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# A New Robust DEA Method to Recognize the Anchor Points in the Presence of Uncertain Data

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**Abstract.**One of the attractive issues in Data Envelopment Analysis (DEA) literature is to find the anchor points of the production possibility set (PPS). Each extreme efficient unit which is located on the intersection of the strong and weak efficient frontiers of the PPS, is called an anchor point. In the other word, a decision making unit (DMU) is an anchor point, if there is at least one supporting hyperplane at the unit under consideration, in the situation that some components of its gradient vector are equal to zero, and so some input or output factors do not play any role in the performance of that unit. This study presents a new method to identify the anchor points of the PPS under the variable returns to scale (VRS) assumption and in the presence of the uncertain data. The proposed method is based on the robust optimization technique and finding the weak and strong defining supporting hyperplanes passing through the unit under evaluation. The potentially of

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the proposed method is illustrated by a data set, includes 20 banks in Iran.

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# 1 Introduction

Data envelopment analysis (DEA) is a powerful method to assess the efficiency of a set of decision making units (DMUs) with multiple inputs and outputs. See Charnes et al. [10], Banker et al. [4], Fare et al. [17] and Zhu [45]. DEA models assign an efficiency score between 0 and 1 to each unit. An efficient unit is a DMU with the efficiency score of 1. DEA technique builds a set, namely the production possibility set (PPS) to estimate the production function and evaluate the DMUs. For more details about the different PPSs, see Olesen and Petersen [33], Amirteimoori and Kordrostami [2], Davtalab Olyaie et al. [12, 13], Hosseinzadeh Lotfi et al. [19], Jahanshahloo et al. [21], Jahanshahloo et al. [22], Jahanshahloo et al. [23] and Jahanshahloo et al. [24].

The anchor points set is an important subset of the extreme efficient units set in each PPS. A DMU is an anchor point, if its inputs and outputs can be increased and decreased, respectively, without the DMU entering the PPS. In the other word, an anchor point is an extreme efficient unit located on the intersection of the strong and weak efficient frontiers of the PPS. In fact, for anchor points, there is at least one supporting hyperplane in which some components of the gradient vector are equal to zero, and so some input or output factors do not have any role in the performance of the unit under evaluation.

Anchor points were first named and identified by Allen and Thanassoulis (2004). In the DEA literature, the DEA efficient frontier can be extended by using the concept in Thanassoulis and Allen (1998) for the generation of unobserved DMUs. Anchor points also play an important role in Rouse (2004) which identified the prices for health care services. Bougnol (2001) pointed out that the anchor points are the only DMUs that are efficient for multiple constituencies where a constituency is a specific assignment as either input or output to each attribute. It should be noted that, the anchor points may distort DEA analysis. For example, in the common two-stage DEA approach where the formulation ignores non-Archimedean constants, anchor points may be confused with weakly efficient units. In general, it is difficult to identify anchor points conclusively. However, some sufficient conditions can be obtained for identification of the anchor points by using the optimal solutions of the multiplier form of DEA models. So, determining the anchor points of the PPS have attracted the attentions of many scholars in the DEA literature, because they play an essential role in the DEA theory and its applications.

Recently, Bani et al. [3] presented a method, based on searching the weak supporting hyperplane passing through the unit under evaluation, to find the anchor points of the PPS of BCC model. In this paper, inspired by the method of Bani et al. [3], we present a method with emphasis on using the definition of the anchor points to find them. Regarding the definition of the anchor points, if there are a weak and a strong defining supporting hyperplanes which the DMU under evaluation is located on them, then this DMU is an anchor point. Hence, the proposed method tries to determine the weak and strong defining supporting hyperplanes passing through the unit under evaluation. Given that the proposed method exactly uses the definition of anchor points to specify them, therefore, it is an efficient and very powerful method compared to the existing methods in the DEA literature.

The classical DEA models deal with the situation that the input and output values are deterministic. However, in many real-world applications, some imprecise data may be exist due to some sources such as the incomplete information, errors in measurements and so on. Imprecise data may lead to some challenges in applying the DEA technique, mostly resulting in a nonlinear DEA model. Given that the decision analysis based on the uncertain data is inevitable in many real-world applications, therefore, many researchers have focused on the evaluation of units in the presence of imprecise data. For example, Cooper et al. [11] developed Imprecise Data Envelopment Analysis (IDEA) method. Kim et al. [26] presented a method to deal with the partial data in DEA. Generally, there are three different approaches to deal with imprecise data in DEA, i.e. fuzzy approaches, stochastic methods, and robust optimization techniques. See Emrouznejad et al [16] for a review on fuzzy DEA approaches, Olesen and Petersen [34] for a review on stochastic DEA methods and Peykani et al [36] for a review on robust DEA methods. One of the most well known approaches to formulate the optimization problems under the data uncertainty is robust Optimization (RO) technique. This method determines an optimal solution for the problem which is the best for all realizations of the uncertain data. For more studies about robust optimization technique, see Ben-Tal and Nemirovski [5, 6, 7] and Bertsimas and Sim [8].

According to the above discussion, the concept of anchor point was used in DEA for the generation of unobserved DMUs in order to extend the DEA efficient frontier and so, this concept plays a critical role in the DEA theory and its applications. Hence, this study focuses on finding the anchor points in the different production possibility sets in the presence of uncertain data. For this purpose, we consider the situation that the inputs and outputs are reported as interval data and the production possibility set under the variable returns to scale (VRS) assumption and propose a new approach to identify the anchor points of this PPS in this situation. However, the proposed method can be easily developed to the different production possibility sets.

The proposed method exactly uses the definition of the anchor points to provide the approach in the case of data uncertainty and so, it is based on the robust optimization technique and finding the weak and strong defining supporting hyperplanes passing through the unit under evaluation. The use of the proposed approach is very simple and the anchor points can be easily identified by solving two simple models. In addition, the proposed approach is such that in addition to determining the anchor points, it also finds two important defining supporting hyperplanes on the PPS, which can be used in many problems in DEA, and this is the main contribution of the proposed method compared to the existing methods in the DEA literature. Regarding the definition of the anchor points, if there are a weak and a strong defining supporting hyperplanes which the DMU under evaluation is located on them, then this DMU is an anchor point.

The rest of the paper is organized as follows: Section 2 presents prior relevant research in the area of anchor points and robust DEA.

Section 3 contains some preliminaries. Section 4 proposes an approach to determine the anchor points in the case of data uncertainty. A numerical example is provided in Section 5. Section 6 concludes the paper.

# 2 Literature Review

## 2.1 Anchor points

Literature of the methods for identifying the anchor points is reviewed in the following. Bougnol and Dulá [10] used the geometric properties of the anchor points and proposed a method to find them. Hosseinzadeh Lotfi et al. [18] proposed an algorithm which could determine a main part of anchor points without solving any model and presented a model to find the remaining anchor points. Jahanshahloo et al. [24] suggested an optimized routing (DEA-OR) algorithm for wireless sensor nodes. DEA-OR algorithm could find the solution for energy efficient transmission and route failure recovery by considering the distance as the basic factor.

Hosseinzadeh and Elahi Moghaddam [20] used the characteristics of the anchor points and presented some methods to identify them. Mostafaee and Soleimani-Damaneh [30] applied the sensitivity analysis techniques and presented an approach to identify the anchor points. Soleimani-Damaneh and Mostafaee [42] introduced the extreme efficient and the anchor points in a non-convex production technology and suggested an algorithm to recognize the anchor points in FDH model. Krivonozhko et al. [29] introduced the terminal units and established some relationships between the terminal units and other sets of units and also, they developed an algorithm to improve the frontier of the PPS.

Jayekumar and Nagarajan [25] introduced the anchor points and defined the unobserved units by using the anchor points in order to extend the efficiency frontier. Zemtsov and Kotsemir [44] applied DEA to determine the relationship between the results of the patenting and resources of a regional innovation system (RIS) and used the DEA technique to compare the regions to one another over time. Mostafaee and Sohraiee [32] used the supporting hyperplanes to provide a definition for the exterior units and presented a model to discover them and also, they suggested some different definitions of the anchor points and demonstrated the relationship between the exterior units set, the terminal and the anchor points. Koushki and Soleimani-Damaneh [28] introduced the concept of anchor points for the multi-objective optimization problems and presented two approaches to recognize the anchor points.

Shadab et al. [41] developed an algorithm to identify the connection between the anchor points and congestion by using the geometric properties of the anchor points. Given that each anchor point is a an extreme efficient DMU, so finding the extreme efficient units can be useful to recognize the anchor points. Hence, there are several algorithms to find the extreme efficient points in the literature of DEA. Bani et al. [3] presented a method, based on searching the weak supporting hyperplane passing through the unit under evaluation, to find the anchor points of the PPS of BCC model. Akbarian [1] used the super-efficiency model to find all extreme efficient units and anchor points under the free disposability assumption.

### 2.2 Robust DEA

Many scholars incorporated the robust optimization technique into DEA. This section reviews some related paper in this subject. Wang and Wei [43] applied the approach of Ben-Tal and Nemirovski [7] in DEA and developed the robust formulation for the multiplier form of CCR model in the case of data uncertainty and provided a ranking method. Sadjadi and Omrani [37] presented the robust formulation of the multiplier form of CCR model based on the method of Ben-Tal and Nemirovski [7] and the method of Bertsimas et al. [8]. Sadjadi and Omrani [38] used the method of Bertsimas et al. [8] to develop a bootstrapped robust model for CCR model. Sadjadi et al. [39] suggested an interactive robust model by using Bertsimas et al.'s approach [8] to determine the targets of units. Omrani [35] focused on finding the common set of weights in DEA by using the goal programming technique and the robust approach of Bertsimas et al. [8].

Ehrgott et al. [15] formulated a DEA model in the case of data uncertainty to determine the maximum possible efficiency score of a unit. Salahi et al. [40] presented the robust counterpart of the CCR model in envelopment and then calculated the robust solutions for common set of weights under interval uncertainties by using the robust efficiency scores of units considering as ideal solutions. Dehnokhalaji et al. [14] proposed the robust counterpart problem for the envelopment form of the DEA model in the case of interval data. Also they developed two methods for ranking the units in the presence of uncertain data and showed that their proposed methods have more benefits compared to some existing approaches in the DEA literature.

Regarding the essential role of the anchor points in the theory and applications of DEA and this fact that in many real-world problems there are some uncertain parameters, this study focuses on developing the method of Khazaeyan et al. [27] for finding the anchor points in the case of data uncertainty.

## 3 Preliminaries and Basic Definitions

Consider n DMUs, $DMU_j, j = 1, \ldots, n$, where each unit uses $m$ different inputs to produce $s$ outputs. $x_{ij}$ and $y_{rj}$, for $i = 1, \cdots, m$ and $r = 1, \cdots, s$, are the $i^{th}$ input and the $r^{th}$ output for $DMU_j$, respectively. Also, assume that $DMU_o$ is the unit under evaluation.

The following PPS under the variable returns to scale (VRS), namely $T_v$, has been introduced by Banker et al. [4]:

$$T_v = \Big\{(x, y) \ | \ x \geq \sum_{j=1}^{n} \lambda_j x_j, \ y \leq \sum_{j=1}^{n} \lambda_j y_j, \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0, \\ j = 1, \cdots, n\Big\}$$

The multiplier form of BCC model is as follows:

$$
\begin{aligned}
\max \quad & \sum_{r=1}^{s} u_r y_{ro} + u_0 \\
s.t. \quad & \sum_{i=1}^{m} v_i x_{io} = 1, \\
& \sum_{i=1}^{m} v_i x_{ij} - \sum_{r=1}^{s} u_r y_{rj} + u_0 \ \geq 0, \quad j = 1, \ldots, n, \\
& u_r \geq 0, \ \ v_i \geq 0, \qquad\qquad\qquad i = 1, \ldots, m, \ r = 1, \ldots, s.
\end{aligned} \tag{1}
$$

where, $v_i, i = 1, \cdots, m$, and $u_r, r = 1, \cdots, s$, are the input and output weights, respectively. If the optimal value of model (1) is equal to 1 and there is at least one optimal solution for model (1), such as $(u_1^*, \cdots, u_s^*, v_1^*, \cdots, v_m^*)$, whose all components are strictly positive, then $DMU_o = (x_o, y_o)$ is BCC-efficient.

**Definition 3.1.** *A hyperplane* $H = \{(x, y)|u^t y - v^t x + u_0 = 0\}$ *is a supporting hyperplane of* $T_v$ *at* $DMU_o = (x_o, y_o)$*, if and only if for all* $(x, y) \in T_v$*, we have* $u^t y_o - v^t x_o + u_0 = 0$ *and* $u^t y - v^t x + u_0 \leq 0$.

A hyperplane H is the strong supporting hyperplane if $(u, v) > 0$ and it is the weak supporting hyperplane if some components of $(u, v)$ are equal to zero.

The anchor points of $T_v$ has been introduced by Bougnol and Dulá [10] as follows:

**Definition 3.2.** *Suppose that* $DMU_o = (x_o, y_o)$ *is a BCC-efficient unit. It is an anchor point if and only if it is located on a supporting hyperplane of* $T_v$*, namely* $H$*, such that at least one component of* $(u, v)$ *is equal to zero.*

The anchor points set is a subset of the extreme efficient units in $T_v$. This means that, an extreme efficient unit which its inputs can be increased and its outputs can be decreased without entering the PPS, is an anchor point. So, an extreme efficient unit, located on the intersection of the strong efficient and weak efficient frontiers, is called an anchor point. Hence, if a unit is an anchor point, then there is at least one supporting hyperplane which some components of its gradient vector are equal to zero, and so some input or output factor has not any role in the performance of that unit. The next section reviews the method of Khazaeyan et al. [27] to find the anchor points.

### 3.1 The method of Khazaeyan et al. [27]

According to the above discussion, Khazaeyan et al. [27] focused on finding the anchor points by searching both weak and strong defining supporting hyperplanes. In the following, we briefly describe their method. Given that, each anchor point is an extreme efficient unit, so, the first step of their method determines the BCC-efficient DMUs by using model (1) and defines the set $E$ as the set of the BCC-efficient DMUs. After that, the anchor pints are selected among the member of the set $E$. In the next step of their method, regarding each anchor point lies on the weak and strong defining supporting hyperplanes, Khazaeyan et al. [27] proposed some models to find these hyperplanes if they exist. They pointed out that, the weak defining supporting hyperplane has the minimum slope among all the weak supporting hyperplanes on $T_v$ at $DMU_o$ and the strong defining supporting hyperplane has the maximum slope among the strong supporting hyperplanes on $T_v$ at $DMU_o$. With this argument, they presented two multi-objective models for determining the weak and strong defining supporting hyperplanes and presented an approach to convert the multi-objective models into the single-objective models, model (2) and model (3), respectively.

Model (2) finds the input and output weights such that the slope of the corresponding defining supporting hyperplane on $T_v$ at the efficient unit, $DMU_o$, be minimum. Model (2) minimizes the minimum of all the input and output weights. So, if the optimal value of model (2) is equal to zero, then there is a set of weights in which at least one of the input or output weights is equal to zero. Hence, there is a weak defining supporting hyperplane which the unit under evaluation lies on.

The weak defining supporting hyperplane:

$$\begin{aligned}
\min\ & \zeta \\
s.t.\ & \zeta \le u_r, && r = 1, \cdots, s, \\
& \zeta \le v_i, && i = 1, \cdots, m, \\
& \sum_{r=1}^{s} \mu_r + \sum_{i=1}^{m} w_i = 1, \\
& \zeta = \sum_{r=1}^{s} \mu_r u_r + \sum_{i=1}^{m} w_i v_i, \\
& \sum_{i=1}^{m} v_i x_{io} = 1, \\
& \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + u_0 \le 0, && j = 1, \ldots, n, \\
& \sum_{r=1}^{s} u_r y_{ro} + u_0 = 1, \\
& u_r \ge 0,\ \ \mu_r \in \{0,1\}, && r = 1, \ldots, s, \\
& v_i \ge 0,\ \ w_i \in \{0,1\}, && i = 1, \ldots, m.
\end{aligned} \tag{2}$$

The strong defining supporting hyperplane:

$$\begin{aligned}
\max\ & \psi \\
s.t.\ & \psi \le u_r, && r = 1, \cdots, s, \\
& \psi \le v_i, && i = 1, \cdots, m, \\
& \sum_{i=1}^{m} v_i x_{io} = 1, \\
& \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + u_0 \le 0, && j = 1, \ldots, n, \\
& \sum_{r=1}^{s} u_r y_{ro} + u_0 = 1, \\
& u_r \ge 0, && r = 1, \ldots, s, \\
& v_i \ge 0, && i = 1, \ldots, m.
\end{aligned} \tag{3}$$

Similarly, model (3) finds the input and output weights such that the slope of the corresponding defining supporting hyperplane on $T_v$ at the efficient unit, $DMU_o$, be maximum. Model (3) maximizes the minimum of all the input and output weights. So, if the optimal value of model (3) is positive, then there is a set of weights in which all of the input and output weights are positive. Hence, there is a strong defining supporting hyperplane which the unit under evaluation lies on.

In summary, models (2) and (3) determine two defining supporting hyperplanes which $DMU_o$ lies on both of them. Given that the anchor point is an extreme efficient unit which lies on both weak and strong defining hyperplanes, hence, if one of the defining supporting hyperplanes, determined by models (2) and (3), is the weak hyperplane and another is the strong hyperplane, then $DMU_o$ is an anchor point. This means that, if the optimal value of model (2) is equal to zero and the optimal value of model (3) is positive, then the unit under evaluation is an anchor point.

The next section generalizes the method of Khazaeyan et al. [27] to the situation that the input and output of DMUs are reported as interval data.

## 4 Our Proposed Method

This section develops the idea of Khazaeyan et al. [27] to recognize the anchor points into the case of interval data and proposes a new method based on the robust optimization technique and finding the weak and strong defining supporting hyperplanes passing through the unit under evaluation.

In this section, we suppose that the input and output values are not deterministic for all units and $x_{ij} \in \left[x_{ij}^L, x_{ij}^U\right]$ and $y_{rj} \in \left[y_{rj}^L, y_{rj}^U\right]$, for $i = 1, \cdots, m$ and $r = 1, \cdots, s$, where the lower and upper bounds are positive and finite values. Assume that $DMU_o$ is the unit under evaluation. Dehnokhalaji et al. [14] formulated the optimistic counterpart of the multiplier form of the CCR model, which can be extended into the case of VRS as follows:

$$
\begin{aligned}
E_o^{OP} = \max \quad & \sum_{r=1}^{s} u_r y_{ro}^U + u_0 \\
s.t. \quad & \sum_{i=1}^{m} v_i x_{io}^L = 1, \\
& \sum_{r=1}^{s} u_r y_{rj}^L - \sum_{i=1}^{m} v_i x_{ij}^U + u_0 \le 0, \quad j = 1, \ldots, n,\ j \neq o, \\
& \sum_{r=1}^{s} u_r y_{ro}^U - \sum_{i=1}^{m} v_i x_{io}^L + u_0 \le 0, \\
& u_r \ge 0, \ \ v_i \ge 0, \qquad i = 1, \ldots, m,\ r = 1, \ldots, s.
\end{aligned} \tag{4}
$$

According to the definition of the anchor points, an anchor point is an efficient unit, so, the first step of the proposed method determines the efficient units. So, model (4) is solved and the set E is defined as the set of the optimistic efficient DMUs. The next step of the proposed method develops the method of Khazaeyan et al. [27] into the case of interval data to find the weak and strong defining supporting hyperplane which the efficient unit lies on, if there are. It is clear that the weak and the strong defining supporting hyperplane have the minimum and the maximum slopes among all the weak supporting hyperplanes on $T_v$ at $DMU_o$, respectively. So, models (5) and (6) are formulated to determine the gradient vectors of these defining supporting hyperplanes in the case of interval data.

The weak defining supporting hyperplane:

$$
\begin{aligned}
\min_{u_1,\cdots,u_s,v_1,\cdots,v_m \ge 0} \quad & \min\{u_1, \cdots, u_s, v_1, \cdots, v_m\} \\
s.t. \quad & \frac{\sum_{r=1}^{s} u_r y_{rj}^L + u_0}{\sum_{i=1}^{m} v_i x_{ij}^U} \le 1, \qquad j = 1, \ldots, n,\ j \neq o, \\
& \frac{\sum_{r=1}^{s} u_r y_{ro}^U + u_0}{\sum_{i=1}^{m} v_i x_{io}^L} \le 1,
\end{aligned} \tag{5}
$$

$$\frac{\sum_{r=1}^{s} u_r y_{ro}^U + u_0}{\sum_{i=1}^{m} v_i x_{io}^L} = E_o^{OP},$$
$$u_r \geq 0, \qquad r = 1, \dots, s,$$
$$v_i \geq 0, \qquad i = 1, \dots, m.$$

The strong defining supporting hyperplane:

$$\begin{aligned}
\max_{u_1,\cdots,u_s,v_1,\cdots,v_m \geq 0} \quad & \min\{u_1, \cdots, u_s, v_1, \cdots, v_m\} \\
s.t. \quad & \frac{\sum_{r=1}^{s} u_r y_{rj}^L + u_0}{\sum_{i=1}^{m} v_i x_{ij}^U} \leq 1, \qquad j = 1, \dots, n,\ j \neq o, \\
& \frac{\sum_{r=1}^{s} u_r y_{ro}^U + u_0}{\sum_{i=1}^{m} v_i x_{io}^L} \leq 1, \\
& \frac{\sum_{r=1}^{s} u_r y_{ro}^U + u_0}{\sum_{i=1}^{m} v_i x_{io}^L} = E_o^{OP}, \\
& u_r \geq 0, \qquad r = 1, \dots, s, \\
& v_i \geq 0, \qquad i = 1, \dots, m.
\end{aligned} \tag{6}$$

Given that model (4) is feasible and bounded and models (5) and (6) try to determine the weak and strong defining supporting hyperplanes by using the optimal solutions of model (4), therefore, models (5) and (6) are always feasible. Models (5) and (6) are non-linear programming problems which can be converted to the LP models (7) and (8), respectively, by using the Charnes and Cooper transformation as follows:

The weak defining supporting hyperplane:

$$
\begin{aligned}
\min_{u_1,\cdots,u_s,v_1,\cdots,v_m\geq 0} \quad & \min\{u_1,\cdots,u_s,v_1,\cdots,v_m\} \\
s.t. \quad & \sum_{i=1}^{m} v_i x_{io}^{L} = 1, \\
& \sum_{r=1}^{s} u_r y_{rj}^{L} - \sum_{i=1}^{m} v_i x_{ij}^{U} + u_0 \leq 0, \quad j = 1,\ldots,n, j \neq o, \\
& \sum_{r=1}^{s} u_r y_{ro}^{U} - \sum_{i=1}^{m} v_i x_{io}^{L} + u_0 \leq 0, \\
& \sum_{r=1}^{s} u_r y_{ro}^{U} + u_0 = E_o^{OP}, \\
& u_r \geq 0, \qquad r = 1,\ldots,s, \\
& v_i \geq 0, \qquad i = 1,\ldots,m.
\end{aligned} \tag{7}
$$

The strong defining supporting hyperplane:

$$
\begin{aligned}
\max_{u_1,\cdots,u_s,v_1,\cdots,v_m\geq 0} \quad & \min\{u_1,\cdots,u_s,v_1,\cdots,v_m\} \\
s.t. \quad & \sum_{i=1}^{m} v_i x_{io}^{L} = 1, \\
& \sum_{r=1}^{s} u_r y_{rj}^{L} - \sum_{i=1}^{m} v_i x_{ij}^{U} + u_0 \leq 0, \quad j = 1,\ldots,n, j \neq o, \\
& \sum_{r=1}^{s} u_r y_{ro}^{U} - \sum_{i=1}^{m} v_i x_{io}^{L} + u_0 \leq 0, \\
& \sum_{r=1}^{s} u_r y_{ro}^{U} + u_0 = E_o^{OP}, \\
& u_r \geq 0, \qquad r = 1,\ldots,s, \\
& v_i \geq 0, \qquad i = 1,\ldots,m.
\end{aligned} \tag{8}
$$

As we said before, models (5) and (6) are always feasible, so, models (7) and (8), obtained by applying the Charnes and Cooper transformation on models (5) and (6), are always feasible. The aim of model (7) is to find the input and output weights such that the slope of the corresponding defining supporting hyperplane on $T_v$ at the optimistic efficient unit be minimum. If the optimal value of model (7) is equal to zero, then there is a set of weights in which at least one of the input or output weights is equal to zero. So, there is a weak defining supporting hyperplane which the unit under evaluation lies on. Similarly, the aim of model (8) is to find the input and output weights such that the slope of the corresponding defining supporting hyperplane on $T_v$ at the optimistic efficient unit be maximum. If the optimal value of model (8) is positive, then there is a set of weights in which all of the input and output weights are positive. Hence, there is a strong defining supporting hyperplane which the unit under evaluation lies on. Then, we use the idea of Khazaeyan et al. [27] to convert models (7) and (8) to the single-objective models (9) and (10), respectively:

The weak defining supporting hyperplane:

$$
\begin{array}{lll}
\min & \zeta & \\
s.t. & \zeta \leq u_r, & r = 1, \cdots, s, \\
& \zeta \leq v_i, & i = 1, \cdots, m, \\
& \sum_{r=1}^{s} \mu_r + \sum_{i=1}^{m} w_i = 1, & \\
& \zeta = \sum_{r=1}^{s} \mu_r u_r + \sum_{i=1}^{m} w_i v_i, & \\
& \sum_{i=1}^{m} v_i x_{io}^{L} = 1, & \\
& \sum_{r=1}^{s} u_r y_{rj}^{L} - \sum_{i=1}^{m} v_i x_{ij}^{U} + u_0 \leq 0, & j = 1, \ldots, n, j \neq o, \\
& \sum_{r=1}^{s} u_r y_{ro}^{U} - \sum_{i=1}^{m} v_i x_{io}^{L} + u_0 \leq 0, & \\
& \sum_{r=1}^{s} u_r y_{ro}^{U} + u_0 = E_o^{OP}, &
\end{array}
\tag{9}
$$

$$\begin{aligned} & u_r \geq 0, \mu_r \in \{0,1\} && r=1,\ldots,s, \\ & v_i \geq 0, w_i \in \{0,1\} && i=1,\ldots,m. \end{aligned}$$

The strong defining supporting hyperplane:

$$\begin{aligned} \max \quad & \psi \\ s.t. \quad & \psi \leq u_r, && r=1,\cdots,s, \\ & \psi \leq v_i, && i=1,\cdots,m, \\ & \sum_{i=1}^{m} v_i x_{io}^{L} = 1, \\ & \sum_{r=1}^{s} u_r y_{rj}^{L} - \sum_{i=1}^{m} v_i x_{ij}^{U} + u_0 \leq 0, && j=1,\ldots,n, j \neq o, \\ & \sum_{r=1}^{s} u_r y_{ro}^{U} - \sum_{i=1}^{m} v_i x_{io}^{L} + u_0 \leq 0, \\ & \sum_{r=1}^{s} u_r y_{ro}^{U} + u_0 = E_o^{OP}, \\ & u_r \geq 0, && r=1,\ldots,s, \\ & v_i \geq 0, && i=1,\ldots,m. \end{aligned} \tag{10}$$

Models (9) and (10) find two defining supporting hyperplanes which $DMU_o$ lies on both of them. According to the definition of the anchor points, a DMU is an anchor point if it is an extreme efficient unit which lies on both weak and strong defining hyperplanes, hence, if one of the defining supporting hyperplanes, determined by models (9) and (10), is the weak hyperplane and another is the strong hyperplane, then $DMU_o$ is an anchor point. This means that, if the optimal value of model (9) is equal to zero and the optimal value of model (10) is non-zero, then the unit under evaluation is an anchor point.

Now, we summarize the proposed method in an algorithm for more clarity.

**Algorithm**

**Step 1.** Solve model (4) to find the optimistic efficient units in the presence of interval data and define the set E as the set of the optimistic efficient units.

**Step 2.** Solve models (9) and (10) for all $DMU_o, o \in E$, to find two defining supporting hyperplanes with the minimum and maximum slopes among the supporting hyperplanes on $T_v$ at $DMU_o$.
**Step 3.** If the optimal value of model (9) is equal to zero and the optimal value of model (10) is positive, then this unit is an anchor point. Otherwise, it is not an anchor point.
**Step 4.** End

In summary, in this section, a new method for finding the anchor points of $T_v$, in the case of data uncertainty, has been proposed which uses the definition of the anchor points to ensure that it finds all that units. Given that, each anchor point is an extreme efficient unit which is located on the intersection of the strong efficient frontier and the weak efficient frontier. So, we focused on finding the anchor points by searching both weak and strong defining supporting hyperplanes in the PPS with interval data. For this purpose, we developed the method of Khazaeyan et al. [27] to the situation that the data can be reported as interval data and proposed two uncertain multi-objective models for determining the weak and strong defining supporting hyperplanes. Then, the uncertain models have been converted into the certain models by using the robust optimization technique. Hence, in addition to find the anchor points, we can determine two of the most important defining supporting hyperplanes, which can be used in the different problems in the DEA literature. Finally, two multi-objective certain models were transformed into the single objective models by using some techniques. If the proposed method finds the weak and strong defining supporting hyperplanes, which the unit under evaluation is located on them, then this unit is an anchor point.

## 5 Numerical Examples

This section provides two numerical examples for illustrating the potential of the proposed method.

**Example 5.1.** Consider five DMUs with interval data. Each DMU consumes one input to produce one output. The second and third columns of Table 1 report the data and figure 1 shows the PPS. Columns 4 and 5 of Table 1 show the optimistic efficiency score of units and the status

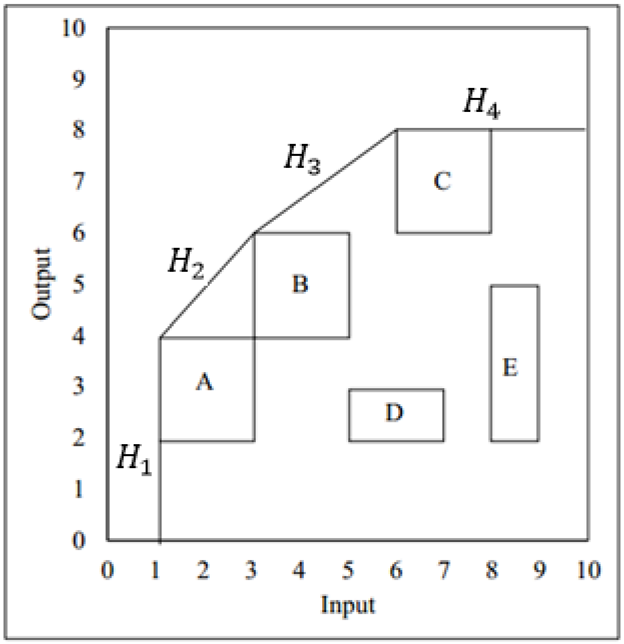


**Figure 1:** The PPS for five DMUs in Example 5.1.

of units, respectively. As we see, the optimistic efficiency score of units $A, B$ and C are equal to 1 and hence, they are efficient. So, $E = A, B, C$ is the set of efficient units. The optimistic efficiency score of unit $D$ and $E$ are less than 1 and hence, they are inefficient. The next step of the proposed method solves models (9) and (10) to find the weak and strong defining supporting hyperplanes on $T_v$ at $DMU_o, o \in E$, if they exist. The optimal value of these models are reported in Table 2. Columns 2 and 3 of Table 2 report the optimal value of models (9) and (10), respectively. If $\zeta^* = 0$ and $\psi^* > 0$, then the unit under evaluation is located on the weak and strong defining supporting hyperplanes and so this unit is an anchor point. The last column of this table reports the status of each unit.

Therefore, the anchor points are the units $A$ and $C$. Given that each anchor point is located on weak and strong defining supporting hyperplanes, hence, we can report the hyperplanes which at least one

**Table 1:** The data and obtained results for five DMUs in Example 5.1.

| DMU | Input | Output | $E_o^{OP}$ | Efficient |
|---|---|---|---|---|
| $A$ | $[1, 3]$ | $[2, 4]$ | 1.00 | Yes |
| $B$ | $[3, 5]$ | $[4, 6]$ | 1.00 | Yes |
| $C$ | $[6, 8]$ | $[6, 8]$ | 1.00 | Yes |
| $D$ | $[5, 7]$ | $[2, 3]$ | 0.20 | No |
| $E$ | $[8, 9]$ | $[2, 5]$ | 0.25 | No |

**Table 2:** The optimal value of models (9) and (10).

| DMU | $\zeta^*$ | $\psi^*$ | Anchor point |
|---|---|---|---|
| $A$ | 0.0000 | 0.7019 | Yes |
| $B$ | 0.5293 | 0.7019 | No |
| $C$ | 0.0000 | 0.5293 | Yes |
| $D$ | - | - | - |
| $E$ | - | - | - |

anchor point is located on them. Figure 1 shows that there are four defining supporting hyperplanes, $H_1, H_2, H_3$ and $H_4$, which $H_2, H_3$ are strong defining supporting hyperplanes and $H_1, H_4$ are weak defining supporting hyperplanes. It is also clear from Figure 1 that unit $A$ and unit $C$ are the efficient units which are located on the weak and strong defining supporting hyperplanes and so, these units are anchor points.

**Example 5.2.** This example uses a the dataset, includes 20 Iranian banks with three inputs, number of staffs ($x_1$), computer terminals ($x_2$) and space ($x_3$) to produce three outputs, deposits ($y_1$), loans ($y_2$) and charge ($y_3$). The data is summarized in tables 3, 4 and 5.

The first step of the proposed method determines the optimistic efficiency score of units by solving model (4). The results are reported in the last column of Table 4. As can be seen in tables 4 and 5, the set of efficient units is $E = \{1, 2, 3, 4, 7, 8, 9, 12, 15, 17\}$. Next, models (9) and (10) are solved to find the defining supporting hyperplanes on

**Table 3:** The inputs of Iranian banks in Example 5.2.

| DMU | $x_1^L$ | $x_1^U$ | $x_2^L$ | $x_2^U$ | $x_3^L$ | $x_3^U$ |
|---|---|---|---|---|---|---|
| 1 | 0.7602 | 1.1404 | 0.56 | 0.84 | 0.1240 | 0.1860 |
| 2 | 0.6370 | 0.9554 | 0.48 | 0.72 | 0.8000 | 1.2000 |
| 3 | 0.6386 | 0.9578 | 0.60 | 0.90 | 0.4100 | 0.6150 |
| 4 | 0.6921 | 1.0381 | 0.44 | 0.66 | 0.1680 | 0.2520 |
| 5 | 0.6521 | 0.9781 | 0.68 | 1.02 | 0.2140 | 0.3210 |
| 6 | 0.6733 | 1.0099 | 0.52 | 0.78 | 0.4000 | 0.6000 |
| 7 | 0.5751 | 0.8627 | 0.48 | 0.72 | 0.2800 | 0.4200 |
| 8 | 0.6282 | 0.9424 | 0.60 | 0.90 | 0.0960 | 0.1440 |
| 9 | 0.3805 | 0.5707 | 0.48 | 0.72 | 0.1080 | 0.1620 |
| 10 | 0.5426 | 0.8138 | 0.44 | 0.66 | 0.4080 | 0.6120 |
| 11 | 0.5690 | 0.8534 | 0.80 | 1.20 | 0.2440 | 0.3660 |
| 12 | 0.6490 | 0.9736 | 0.52 | 0.78 | 0.2040 | 0.3060 |
| 13 | 0.5269 | 0.7903 | 0.68 | 1.02 | 0.2720 | 0.4080 |
| 14 | 0.7810 | 1.1716 | 0.64 | 0.96 | 0.4320 | 0.6480 |
| 15 | 0.5476 | 0.8214 | 0.76 | 1.14 | 0.3600 | 0.5400 |
| 16 | 0.4902 | 0.7352 | 0.72 | 1.08 | 0.4200 | 0.6300 |
| 17 | 0.8000 | 1.2000 | 0.48 | 0.72 | 0.1640 | 0.2460 |
| 18 | 0.5070 | 0.7604 | 0.52 | 0.78 | 0.1880 | 0.2820 |
| 19 | 0.2972 | 0.4458 | 0.56 | 0.84 | 0.1896 | 0.2844 |
| 20 | 0.4662 | 0.6992 | 0.44 | 0.66 | 0.4000 | 0.6000 |

**Table 4:** The outputs and the optimistic efficiency of Iranian banks in Example 5.2.

| DMU | $y_1^L$ | $y_1^U$ | $y_2^L$ | $y_2^U$ | $y_3^L$ | $y_3^U$ | $E_o^{op}$ |
|---|---|---|---|---|---|---|---|
| 1 | 0.1520 | 0.2280 | 0.4171 | 0.6257 | 0.2341 | 0.3511 | 1.0000 |
| 2 | 0.1813 | 0.2719 | 0.5019 | 0.7529 | 0.3699 | 0.5549 | 1.0000 |
| 3 | 0.1826 | 0.2740 | 0.7762 | 1.1644 | 0.2085 | 0.3127 | 1.0000 |
| 4 | 0.1542 | 0.2312 | 0.5059 | 0.7589 | 0.8000 | 1.2000 | 1.0000 |
| 5 | 0.1866 | 0.2800 | 0.5777 | 0.8665 | 0.1970 | 0.2956 | 0.9453 |
| 6 | 0.1655 | 0.2483 | 0.4820 | 0.7230 | 0.4551 | 0.6827 | 0.8901 |

**Table 5:** Continued Table 4.

| DMU | $y_1^L$ | $y_1^U$ | $y_2^L$ | $y_2^U$ | $y_3^L$ | $y_3^U$ | $E_o^{op}$ |
|---|---|---|---|---|---|---|---|
| 7 | 0.1459 | 0.2189 | 0.7200 | 1.0800 | 0.5726 | 0.8590 | 1.0000 |
| 8 | 0.1000 | 0.1500 | 0.1872 | 0.2808 | 0.2382 | 0.3572 | 1.0000 |
| 9 | 0.0641 | 0.0961 | 0.2914 | 0.4372 | 0.1951 | 0.2927 | 1.0000 |
| 10 | 0.0654 | 0.0982 | 0.1468 | 0.2202 | 0.0389 | 0.0583 | 0.6471 |
| 11 | 0.1694 | 0.2540 | 0.2543 | 0.3815 | 0.3225 | 0.4837 | 0.8527 |
| 12 | 0.0982 | 0.1472 | 0.7380 | 1.1070 | 0.5023 | 0.7535 | 1.0000 |
| 13 | 0.1404 | 0.2106 | 0.5162 | 0.7742 | 0.2084 | 0.3126 | 0.9564 |
| 14 | 0.1154 | 0.1732 | 0.4114 | 0.6172 | 0.1946 | 0.2920 | 0.7432 |
| 15 | 0.8000 | 1.2000 | 0.2334 | 0.3500 | 0.0786 | 0.1178 | 1.0000 |
| 16 | 0.0921 | 0.1381 | 0.3217 | 0.4825 | 0.3713 | 0.5569 | 0.8759 |
| 17 | 0.0720 | 0.1080 | 0.8000 | 1.2000 | 0.1291 | 0.1937 | 1.0000 |
| 18 | 0.0473 | 0.0709 | 0.2794 | 0.4190 | 0.0542 | 0.0814 | 0.9105 |
| 19 | 0.0308 | 0.0462 | 0.1518 | 0.2278 | 0.0890 | 0.1334 | 0.9185 |
| 20 | 0.0881 | 0.1321 | 0.4916 | 0.7374 | 0.6114 | 0.9172 | 0.9324 |

$T_v$ at $DMU_o, o \in E$. The obtained results are summarized in Table 6. Columns 2 and 3 of this table report the optimal value of models (9) and (10), respectively. If $\zeta^* = 0$ and $\psi^* > 0$, then the unit under evaluation is located on the weak and strong defining supporting hyperplanes and so this unit is an anchor point. The last column of this table reports the status of each unit.

Therefore, the anchor points are the units 1, 2, 7, 8, 12, 15 and 17. Given that each anchor point is located on weak and strong defining supporting hyperplanes, hence, we can report the hyperplanes which at least one anchor point is located on them. The weak and strong defining hyperplanes, determined by models (9) and (10), are summarized in Table 7 and Table 8. In each row of these tables, the first hyperplane is the weak defining supporting hyperplane on $T_v$ at the anchor point and the second hyperplane is the strong defining supporting hyperplane on $T_v$ at the anchor point, determined by models (9) and (10), respectively.

**Table 6:** The optimal value of models (9) and (10).

| DMU | $\zeta^*$ | $\psi^*$ | Anchor point |
|---|---|---|---|
| 1 | 0.0000 | 0.6853 | YES |
| 2 | 0.0000 | 0.7421 | YES |
| 3 | 0.2732 | 0.6140 | NO |
| 4 | 0.3118 | 0.7964 | NO |
| 5 | - | - | - |
| 6 | - | - | - |
| 7 | 0.0000 | 0.5418 | YES |
| 8 | 0.0000 | 0.4225 | YES |
| 9 | 0.3212 | 0.5103 | NO |
| 10 | - | - | - |
| 11 | - | - | - |
| 12 | 0.0000 | 0.7163 | YES |
| 13 | - | - | - |
| 14 | - | - | - |
| 15 | 0.0000 | 0.2114 | YES |
| 16 | - | - | - |
| 17 | 0.0000 | 0.6452 | YES |
| 18 | - | - | - |
| 19 | - | - | - |
| 20 | - | - | - |

**Table 7:** The weak and the strong defining supporting hyperplane on $T_v$ at anchor points.

| DMU | Weak hyperplane and strong hyperplane |
|---|---|
| 1 | $0.2311y_1^U + 0.0314y_2^U - 0.5421x_2^L - 0.2312 = 0$ |
| | $0.0142y_1^U + 0.4344y_2^U + 0.0231y_3^U - 0.0014x_1^L - 0.2341x_2^L - 0.2298x_3^L - 0.1225 = 0$ |

**Table 8:** The weak and the strong defining supporting hyperplane on $T_v$ at anchor points.

| DMU | Weak hyperplane and strong hyperplane |
|---|---|
| 2 | $0.4381y_2^U - 0.6432x_1^L - 0.1230x_2^L + 0.1389 = 0$ <br> $0.0784y_1^U + 0.5042y_2^U + 0.3193y_3^U - 0.2345x_1^L - 0.3212x_2^L - 0.0015x_3^L - 0.2734 = 0$ |
| 7 | $0.5124y_1^U + 0.2354y_3^U - 0.6032x_1^L + 0.0325 = 0$ <br> $0.6119y_1^U + 0.5438y_2^U + 0.4230y_3^U - 0.1325x_1^L - 0.2221x_2^L - 0.1398x_3^L - 0.9836 = 0$ |
| 8 | $0.4026y_2^U + 0.0059y_3^U - 0.4031x_1^L - 0.5113x_3^L + 0.1872 = 0$ <br> $0.0876y_1^U + 0.5439y_2^U + 0.8125y_3^U - 0.3091x_1^L - 0.5112x_2^L - 0.0014x_3^L + 0.0449 = 0$ |
| 12 | $0.5678y_1^U + 0.9113y_2^U - 0.3915x_1^L - 0.5162 = 0$ <br> $0.6584y_1^U + 0.4251y_2^U + 0.1238y_3^U - 0.2410x_1^L - 0.4563x_2^L - 0.1032x_3^L - 0.2460 = 0$ |
| 15 | $0.4764y_1^U + 0.3459y_2^U - 0.5421x_2^L - 0.2807 = 0$ <br> $0.0871y_1^U + 0.3569y_2^U + 0.8921y_3^U - 0.0142x_1^L - 0.6219x_2^L - 0.1459x_3^L + 0.1984 = 0$ |
| 17 | $0.4568y_2^U - 0.2543x_1^L - 0.3429x_2^L - 0.1801 = 0$ <br> $0.4901y_1^U + 0.1458y_2^U + 0.6325y_3^U - 0.1235x_1^L - 0.0235x_2^L - 0.0056x_3^L - 0.2394 = 0$ |

## 6 Conclusion

One of the most well-known set in each production possibility set (PPS) is the set of anchor points. This paper focused on finding the anchor points with interval data in the PPS under the variable returns to scale assumption and presented a new method to find the anchor points. For this purpose, Regarding the definition of the anchor points, the proposed method was based on finding the weak and strong defining supporting hyperplanes passing through the unit under evaluation. For this purpose, we developed the method of Khazaeyan et al. [27] and proposed two models for determining the supporting hyperplanes with the minimum and maximum slopes among the supporting hyperplanes on the PPS at the unit under assessment in the presence of interval data. For the first time, we proposed a method to determine the anchor points with interval data by using the weak and strong defining supporting hyperplanes. The potentially of the proposed method was illustrated by a numerical example, reported in the DEA literature.

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