x)$. Since g is monotone then the sequence $\{g^n(x)\}$ is increasing. Also by continuity of μ_X one can see that

$$\mu_X(\vee_{n=0}^{\infty} g^n(x)) = \lim_{n\to\infty} \mu_X(g^n(x), t) \geq \lim_{n\to\infty} \mu_X(x, \frac{t}{c^n}) = H_0.$$

This means that $x^* = \vee_{n=0}^{\infty} g^n(x) \in ker\mu_X$ and by Theorem 3.4, $x^* \in maxD$. Also, $g^n(x) \leq g^{n+1}(x)$, for each $n \geq 0$. Then

$$x^* = \vee_{n=0}^{\infty} g^n(x) \leq \vee_{n=0}^{\infty} g^{n+1}(x) = g(\vee_{n=0}^{\infty} g^n(x)) = g(x^*),$$

and since $x^* \in maxD$, we implies that $x^* = g(x^*)$ is a fixed point of g. Suppose that $x \in ker\mu_X$ and $x \leq x^*$. Then for each $n \geq 0$, $g^n(x) \leq g(x^*) = x^*$ since g is monotone. Furthermore

$$\lim_{n\to\infty} \mu_X g^n(x) = \mu_X(\lim_{n\to\infty} g^n(x)) = \mu_X x^* = H_0.$$

As we had supposed that μ_X is a $p-$measurement and using Theorem 3.4, we imply that $x^* = \vee_{n=0}^{\infty} g^n(x)$. This argument shows in practice that x^* is an attractive point in a neighborhood related to μ_X, defined by (1), also this completes the proof $i) \Rightarrow ii)$. For $ii) \Rightarrow i)$, suppose that $x^{'}$ be another fixed point of g. According to $ii)$, one can find $z \in D$ such that $z \leq x^*, x^{'}$. By what we have shown in above, $\vee_{n=0}^{\infty} g^n(z) = x^* = x^{'}$. Thus $x^{'} = x^*$ and x^* is unique. □

During the previous discussions, a subtle and fundamental point can be seen. Neighborhoods generated by relation (1) are the basis of a new topology on D which we call the topology generated by μ_X (or μ_X- topology). What does this topology have to do with Scott's topology? Are there conditions in which the two coincide? If so, the approximation given by this measurement would give a very good assessment of what is going on the structure of D. Next lemma examines this issue.

Lemma 3.6. *Suppose that (D, μ_X) be any $p-$measurable domain, $\{y_k\}$ a sequence in D and $y \in D$. The following assertions are equivalent:*

i) $y_k \xrightarrow{\mu_X} y$,

ii) There exists $n_0 \in \mathbb{Z}^+$ such that for each $k \geq n_0$, $y_k \leq y$ and $y_k \xrightarrow{Scott} y$.

Proof. $i) \Longrightarrow ii)$ is obvious, because each Scott open set is $\mu-$open. Also choose $a \ll y$. Then $\Uparrow a \cap \downarrow y$ is $\mu-$open and will include all but a finite number of y_k. Now to prove $ii) \Longrightarrow i)$, take a $\mu-$open set O contains y. Then there exists $a \ll y$ such that $\Uparrow a \downarrow y \subseteq O$. But $y_k \xrightarrow{Scott} y$, so there exists $n_0 \in \mathbb{Z}^+$ such that $y_k \in \Uparrow a$, $\forall k \geq n_0$. So $y_k \leq a$, $\forall k \geq n_0$. But $a \ll y$ and implies that $y_k \leq y$, $\forall k \geq n_0$, and then y is an upper bound for $\{y_k\}_{k \geq n_0}$. Now from (1) and Definition 3.3, we observe that $y_k \xrightarrow{\mu_X} y$. □

Remark 3.7. *According to the previous lemma, there are good criteria to check the convergent sequences with respect to μ_X: For any sequence $\{y_n\}$ in D, $y_n \overset{\mu_X}{\to} y$ iff there exists $n_0 \in \mathbb{Z}^+$ such that $y_n \geq y$, $\forall n \geq n_0$ and also $\lim_{n \to \infty} \mu_X y_n = \mu_X y$. In addition, another important point can be inferred during the proof: If μ_X is Scott continuous then $\mu_X \to \sigma_S$ iff the set $\{\mu_X^{\epsilon}(y) : y \in D, \epsilon > 0\}$ forms a base for μ_X-topology on D.*

Contraction conditions in the absence of derivation are among the best conditions to reach a stable equilibrium point. In probabilistic metric spaces, these conditions have been analyzed in detail by various authors and important results have been obtained ([6], [7], [9] and [12]). But what we want to do here is to examine the conditions of contraction in the absence of any probabilistic metric. Imposing a topology on D in which the convergence can be studied is so vital. Naturally, we have two topology in this regard, the Scott topology and the topology induced by μ_X. Of the two, it makes sense to choose μ_X-topology because it has a constructive topology.

If (V, ν, τ) is a Menger space, the linear structure on V can be used to set a new building on BV (The set of balls at origin) with the following operations:

i) $B_0(\epsilon_1) + B_0(\epsilon_2) = B_0(\epsilon_1 + \epsilon_2)$

ii) $\lambda B_0(\epsilon) = B_0(\epsilon\lambda)$

Where $0 < \lambda < 1$ and $\epsilon \in \mathbb{R}^+$. Since there is a bijection between $\mathbb{R}$ and $(0, 1)$ one can think as $\lambda \in \mathbb{R}$. O_{BV} can be thought of as ball with zero radius. Also considering that all balls are nested and with zero center, the difference between the objects can be considered as follows:

$$B_0(\epsilon_2) - B_0(\epsilon_1) = B_0(\epsilon_2 - \epsilon_1) = B_0(\epsilon_1),$$

for $0 < \epsilon_1 < \epsilon_2 < 1$. A simple calculation shows that BV is actually a domain.

Definition 3.8. *Suppose that g is a monotone self–mapping on the $p-$measurable domain (BV, μ_X). g is called a contraction map if there exists $0 < c < 1$ such that for all $x, y \in BV$ and $y \leq x$, the following inequality holds:*

$$(\mu_X g(y) - \mu_X g(x)\, ,\, t) \geq (\mu_X x - \mu_X y\, ,\, \frac{t}{c}), \forall t > 0. \tag{3}$$

If $c \leq 1$, we will call it a non-expansive map.

Although this definition my seem a little unusual for general domains, it can be nicely interpreted in our particular context. Without any ambiguity it can be assumed that $x = B_0(\epsilon_1)$, $y = B_0(\epsilon_2)$ and $\epsilon_2 \leq \epsilon_1$. Then in LPD, we have

$$\mu_X x - \mu_X y = \mu_X B_0(\epsilon_2) - \mu_X B_0(\epsilon_1) = H(\epsilon_2) - H(\epsilon_1) = U_{[\epsilon_2,\epsilon_1]},$$

where

$$U_{[a,b]}(t) = \begin{cases} 1 & a \leq t \leq b, \\ 0 & otherwise. \end{cases}$$

Also, take $g(x) = B_0(\beta_1)$, $g(y) = B_0(\beta_2)$ with $\beta_2 \leq \beta_1$, then

$$\mu_X g(y) - \mu_X g(x) = H_{\beta_2} - H_{\beta_1} = U_{[\beta_2,\beta_1]}.$$

So the inequality (3) can be written as follows:

$$U_{[\beta_2,\beta_1]}(ct) \geq U_{[\epsilon_2,\epsilon_1]}(t)\,,\ \forall\ 0 < c < 1\,,\ \forall t \in \mathbb{R}. \tag{4}$$

The physical meaning of inequality (4) is that, g brings us closer to the starting point of process over the time, at least as much as ct, and this so vital in the dynamics of the space. Also in these conditions the inequality (3) can be reduced to (2). Given that the condition of continuity is relatively important and in other discussions usually comes from contraction, we need to investigate the similar result here.

Lemma 3.9. *Any contraction g on (BV, μ_X) is continuous.*

Proof. According to remark 3.7, it is enough to follow the proof by convergent sequences in BV. Suppose that $y_n \xrightarrow{\mu_X} y$ in BV. Then one can assume that $y_n \leq y$. Take

$$D(x,y,t) = (\mu_X x - \mu_X y, t),$$

Since g is a contraction, we have

$$D(g(y_n), g(y), ct) \geq D(y_n, y, t). \tag{5}$$

As $y_n \xrightarrow{\mu_X} y$, the right side of (5) tends to H_0. But H_0 is the maximal element in LPD_X and so

$$\lim_{n\to\infty} D(g(y_n), g(y), t') = H_0,$$

where $t' = ct$ is another parametrization with respect to t. These means $g(y_n) \xrightarrow{\mu_X} g(y)$, and by Remark 3.7, the proof is complete. □

Here two goals are in important priority for us, achieving stable equilibrium points and using constructive algorithms. So we are moving in the direction of attaining both goals in the current situation. These points are often generated by iterative methods.

Theorem 3.10. *Suppose that g be a contraction on (BV, μ_X). Then*

$$Fix(g) = \vee_{n=p}^{\infty} g^n(\perp_{BV}),$$

is the unique fixed point of g. Moreover take $y_0 = Fix(g)$ and define

$$U_A(y_0) = \{x \in BV : x \leq y_0\}.$$

Then for each $x \in U_A(y_0)$, we have $\vee_{n=p}^{\infty} g^n(x) = y_0$, and hence y_0 is an attractive point for g on local area $U_A(y_0)$. Finally y_0 is a limit point in μ_X–topology.

Proof. By previous lemma, g is Scott continuous and so

$$g(y_0) = g(\vee_{n=0}^{\infty} g^n(\perp_{BV})) = \vee_{n=0}^{\infty} g^{n+1}(\perp_{BV}) = y_0.$$

This shows that $y_0 \in Fix(g)$. The construction method of y_0 shows that it is unique. Now let $x \in U_A(y_0)$. Then $\perp_{BV} \leq x \leq y_0$. For each $n \in \mathbb{Z}^+$, $g^n(\perp_{BV}) \leq g^n(x) \leq g^n(y_0)$. Then

$$y_0 = \vee_{n=0}^{\infty} g^n(\perp_{BV}) \leq \vee_{n=0}^{\infty} g^n(x) \leq \vee_{n=0}^{\infty} g^n(y_0) = y_0.$$

This implies that $\vee_{n=0}^{\infty} g^n(x) = y_0$. An easy computation shows that $U_A(y_0) = y_0 \downarrow$. By Remark 3.7, $U_A(y_0)$ is a μ_X–open set and hence y_0 is an attractor point in the μ_X–topology. □

4 Application: Breaking The System in Random Directions and Computing the Spectral–base Equilibriums.

In a one dimensional system, a single particle at the atomic scale behaves like a wave. It is known as wave–particle duality. The quantum state of a system by wave–particle idea is described by a complex function ψ, which depends on the coordinate x and time simultaneously. The wave function encodes all the information about the system in a probabilistic sense. This means that $|\psi(x,t)|^2$ is a probability per unit length or probability density and the total probability of finding the particle somewhere along the real axis must be unity. i.e.

$$\parallel \psi \parallel^2 = \int_{-\infty}^{+\infty} |\psi(x,t)| dx = 1.$$

So any function with final integral along the real axis can be normalized by multi-plying by an appropriate constant and two wave functions differing by an arbitrary scale $c \in \mathbb{C}$ describe the same physical system (see [16], [31] and [32]). The combination of these explanations convinces us that a probabilistic measurement can provide a suitable mathematical model of analysis for quantum systems. Especially in situations where the background spaces are normed or inner product spaces the basic properties can provide more comprehensive analysis of particle behavior. It is also obvious that in the case that x is a vector in X^n(especially $\mathbb{R}^n$), the advantages of this model can be used in probabilistic product spaces. Since our study based on discrete systems, the following example further illustrates the modeling used in this paper.

Example 4.1. Consider a particle in a discretized space. Since it can be only in a finite number of positions along the real axis, we check out the situation with 6 points along the real axis labeled with $0, \cdots, 5$ as shown in Figure 2. The distance between the successive points is ϵ.

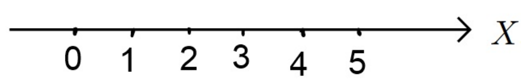

Figure 2: Real axis labeled with $0, \cdots, 5$

By probabilistic model of wave function, the probability for the particle to be at x_i at time t is given by $|\psi(x_i,t)|^2\epsilon$. Hence by defining

$$\psi_i = \sqrt{\epsilon}\psi(x_i,t), \ i = 0, \cdots, 5,$$

the whole information about this system is encoded in a six–dimensional complex vector:

$$|\psi >= (\psi_0, \cdots, \psi_5). \tag{6}$$

The values of ψ at different spatial points are component of the state vector.

Here, we see that the state of a system is represented by a vector. Also the wave function for a continuous system can be seen as the limit of the discretized case when the number of points goes to infinity and the distance ϵ becomes infinitely small. So in (6), we obtain a vector with infinite number of components encoded in a continuous function $\psi(x,t)$. It can be assumed that in a continuous state, we have an infinite dimensional vector space with wave function as its base. For each point x on the real axis, $\psi(x,t)$ has one coordinate in terms of the wave function. According to super position principle, state vectors can be combined linearly to obtain new admissible quantum states, i.e. for any two quantum states ψ_1 and ψ_2 and $c_1, c_2 \in \mathbb{C}$,

$$\psi(x,t) = c_1\psi_1(x,t) + c_2\psi_2(x,t),$$

is also a possible state of the system. The set of all possible quantum state forms a vector space ([3], [4] and [10]).

In this section, our main goal will be on time evolution and stability of a quantum mechanical system. As a consequence of super position principle, the evolution of a system can be determined by a linear equation $L\psi = 0$, where L is a linear operator [25]. one of our main goals is to develop this equation in probabilistic spaces. With help of this, a new concept of stability in dynamic systems on probabilistic spaces can be introduced. The dynamical systems evolve in time. In classical mechanical systems, they are described by Newton's laws of motion, while in the classical electromagnetic field, it is done by Maxwell's equations. The unification of both is provided in terms of Hamilton's principle, a universal way of stating the laws of classical dynamics. In all cases, we need a given state $|\psi(0) >$ of a quantum system at some initial time $t = 0$. The quantum law of evolution tells us what the state will be at some other time t.

Imagine that a state vector is a repository of information we know about the system. What is required to set a general physical law that tells us how this information evolves in the time in response to the other particular variables? Although the details of this law will vary form system to system, but it turns out that the law of evolution can be written in a way that holds true for all systems. Stationary states are generally used to identify those states of a quantum system that do not change in time. The more precise meaning of this description is that if we apply a measurement to describe the spatial property of the system, the probabilities of its outcomes must be the same no matter at what time the measurement is made. If we put $|\psi(0)>$ as the initial state of the system and $|\psi(t)>$ be the state at some other time t, the latter is supposed to represent the system in the same physical state, the most that these to states can differ is a multiplicative factor $u(t)$. So one cane write this equation by:

$$|\psi(0)>= U(t)|\psi(0)> . \tag{7}$$

Equation (7) is an integral equation relating the state at time zero with the state at time t. In Schrödinger equation, there will be a differential equation that provides the same information:

$$ih\frac{\partial\psi(x,t)}{\partial t} = \mathcal{H}\psi(x,t), \tag{8}$$

where $\mathcal{H}$ is the system's Hamiltonian and h is the Plank constant. The solution of (8) gives the wave function $\psi(x,t)$ at any latter time t when $\psi(x,0)$ is known. So equation (8) also says that the stationary states happen when the left side of (8) is zero. In other words, any solution for this case is trapped in the kernel of Hamilton operator, which is also a linear operator. Equation (8) is the main feature of our work in this study. From this equation one can infer that if there is an operator (even nonlinear) on a linear space that can describe the evolution of a system, the stationary points of that system are in fact the kernel elements. Equations (7) and (8) can be examined from several angles:

1. The notion of these equations is essentially based on linear operators on Hilbert spaces.

2. The condition of derivability in equation (8) is relatively hard to materialize.

3. The equilibrium condition is more time-dependent and does not show clear dependency on the geometric position of the point.

The results of the previous section help us to imagine the evolution model of a system as a non–linear function. The maximum condition imposed on this function is μ_X–continuity. Equilibrium is not inherent in nature and depends on the measurement assigned to the system. In addition, the theoretical results obtained by this article, help us to study it by spectrums under space-time conditions. For example, we re–examine the theorem 3.5 in situation where $X = \mathbb{R}^n$ and $\|.\|$ is the Euclidean norm on X.

Theorem 4.2. *Let D be a domain induced by the Euclidean normed space $(\mathbb{R}^n, \|.\|)$ and put*

$$\mu_X(t) = H_{\|a\|}(t), \ \forall a \in \mathbb{R}^n, \ t \in \mathbb{R},$$

and suppose that g be an evolution model related to a dynamical system on $\mathbb{R}^n$ that has the monotonicity and returner properties. Then the given dynamical system has a stable equilibrium $x^* \in D$. The constructive method to calculate x^* is $x^* = \bigvee_{n=0}^{\infty} g^n(\bot)$, where $\bot = B_0(0_{\mathbb{R}^n})$.

It is noteworthy that each equilibrium point represents a spectrum of points in $\mathbb{R}^n$. This is more consistent with quantum models, where equilibrium generally occurs at the scale of a space-time spectrum without more geometric description.

5 Work Path Development

In order to increase the accuracy, any space can be evaluated with different measurements. Suppose that such a space can be decomposed into different components under different measurements. Naturally, it will be possible to generalize the results to product spaces. We summarize this idea in Figure 3. Then $BX = B_1 \times B_2 \times \cdots \times B_k$ with $\mu_{BX} = \mu_1 \times \mu_2 \times \cdots \mu_k$ has some local stable equilibrium points under constrains on any evolutionary system.

Conclusion: In this research, the structure of probabilistic domains was generally introduced. Also, by considering a constructive measurement one could able to achieve useful statements related to equilibrium theory. The theoretical results helped to reach a more comprehensive notion

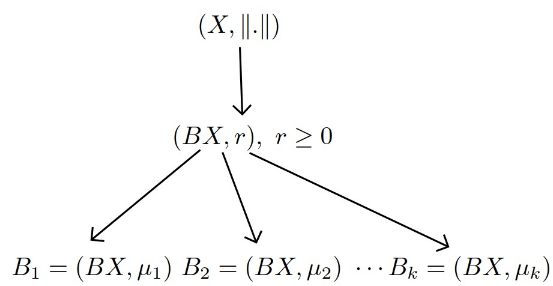

Figure 3: An evaluated space with different measurements

called spectral–base equilibrium. This helps to open a view framework in quantum theory. As a work plan for the future of this research, it is possible to obtain more valuable information about the chaotic behavior of the system by using product measurements on product spaces.

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