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Finite k-Projective Dimension and Generalized Auslander-Buchsbaum Inequality and Intersection Theorem

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Abstract. Let R be a commutative Noetherian ring, M be a finitely generated R-module and $\mathfrak{a}$ be an ideal of R. For an arbitrary integer $k \geq -1$, we introduce the concept of k-projective dimension of M denoted by $k\text{-pd}_R M$. We show that the finite k-projective dimension of M is at least $k\text{-depth}(\mathfrak{a}, R) - k\text{-depth}(\mathfrak{a}, M)$. As a generalization of the Intersection Theorem, we show that for any finitely generated R-module N, in certain conditions, $k\text{-pd}_R M$ is nearer upper bound for $\dim N$ than $\text{pd}_R M$. Finally, if M is k-perfect, $\dim N \leq k\text{-grade} M$ that generalizes the Strong Intersection Theorem.

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1 Introduction

Throughout this paper, R denotes a commutative Noetherian ring with non-zero identity, M denotes a finitely generated R-module, and $k \geq -1$ is an arbitrary integer. For a subset T of Spec$R$, we set

$$(T)_{>k} := \{\mathfrak{p} \in T \mid \dim R/\mathfrak{p} > k\},$$
$$(T)_{\leq k} := \{\mathfrak{p} \in T \mid \dim R/\mathfrak{p} \leq k\}.$$

This paper is essentially devoted to generalize some interesting conjectures in commutative algebra, which deal with the concept of depth of a module. An effective instrument for the computation of the depth of a module is the Auslander-Buchsbaum Formula which is related to its projective dimension. There are various generalizations of the depth of a module. The notion of k-regular sequence was introduced by Chinh and Nhan [4] which is an extension of the well-known notion of filter regular sequence introduced by Schenzel, Trung, and Cuong [8] and the notion of regular sequence as well. So it is important to know what does happen to the Strong Intersection Theorem for filter regular sequences or even more general for k-regular sequences ($k \geq -1$).

To generalize the Intersection Theorem, we need to generalize the Auslander-Buchsbaum inequality to derive a relation between the k-depth of an arbitrary ideal $\mathfrak{a}$ of R and the k-projective dimension on a not necessarily local ring. For this purpose, firstly, in section 2, we introduce the concept of k-projective dimension which is an extension of the well-known notion of projective dimension (for $k \geq -1$). Then we give some properties of k-projective dimension which we shall need in the sequel.

In section 3, we are concerned with the k-projective dimension of M/xM, where x is a k-regular element on both R-module M and R. The main result of this section is the following theorem.

Theorem 1. *(Generalized Second Change of Rings Theorem). Let M be an R-module and x be a poor k-regular element on both M and R. Then k-$\mathrm{pd}_R M \geq k$-$\mathrm{pd}_{R/xR} M/xM$ (see Theorem 3.2).*

As a consequence of this theorem, we get that k-$\mathrm{pd}_R M/xM \leq 1 + k$-$\mathrm{pd}_R M$ for any poor k-reqular element x on both M and R (see Corollary 3.3).

Classically, there exist two theorems which relate two k-projective dimensions of M and M/xM over two rings R and R/xR. Most of these properties are familiar results for the case $k=-1$, cf. [3], [7], and [10]. The main generalizations of this paper appear in section 4. Recall that the projective dimension of an R-module is related to its depth as follows.

Auslander-Buchsbaum Formula. ([3, Theorem 1.3.3]). Let R be a local ring and M be a non-zero finitely generated R-module of finite projective dimension. Then

$$\mathrm{pd}_R M + \mathrm{depth} M = \mathrm{depth} R.$$

The following theorem presents a generalized inequality of Auslander-Buchsbaum Formula for an arbitrary ideal and any Noetherian ring.

Theorem 2. *Let M be an R-module with finite k-projective dimension, and $\mathfrak{a}$ be an ideal of R such that $(\mathrm{Supp} R/\mathfrak{a})_{>k} \neq \emptyset$. Then k-$\mathrm{pd}_R M \geq k$-$\mathrm{depth}(\mathfrak{a}, R) - k$-$\mathrm{depth}(\mathfrak{a}, M)$ (see Theorem 4.1).*

Conjecture (The Dimension Inequality). Let R be a local ring, M and N be finitely generated R-modules with $\ell(M \otimes_R N) < \infty$. Assume that $\mathrm{pd}_R M < \infty$. Then

$$\dim M + \dim N \leq \dim R.$$

Serre in [9] proved the above cojecture for a regular local ring R, for finitely generated R-modules M and N with $\ell(M \otimes_R N) < \infty$. Since R is a regular local ring, $\dim R = \mathrm{depth} R$. Therefore by Auslander-Buchsbaum Formula, we have $\dim N \leq \mathrm{pd}_R M$. As a regular local ring has finite projective dimension, in 1973, Peskine and Szpiro used the above formulation to generalize Serre's Theorem, replacing the condition that R is regular local ring with the much weaker condition that $\mathrm{pd}_R M < \infty$.

The Intersection Theorem. [6] Let R be a local ring and let M and N be finitely generated R-modules with $\ell(M \otimes_R N) < \infty$. Assume that $\mathrm{pd}_R M < \infty$. Then $\dim N \leq \mathrm{pd}_R M$.
One of our main results in this paper is to show that $\dim N$ is at most k-projective dimension of M which is nearer upper bound for $\dim N$ than projective dimension of M, (see Example 2.12 and Theorem 4.6).

Theorem 3. *(Generalized Intersection Theorem). Let R be a k-Cohen-Macaulay ring. Let M and N be finitely generated R-modules and $\mathfrak{a}$ be an ideal of R with $(\mathrm{Supp}R/\mathfrak{a})_{>k} \neq \emptyset$ such that $\dim M + \dim N \leq k\text{-}\mathrm{ht}_R\mathfrak{a}$. Assume that $k\text{-}\mathrm{pd}_R M < \infty$. Then $\dim N \leq k\text{-}\mathrm{pd}_R M$ (see Theorem 4.6).*

For a perfect R-module M the following conjecture follows immediately from the Intersection Theorem. Also, for k-perfect modules, we prove a generalization of the following conjecture (see Theorem 4.11).

The Strong Intersection Conjecture.[2] Let R be a local ring and let M and N be finitely generated R-modules with $\ell(M \otimes_R N) < \infty$. Assume that $\mathrm{pd}_R M < \infty$. Then

$$\dim N \leq \mathrm{grade} M.$$

Theorem 4. *Let R be a k-Cohen-Macaulay ring. Let M and N be finitely generated R-modules and $\mathfrak{a}$ be an ideal of R with $(\mathrm{Supp}M/\mathfrak{a}M)_{>k} \neq \emptyset$, such that $\dim M + \dim N \leq k\text{-}\mathrm{ht}_R\mathfrak{a}$. Assume that M is k-perfect with $k\text{-}\mathrm{pd}_R M < \infty$. Then $\dim N \leq k\text{-}\mathrm{grade} M$ (see Theorem 4.11).*

Ischebeck Inequality. Let R be a local ring and M be a non-zero R-module. It is well-known that for any $\mathfrak{p} \in \mathrm{Ass}M$, $\mathrm{depth} M \leq \dim R/\mathfrak{p}$. Also for any prime ideal $\mathfrak{p}$ of R,

$$\mathrm{depth} M \leq \mathrm{depth}(\mathfrak{p}, M) + \dim R/\mathfrak{p}.$$

In a Noetherian ring R, we say that $(*)$ holds for the ideal $\mathfrak{a}$ of R with $(\mathrm{Supp}R/\mathfrak{a})_{>k} \neq \emptyset$, if for any $\mathfrak{p} \in \mathrm{Ass}R$,

$$k\text{-}\mathrm{depth}(\mathfrak{a}, R) \leq \dim R/\mathfrak{p}.$$

The Grade Conjecture. ([6, Conjecture (f) of Chapter II]) Let R be a local ring and M be a finitely generated R-module with $\mathrm{pd}_R M < \infty$. Then

$$\mathrm{grade} M + \dim M = \dim R.$$

To generalize the Grade Conjecture, we prove the following Theorem (see Theorem 4.8).

Theorem 5. *Let R be a Noetherian ring and $\mathfrak{a}$ be an ideal of R and M be a finitely generated R-module with $(\mathrm{Supp}M/\mathfrak{a}M)_{>k} \neq \emptyset$. If $(*)$ holds for $\mathfrak{a}$, then we have the double inequality:*

$$k\text{-}\mathrm{depth}(\mathfrak{a}, R) \leq k\text{-}\mathrm{grade} M + \dim M \leq \dim R.$$

2 Preliminaries

In this section, we shall deal with a particular generalization of the concept of projective dimension called k-projective dimension. First, we recall the concept of k-regular sequences as introduced by Chinh and Nhan in [4].

Definition 2.1. *[1] Let M be an R-module. A sequence $a_1, ..., a_n$ of elements of R is called a poor k-regular M-sequence whenever $a_i \notin \mathfrak{p}$ for all*

$$\mathfrak{p} \in \mathrm{Ass}(M/\sum_{j=1}^{i-1} a_j M),\ \dim R/\mathfrak{p} > k$$

for all $i = 1, ..., n$. Moreover, if $\dim(M/\sum_{i=1}^{n} a_i M) > k$, $a_1, ..., a_n$ is called a k-regular M-sequence. An element a of R is called a k-regular M-element if $a \notin \mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Ass}M$ satisfying $\dim R/\mathfrak{p} > k$.

In the case $k = -1$ ($k = 0$ and $k = 1$, resp.), we deal with the familiar concept of regular M-sequenece (filter regular M-sequence and generalized regular M-sequence, resp.). (See [8] and [5], resp.)

Definition 2.2. *[1]. Let M be an R-module and $\mathfrak{a}$ be an ideal of R with $(\mathrm{Supp}M/\mathfrak{a}M)_{>k} \neq \emptyset$. Then we denote the length of any maximal k-regular M-sequence contained in $\mathfrak{a}$ by $k\text{-depth}(\mathfrak{a}, M)$.*

In the case $k = -1$ ($k = 0$ and $k = 1$, resp.), this is the usual notion $\mathrm{depth}(\mathfrak{a}, M)$ (f-$\mathrm{depth}(\mathfrak{a}, M)$, g-$\mathrm{depth}(\mathfrak{a}, M)$, resp.)

Remark 2.3. Let $\mathfrak{a}$ be an ideal with $(\mathrm{Supp}M/\mathfrak{a}M)_{>k} \neq \emptyset$. For all $\mathfrak{p} \in \mathrm{Supp}(M/\mathfrak{a}M) > k$, we have

$$k < \dim R/\mathfrak{p} \leq \dim M/\mathfrak{a}M.$$

Therefore, whenever we are concerned with $k\text{-depth}(\mathfrak{a}, M)$, we should know that $-1 \leq k < \dim M/\mathfrak{a}M$.

By the above definition, we have the following lemmas.

Lemma 2.4. *Let M be an R-module and $\mathfrak{a}$ be an ideal of R with $(\mathrm{Supp}M/\mathfrak{a}M)_{>k} \neq \emptyset$. Then*

$$k\text{-depth}(\mathfrak{a}, M) = \min\{\text{depth}(\mathfrak{a}R_{\mathfrak{q}}, M_{\mathfrak{q}}) \mid \mathfrak{q} \in (\text{Supp}M/\mathfrak{a}M)_{>k}\}.$$

Lemma 2.5. *Let M be an R-module and $\mathfrak{a}$ be an ideal of R with $(\text{Supp}M/\mathfrak{a}M)_{>k} \neq \emptyset$. Then*

$$\text{depth}(\mathfrak{a}, M) \leq k\text{-depth}(\mathfrak{a}, M) \leq \dim M.$$

Proposition 2.6. *([1, Proposition 2.7]). Let M be an R-module, $\mathfrak{a}$ be an ideal with $(\text{Supp}M/\mathfrak{a}M)_{>k} \neq \emptyset$ and $x \in \mathfrak{a}$ be a k-regular M-element. Then*

$$k\text{-depth}(\mathfrak{a}/xR, M/xM) = k\text{-depth}(\mathfrak{a}, M/xM) = k\text{-depth}(\mathfrak{a}, M) - 1.$$

Proposition 2.7. *([1, Proposition 2.8]). Let M be an R-module, $\mathfrak{a}$ be an ideal with $(\text{Supp}M/\mathfrak{a}M)_{>k} \neq \emptyset$. Then*

$$k\text{-depth}(\mathfrak{a}, M) = \min\{\ k\text{- depth}(\mathfrak{p}, M) \mid \mathfrak{p} \in V(\mathfrak{a})\}.$$

Following, we introduce the concept of k-projective dimension as a generalization of projective dimension of an R-module.

Definition 2.8. *Let M be an R-module. The k-projective dimension of M denoted by $k\text{-pd}_R M$, is defined*
$k\text{-pd}_R M = \sup\{i \in \mathbb{N}_0 | \exists N, (\text{Supp}N)_{>k} \neq \emptyset \ s.t \ \dim\text{Ext}^i_R(M, N) > k\}$,
if sup *exists; otherwise, we define $k\text{-pd}_R M = -\infty$.*
By this convention, a zero module has k-projective dimension $-\infty$. It is clear that if $(\text{Supp}M)_{>k} = \emptyset$, then $k\text{-pd}_R M = -\infty$.

In the case $k = -1$, the notion of $k\text{-pd}_R M$ is the same as $\text{pd}_R M$, the projective dimension of M.

By the above definition, we have the following results.

Lemma 2.9. *Let M be an R-module and t be a non-negative integer. Then $k\text{-pd}_R M \leq t$ if and only if $\dim\text{Ext}^i_R(M, N) \leq k$, for all $i > t$ and all R-modules N with $(\text{Supp}N)_{>k} \neq \emptyset$.*

Lemma 2.10. *Let M be an R-module, and t be a non-negative integer. If for every $\mathfrak{p} \in (\text{Spec}R)_{>k}$, $\dim\text{Ext}^i_R(M, R/\mathfrak{p}) \leq k$, for all $i > t$, then $k\text{-pd}_R M \leq t$.*

Remark 2.11. For every R-module M, if $j \geq k$ is an integer, then $j\text{-pd}_R M \leq k\text{-pd}_R M$. Let $k\text{-pd}_R M = t$ be an integer. Then by Lemma 2.9, $\dim \mathrm{Ext}^i_R(M, N) \leq k$ for all $i > t$ and all R-modules N with $(\mathrm{Supp} N)_{>k} \neq \emptyset$. So that for all $j \geq k$, we have $\dim \mathrm{Ext}^i_R(M, N) \leq j$. Therefore $j\text{-pd}_R M \leq k\text{-pd}_R M$, as desired. Specially, $j\text{-pd}_R M \leq \mathrm{pd}_R M$ for all $j \geq -1$.

Example 2.12. It is notable that, the k-projective dimension of an R-module is not necessarily equal to its projective dimension. It is clear that $\mathbb{Z}_2$ is not projective module over $\mathbb{Z}$, in fact $\mathrm{pd}_{\mathbb{Z}} \mathbb{Z}_2 = 1$; but 0-$\mathrm{pd}_{\mathbb{Z}} \mathbb{Z}_2 \neq 1$.

In the following, we give some properties of k-projective dimension which we shall need in the sequel.

Lemma 2.13. *For every R-module M and for all $\mathfrak{p} \in \mathrm{Spec} R$, we have*

$$k\text{-pd}_{R_\mathfrak{p}} M_\mathfrak{p} \leq k\text{-pd}_R M.$$

Proof. We may assume that $k\text{-pd}_R M = t$ is an integer. Let $\mathfrak{p} \in \mathrm{Spec} R$ and X be an arbitrary $R_\mathfrak{p}$-module with $(\mathrm{Supp}_{R_\mathfrak{p}} X)_{>k} \neq \emptyset$. Then there is an R-module N such that $X = N_\mathfrak{p}$. By Lemma 2.9, $\dim \mathrm{Ext}^i_R(M, N) \leq k$ for all $i > t$, and so $\dim \mathrm{Ext}^i_{R_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}) \leq k$ for all $i > t$. It means that $k\text{-pd}_{R_\mathfrak{p}} M_\mathfrak{p} \leq t$. $\square$

Before the following lemma, we remind that for any R-module X, there exists a maximal ideal $\mathfrak{m}$ in $\mathrm{Supp} X$ such that $\dim_R X = \dim_{R_\mathfrak{m}} \mathrm{X}_\mathfrak{m}$.

Proposition 2.14. *Let M be a non-zero R-module. Then*

$$k\text{-pd}_R M = \max\{k\text{-pd}_{R_\mathfrak{m}} M_\mathfrak{m} \mid \mathfrak{m} \in \mathrm{Supp} M \cap \mathrm{Max} R\}.$$

Proof. By Lemma 2.13, $k\text{-pd}_{R_\mathfrak{m}} M_\mathfrak{m} \leq k\text{-pd}_R M$ for all $\mathfrak{m} \in \mathrm{Supp} M \cap \mathrm{Max} R$. Let $k\text{-pd}_R M = t$. Then there is an R-module N with $(\mathrm{Supp} N)_{>k}$ $\neq \emptyset$ such that $\dim \mathrm{Ext}^t_R(M, N) > k$.
Assume that $\mathfrak{m}$ is a maximal ideal in $\mathrm{Supp} \mathrm{Ext}^t_R(M, N)$ such that $\dim_R \mathrm{Ext}^t_R(M, N) = \dim_{R_\mathfrak{m}} \mathrm{Ext}^t_{R_\mathfrak{m}}(M_\mathfrak{m}, N_\mathfrak{m})$. Hence $k\text{-pd}_{R_\mathfrak{m}} M_\mathfrak{m} \geq t$ and so $k\text{-pd}_R M = k\text{-pd}_{R_\mathfrak{m}} M_\mathfrak{m}$, as required. $\square$

Now, we are ready to get a relation between the concept of projective dimension and k-projective dimension.

Proposition 2.15. *Let M be an R-module. Then*

$$k\text{-pd}_R M = \sup\{\text{pd}_{R_\mathfrak{p}} M_\mathfrak{p} \mid \mathfrak{p} \in (\text{Supp}M)_{>k}\}.$$

Proof. First assume that $k\text{-pd}_R M = t$ is an integer and $\mathfrak{p} \in (\text{Supp}M)_{>k}$. We show that $\text{Ext}^i_{R_\mathfrak{p}}(M_\mathfrak{p}, X) = 0$ for all $i > t$ and all $R_\mathfrak{p}$-modules X. Let $i > t$ and N be an R-module such that $X = N_\mathfrak{p}$. Assume that $\text{Ext}^i_{R_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}) \neq 0$. Then $\mathfrak{p} \in \text{SuppExt}^i_R(M, N)$ and so $\dim R/\mathfrak{p} \leq \dim\text{Ext}^i_R(M, N) \leq k$ which is a contradiction. Thus $\text{pd}_{R_\mathfrak{p}} M_\mathfrak{p} \leq t$, for all $\mathfrak{p} \in (\text{Supp}M)_{>k}$. On the other hand, as $k\text{-pd}_R M = t$, there exists an R-module N' with $(\text{Supp}N')_{>k} \neq \emptyset$ such that $\dim\text{Ext}^t_R(M, N') > k$. Let $\mathfrak{q} \in \text{SuppExt}^t_R(M, N')$ be such that $\dim R/\mathfrak{q} > k$. Then $\text{Ext}^t_{R_\mathfrak{q}}(M_\mathfrak{q}, N'_\mathfrak{q}) \neq 0$ and so $\text{pd}_{R_\mathfrak{q}} M_\mathfrak{q} \geq t$. But, as we have seen, $\text{pd}_{R_\mathfrak{q}} M_\mathfrak{q} \leq t$. Therefore $\text{pd}_{R_\mathfrak{q}} M_\mathfrak{q} = k\text{-pd}_R M$. Finally, if $k\text{-pd}_R M = -\infty$, then $\dim\text{Ext}^i_R(M, N) \leq k$ for all $i \geq 0$ and all R-modules N with $(\text{Supp}N)_{>k} \neq \emptyset$. Therefore for all $\mathfrak{p} \in (\text{Supp}M)_{>k}$, $\dim\text{Ext}^i_{R_\mathfrak{p}}(M_\mathfrak{p}, X) \leq k$ for all $i \geq 0$ and all $R_\mathfrak{p}$-modules X with $(\text{Supp}X)_{>k} \neq \emptyset$. That means $\text{pd}_{R_\mathfrak{p}} M_\mathfrak{p} = -\infty$ for all $\mathfrak{p} \in (\text{Supp}M)_{>k}$. $\square$

At the end of this section, we peruse the behaviour of k-depth and k-pd along exact sequences, which are needed in the next sections, and all come from definitions.

Lemma 2.16. *Let $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be a short exact sequence of R-modules, and $\mathfrak{a}$ be an ideal of R. Then*

(i) $k\text{-depth}(\mathfrak{a}, A) \geq \min\{k\text{-depth}(\mathfrak{a}, B), k\text{-depth}(\mathfrak{a}, C) + 1\}$.

(ii) $k\text{-depth}(\mathfrak{a}, B) \geq \min\{k\text{-depth}(\mathfrak{a}, A), k\text{-depth}(\mathfrak{a}, C)\}$.

(iii) $k\text{-depth}(\mathfrak{a}, C) \geq \min\{k\text{-depth}(\mathfrak{a}, B), k\text{-depth}(\mathfrak{a}, A) - 1\}$.

Corollary 2.17. *Let $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be a short exact sequence of R-modules and $\mathfrak{a}$ be an ideal of R. If* $k\text{-depth}(\mathfrak{a}, B) > \min\{k\text{-depth}(\mathfrak{a}, A), k\text{-depth}(\mathfrak{a}, C)\}$, *then* $k\text{-depth}(\mathfrak{a}, A) = k\text{-depth}(\mathfrak{a}, C) + 1$.

Lemma 2.18. *Let $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be a short exact sequence of R-modules. Then*

(i) $k\text{-pd}_R A \leq \max\{k\text{-pd}_R B, k\text{-pd}_R C - 1\}$.

(ii) $k\text{-pd}_R B \leq \max\{k\text{-pd}_R A, k\text{-pd}_R C\}$.

(iii) $k\text{-pd}_R C \leq \max\{k\text{-pd}_R B, k\text{-pd}_R A + 1\}$.

Corollary 2.19. *Let $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be a short exact sequence of R-modules. If $k\text{-pd}_R B < \max\{k\text{-pd}_R A, k\text{-pd}_R C\}$, then $k\text{-pd}_R A = k\text{-pd}_R C - 1$.*

3 Main Theorems

Theorem 3.1. *(Generalized First Change of Rings Theorem). Let x be a poor k-regular element on R and M be an R/xR-module with $k\text{-pd}_{R/xR} M < \infty$. Then*

$$k\text{-pd}_R M = 1 + k\text{-pd}_{R/xR} M.$$

Proof. For some $\mathfrak{p} \in (\mathrm{Supp} M)_{>k}$, $k\text{-pd}_R M = \mathrm{pd}_{R_\mathfrak{p}} M_\mathfrak{p}$, by Proposition 2.15. If $x \notin \mathfrak{p}$, then $x/1$ is invertible in $R_\mathfrak{p}$. Since $xM = 0$, then $x/1 M_\mathfrak{p} = 0$. That is $M_\mathfrak{p} = 0$, which is a contradiction. Therefore $x \in \mathfrak{p}$. As $x/1$ is non-zerodivisor on $R_\mathfrak{p}$ by [1, Theorem 2.3], and $M_\mathfrak{p} \neq 0$ is an $R_\mathfrak{p}/\frac{x}{1}R_\mathfrak{p}$-module with $\mathrm{pd}_{R_\mathfrak{p}/\frac{x}{1}R_\mathfrak{p}} M_\mathfrak{p} < \infty$, so we have $\mathrm{pd}_{R_\mathfrak{p}} M_\mathfrak{p} = 1 + \mathrm{pd}_{R_\mathfrak{p}/\frac{x}{1}R_\mathfrak{p}} M_\mathfrak{p}$ (e.g. see[10, Theorem 4.3.3]). Therefore $k\text{-pd}_R M \leq 1 + k\text{-pd}_{R/xR} M$. Now, let $\mathfrak{q}/xR \in (\mathrm{Supp}_{R/xR} M)_{>k}$ be such that $k\text{-pd}_{R/xR} M = \mathrm{pd}_{(R/xR)_{\mathfrak{q}/xR}} M_{\mathfrak{q}/xR}$. Thus $k\text{-pd}_{R/xR} M = \mathrm{pd}_{R_\mathfrak{q}/\frac{x}{1}R_\mathfrak{q}} M_\mathfrak{q}$. Again using [10, Theorem 4.3.3], we obtain $\mathrm{pd}_{R_\mathfrak{q}/\frac{x}{1}R_\mathfrak{q}} M_\mathfrak{q} = \mathrm{pd}_{R_\mathfrak{q}} M_\mathfrak{q} - 1 \leq k\text{-pd}_R M - 1$, which completes the proof. □

Theorem 3.2. *(Generalized Second Change of Rings Theorem). Let M be an R-module and x be a poor k-regular element on both M and R. Then*

$$k\text{-pd}_R M \geq k\text{-pd}_{R/xR} M/xM.$$

Proof. If $k\text{-pd}_{R/xR} M/xM = -\infty$, then there is nothing to prove. So we assume that it is finite. Let $k\text{-pd}_{R/xR} M/xM = \mathrm{pd}_{(R/xR)_{\mathfrak{p}/xR}} (M/xM)_{\mathfrak{p}/xR}$, for some $\mathfrak{p}/xR \in (\mathrm{Supp}_{R/xR} M/xM)_{>k}$, by Proposition 2.15. We have $\mathrm{pd}_{R_\mathfrak{p}/(\frac{x}{1}R_\mathfrak{p})} M_\mathfrak{p}/\frac{x}{1}M_\mathfrak{p} \leq \mathrm{pd}_{R_\mathfrak{p}} M_\mathfrak{p}$, by [10, Theorem 4.3.5]. But, by Proposition 2.15, $\mathrm{pd}_{R_\mathfrak{p}} M_\mathfrak{p} \leq k\text{-pd}_R M$. This proves the theorem. □

Corollary 3.3. *Let M be an R-module with k-$\mathrm{pd}_R M < \infty$. If x is a poor k-regular element on both M and R, then*

$$k\text{-}\mathrm{pd}_R M/xM \leq 1 + k\text{-}\mathrm{pd}_R M.$$

Proof. Combine the First and Second Generalized Change of Rings Theorems. □

Corollary 3.4. *Let M be an R-module. Let $x_1, \cdots, x_r$ be a poor k-regular sequence on both M and R. Then*

$$k\text{-}\mathrm{pd}_R M/(x_1, ..., x_r)M \leq r + k\text{-}\mathrm{pd}_R M.$$

Proof. The proof is by induction on r and using Corollary 3.3. □

4 Main Generalizations

The following theorem is a generalization of a part of the Auslander-Buchsbaum Formula. This formula shows the relation between k-depth and k-projective dimension of an R-module.

Theorem 4.1. *Let R be a Noetherian (not necessarily local) ring and M be an R-module with finite k-projective dimension. Let $\mathfrak{a}$ be an ideal of R such that $(\mathrm{Supp} R/\mathfrak{a})_{>k} \neq \emptyset$. Then*

$$k\text{-}\mathrm{pd}_R M \geq k\text{-}\mathrm{depth}(\mathfrak{a}, R) - k\text{-}\mathrm{depth}(\mathfrak{a}, M).$$

Proof. We prove by induction on k-$\mathrm{pd}_R M = n$. If $n = 0$, then $\mathrm{pd}_{R_\mathfrak{p}} M_\mathfrak{p} = 0$, for all $\mathfrak{p}$ in $(\mathrm{Supp} M)_{>k}$, by Proposition 2.15. It means that $M_\mathfrak{p}$ is a direct sum of finite copies of $R_\mathfrak{p}$, for all $\mathfrak{p} \in (\mathrm{Supp} M)_{>k}$ and so $\mathrm{depth}(\mathfrak{a}R_\mathfrak{p}, M_\mathfrak{p}) = \mathrm{depth}(\mathfrak{a}R_\mathfrak{p}, R_\mathfrak{p})$. Therefore by Lemma 2.4,

$$\begin{aligned} k - \mathrm{depth}(\mathfrak{a}, M) &= \min\{\mathrm{depth}(\mathfrak{a}R_\mathfrak{q}, M_\mathfrak{q}) \mid \mathfrak{q} \in (\mathrm{Supp} M/\mathfrak{a}M)_{>k}\} \\ &= \min\{\mathrm{depth}(\mathfrak{a}R_\mathfrak{q}, R_\mathfrak{q}) \mid \mathfrak{q} \in (\mathrm{Supp} M/\mathfrak{a}M)_{>k}\} \\ &\geq \min\{\mathrm{depth}(\mathfrak{a}R_\mathfrak{q}, R_\mathfrak{q}) \mid \mathfrak{q} \in (\mathrm{Supp} R/\mathfrak{a})_{>k}\} \\ &= k - \mathrm{depth}(\mathfrak{a}, R), \end{aligned}$$

as desired.
Now, suppose that $n > 0$ and the result has been proved for smaller

values of n. Hence $\mathrm{pd}_R M > 0$ and there exists an exact sequence $0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$ where F is a free R-module and K is a non-zero R-module. By Corollary 2.19, we have $k\text{-}\mathrm{pd}_R K = k\text{-}\mathrm{pd}_R M - 1$, so by inductive hypothesis, $k\text{-}\mathrm{pd}_R M - 1 \geq k\text{-depth}(\mathfrak{a}, R) - k\text{-depth}(\mathfrak{a}, K)$. If $k\text{-depth}(\mathfrak{a}, M) \geq k\text{-depth}(\mathfrak{a}, K)$, then $k\text{-}\mathrm{pd}_R M - 1 \geq k\text{-depth}(\mathfrak{a}, R) - k\text{-depth}(\mathfrak{a}, K) > k\text{-depth}(\mathfrak{a}, R) - k\text{-depth}(\mathfrak{a}, M) - 1$, so $k\text{-}\mathrm{pd}_R M > k\text{-depth}(\mathfrak{a}, R) - k\text{-depth}(\mathfrak{a}, M)$, as desired.
Now, suppose that $k\text{-depth}(\mathfrak{a}, K) > k\text{-depth}(\mathfrak{a}, M)$. Then $k\text{-depth}(\mathfrak{a}, R) \geq k\text{-depth}(\mathfrak{a}, M)$, by Lemma 2.16(ii). If $k\text{-depth}(\mathfrak{a}, R) = k\text{-depth}(\mathfrak{a}, M)$, then the assertion holds. Now, let $k\text{-depth}(\mathfrak{a}, R) > k\text{-depth}(\mathfrak{a}, M)$, then since $k\text{-depth}(\mathfrak{a}, R) > \min\{k\text{-depth}(\mathfrak{a}, M), k\text{-depth}(\mathfrak{a}, K)\}$, we get $k\text{-depth}(\mathfrak{a}, K) = k\text{-depth}(\mathfrak{a}, M) + 1$, by Corollary 2.17. Therefore

$$\begin{aligned} k\text{-}\mathrm{pd}_R M - 1 &\geq k\text{-depth}(\mathfrak{a}, R) - k\text{-depth}(\mathfrak{a}, K) \\ &= k\text{-depth}(\mathfrak{a}, R) - k\text{-depth}(\mathfrak{a}, M) - 1. \end{aligned}$$

Thus $k\text{-}\mathrm{pd}_R M \geq k\text{-depth}(\mathfrak{a}, R) - k\text{-depth}(\mathfrak{a}, M)$, as desired. □

In case of $k = -1$, we get a relation between the projective dimension of a finitely generated R-module, with the depth of an ideal on an arbitrary (not necessarily local) ring.

Corollary 4.2. *Let M be an R-module with finite projective dimension and $\mathfrak{a}$ be a proper ideal of R. Then*

$$\mathrm{pd}_R M \geq \mathrm{depth}(\mathfrak{a}, R) - \mathrm{depth}(\mathfrak{a}, M).$$

In 1965, Serre [9] proved the following theorem.

Theorem 4.3. *(Dimension Inequality). Let R be a regular local ring, M and N be finitely generated R-modules with $\ell(M \otimes_R N) < \infty$. Then*

$$\dim M + \dim N \leq \dim R.$$

Applying the above inequality, Serre concluded that $\dim N \leq \mathrm{pd}_R M$. This conclusion and Example 2.12, motivates us to show that k-projective dimension of M might be nearer upper bound for $\dim N$ than projective dimension of M (see Theorem 4.6).

Definition 4.4. *Let M be an R-module and $\mathfrak{a}$ be an ideal of R with $(\mathrm{Supp} M/\mathfrak{a}M)_{>k} \neq \emptyset$. The k-height of $\mathfrak{a}$ with respect to M is defined by*

$$k\text{-ht}_M\mathfrak{a} = \min\{\text{ht}_M\mathfrak{p} | \mathfrak{p} \in (\text{Supp}M/\mathfrak{a}M)_{>k}\}.$$

For an ideal $\mathfrak{a}$ *of* R *with* $(\text{Supp}M/\mathfrak{a}M)_{>k} = \emptyset$*, we set* $k\text{-ht}_M\mathfrak{a} = +\infty$*. In the case* $k = -1$*, the notion of* $k\text{-ht}_M\mathfrak{a}$ *is the same as* $\text{ht}_M\mathfrak{a}$*, the height of ideal* $\mathfrak{a}$ *with respect to* M*.*

Definition 4.5. *Let* M *be an* R*-module.* M *is called a* k*-Cohen-Macaulay module (abbreviately* k*-C.M.), whenever either* $k\text{-depth}(\mathfrak{a}, M)$ $= k\text{-ht}_M\mathfrak{a}$ *for all ideal* $\mathfrak{a}$ *of* R *with* $(\text{Supp}M/\mathfrak{a}M)_{>k} \neq \emptyset$ *or* $(\text{Supp}M)_{>k} = \emptyset$.
In the case $k = -1$*,* (-1)*-modules are exactly Cohen-Macaulay modules.*

Theorem 4.6. *(Generalized Intersection Theorem). Let* R *be a* k*-Cohen-Macaulay ring. Let* M *and* N *be finitely generated* R*-modules and* $\mathfrak{a}$ *be an ideal of* R *with* $(\text{Supp}R/\mathfrak{a})_{>k} \neq \emptyset$ *such that* $\dim M + \dim N \leq k\text{-ht}_R\mathfrak{a}$*. Assume that* $k\text{-pd}_R M < \infty$*. Then*

$$\dim N \leq k\text{-pd}_R M.$$

Proof. As R is a k-Cohen-Macaulay ring, by using Theorem 4.1, we have $\dim M + \dim N \leq k\text{-depth}(\mathfrak{a}, M) + k\text{-pd}_R M$. Since $k\text{-depth}(\mathfrak{a}, M) \leq \dim M$, we deduce that $\dim N \leq k\text{-pd}_R M$. $\square$

Definition 4.7. *Let* M *be an* R*-module with* $(\text{Supp}M)_{>k} \neq \emptyset$*. We define* $k\text{-grade}M$ *as* $k\text{-depth}(\text{Ann}M, R)$*. For a proper ideal* $\mathfrak{a}$ *of* R *with* $(\text{Supp}R/\mathfrak{a})_{>k} \neq \emptyset$*, we denote* $k\text{-grade}R/\mathfrak{a}$ *with* $k\text{-grade}\mathfrak{a}$ *that is* $k\text{-depth}(\mathfrak{a}, R)$*. One can see that*

$$k\text{-grade}M = \inf\{i \geq 0 \mid \dim\text{Ext}^i_R(M, R) > k\}.$$

Also, it is clear that 0-grade$M$ (1-grade$M$) is the length of maximal filter (generalized) regular M-sequence in Ann$M$ for k=0 (k=1).

Ischebeck Inequality. Let R be a local ring and M be a non-zero R-module. It is well-known that for any $\mathfrak{p} \in \text{Ass}M$, $\text{depth}M \leq \dim R/\mathfrak{p}$. Also for any prime ideal $\mathfrak{p}$ of R,

$$\text{depth}M \leq \text{depth}(\mathfrak{p}, M) + \dim R/\mathfrak{p}.$$

In a Noetherian ring R, we say that $(*)$ holds for the ideal $\mathfrak{a}$ of R with $(\text{Supp}R/\mathfrak{a})_{>k} \neq \emptyset$, if for any $\mathfrak{p} \in \text{Ass}R$,

$$k\text{-depth}(\mathfrak{a}, R) \leq \dim R/\mathfrak{p}.$$

Clearly, in any local ring (for $k = -1$), $(*)$ holds for the unique maximal ideal which is known as Ischebeck Inequality.
The following theorem holds in a Noetherian ring (not necessarily local) for an arbitrary ideal. The well-known inequalities of Peskine and Szpiro ([6], Lemma 4.8]) followed by this result in local case.

Theorem 4.8. *Let R be a Noetherian ring, $\mathfrak{a}$ be an ideal of R, and M be an R-module such that* $\mathrm{Ann}M \subseteq \mathfrak{a}$ *and* $\mathrm{V}(\mathfrak{a})_{>\mathrm{k}} \neq \emptyset$. *If $(*)$ holds for $\mathfrak{a}$, then we have the double inequality:*

$$k\text{-depth}(\mathfrak{a}, R) \leq k\text{-grade}M + \dim M \leq \dim R.$$

Proof. Let $\mathfrak{p}$ be a prime ideal of $\mathrm{Supp}M$ such that $\dim R/\mathfrak{p} = \dim M$. Then, by Lemma 2.4,

$$\begin{aligned} k - \mathrm{grade}M + \dim M &\leq k - \mathrm{depth}(\mathfrak{p}, R) + \dim R/\mathfrak{p} \\ &\leq \mathrm{depth}R_{\mathfrak{p}} + \dim R/\mathfrak{p} \\ &\leq \dim R_{\mathfrak{p}} + \dim R/\mathfrak{p} \\ &\leq \dim R \end{aligned}$$

which is the inequality on the right.
We prove the inequality on the left by induction on k-grade$M$. Let k-grade$M$ = *0*. Then k-depth$(\mathrm{Ann}M, R)$ = *0*, so there exists a prime ideal $\mathfrak{p} \supseteq \mathrm{Ann}M$ in $(\mathrm{Ass}R)_{>k}$. Satisfying ideal $\mathfrak{a}$ in $(*)$, we have k-depth$(\mathfrak{a}, R) \leq \dim R/\mathfrak{p} \leq \dim M$ as required. Now, suppose that $n > 0$ and the inequality on the left is proved for any finitely generated R-module N with $(\mathrm{Supp}N/\mathfrak{a}N)_{>k} \neq \emptyset$ which k-grade of N is less than n. Let M be an R-module of k-grade n. There exists a k-regular R-element $\alpha \in \mathrm{Ann}M$. Thus k-grade$_{R/\alpha R}M = n - 1$ by Proposition 2.6. Now, by inductive hypothesis k-depth$(\mathfrak{a}, R/\alpha R) \leq k$-grade$_{R/\alpha R}M + \dim_{R/\alpha R}M$, and evidently k-depth$(\mathfrak{a}, R) \leq k$-grade$_R M + \dim_R M$. □

Corollary 4.9. *[6, Lemma 4.8]. Let $(R, \mathfrak{m})$ be a local ring and M be a non-zero R-module. Then*

$$\mathrm{depth}R \leq \mathrm{grade}M + \dim M \leq \dim R.$$

Specially when R is a Cohen-Macaulay ring, the Grade Conjecture holds.

Proof. We immediately deduce the assertion from Theorem 4.8, for $k = -1$. $\square$

Now, we propose the following conjecture, which is a generalization of the Grade Conjecture.

The Generalized Grade Conjecture. Let M be an R-module of finite k-projective dimension and $(\mathrm{Supp}M)_{>k} \neq \emptyset$. Then

$$k\text{-grade}M + \dim M = \dim R.$$

It is well-known that the above conjecture is true for perfect modules in the case $k = -1$.

Definition 4.10. *Let M be an R-module with finite k-projective dimension and $(\mathrm{Supp}M)_{>k} \neq \emptyset$. M is called a k-perfect R-module, whenever*

$$k\text{-pd}_R M = k\text{-grade}M.$$

By definitions 2.8 and 4.7, it is clear that $k\text{-grade}M \leq k\text{-pd}_R M$.

The Generalized Strong Intersection Conjecture. Let R be a k-Cohen-Macaulay ring. Let M and N be finitely generated R-modules and $\mathfrak{a}$ be an ideal of R with $(\mathrm{Supp}M/\mathfrak{a}M)_{>k} \neq \emptyset$, such that $\dim M + \dim N \leq k\text{-ht}_R\mathfrak{a}$. Assume that $k\text{-pd}_R M < \infty$. Then

$$\dim N \leq k\text{-grade}M.$$

This conjecture holds for k-perfect modules, as follows.

Theorem 4.11. *Let R be a k-Cohen-Macaulay ring. Let M and N be finitely generated R-modules and $\mathfrak{a}$ be an ideal of R with $(\mathrm{Supp}M/\mathfrak{a}M)_{>k} \neq \emptyset$, such that $\dim M + \dim N \leq k\text{-ht}_R\mathfrak{a}$. Assume that M is k-perfect with $k\text{-pd}_R M < \infty$. Then*

$$\dim N \leq k\text{-grade}M.$$

Proof. It follows immediately from Theorem 4.6. $\square$

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