On Weak Generalized Amenability of Triangular Banach Algebras

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Abstract. Let $A_1$, $A_2$ be unital Banach algebras and $X$ be an $A_1 - A_2$– module. Applying the concept of module maps, (inner) module generalized derivations and generalized first cohomology groups, we present several results concerning the relations between module generalized derivations from $A_i$ into the dual space $A_i^*$ (for $i = 1, 2$) and such derivations from the triangular Banach algebra of the form $T := \begin{pmatrix} A_1 & X \\ 0 & A_2 \end{pmatrix}$ into the associated triangular $T$– bimodule $T^*$ of the form $T^* := \begin{pmatrix} A_1^* & X^* \\ 0 & A_2^* \end{pmatrix}$. In particular, we show that the so-called generalized first cohomology group from $T$ to $T^*$ is isomorphic to the directed sum of the generalized first cohomology group from $A_1$ to $A_1^*$ and the generalized first cohomology group from $A_2$ to $A_2^*$. Finally, Inspiring the above concepts, we establish a one to one corresponding between weak (resp. ideal) generalized amenability of $T$ and those amenability of $A_i$ ($i = 1, 2$).

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1. Introduction

Let $A$ be a Banach algebra and $M$ be a Banach $A$– bimodule. A module derivation $d : A \to M$ is a linear map which satisfies $d(ab) = d(a)b + ad(b)$ for all $a, b \in A$. The linear space of all bounded derivations from $A$ into $M$
is denoted by $Z^1(A,M)$. As an example, let $x \in M$ and define $d_x : A \to M$ by $d_x(a) := xa - ax$. Then $d_x$ is a module derivation which is called inner. Denoting the linear space of inner derivations from $A$ into $M$ by $N^1(A,M)$, we may consider the quotient space $H^1(A,M) := Z^1(A,M)/N^1(A,M)$, called the first cohomology group from $A$ into $M$.

A linear mapping $\psi : A \to M$ is called a module map if $\psi(ab) = \psi(a)b$. We denote by $B(A,M)$ the set of all bounded linear module maps from $A$ into $M$. Recently, a number of analysts [1, 3, 10] have studied various extended notions of derivations in the context of Banach algebras. For instance, suppose that $x, y \in M$ and define $\delta_{x,y} : A \to M$ by $\delta_{x,y}(a) := xa - ay$, then it is easily seen that $\delta(ab) = \delta(a)b + ad_b(b)$ for every $a, b \in A$. Mathieu [9] called the map of the form $\delta_{x,y}$ an inner generalized derivations. Therefore considering the relation $d(ab) = d(a)b + ad_b(b)$ as a special case of $\delta(ab) = \delta(a)b + ad_b(b)$ for all $a, b \in A$, where $d : A \to M$ is a module derivation, leads the theory of derivations to be extensively developed.

The above consideration motivated the authors in [1] to generalize the notion of derivation as follows:

Let $A$ be a Banach algebra and $M$ be a Banach $A$–module. A linear mapping $\delta : A \to M$ is called module generalized derivation if there exists a module derivation $d : A \to M$ such that $\delta(ab) = \delta(a)b + ad_b(b)$ $(a, b \in A)$. For convenience, we say that a generalized derivation $\delta$ is a $d$– derivation. Following as stated in [1], we call $\delta : A \to M$ an inner generalized derivations if there exist a $\psi \in B(A,M)$ and an element $y \in M$ in such a way that $\delta(a) := \psi(a) - ay$. As an example of the above so-called inner generalized derivation suppose that $x$ is an arbitrary element of $M$ and define $\psi : A \to M$ by $\psi(a) := xa$, then it follows that $\psi$ is a bounded module map and $\delta_{x,y}(a) = \psi(a) - ay$. This shows that the definition of the author in [1] covers the notion of Mathieu. From now on, we base our definition of inner generalized derivation on the interpretation of Abbaspour et al in [1]. The method has been used in [1] shows that a bounded module generalized derivation $\delta : A \to M$ is an inner generalized derivation if and only if there exists an element $y \in M$ such that $\delta$ is a $d_y$– derivation. We denote by $GZ^1(A,M)$ and $GN^1(A,M)$ the linear spaces of all bounded module generalized derivations and inner module generalized derivations from $A$ into $M$, respectively. Also we call the quotient space

$$GH^1(A,M) := GZ^1(A,M)/GN^1(A,M)$$

the generalized first cohomology group from $A$ to $M$. Applying these notations, the above characterization of inner generalized derivations immediately implies that $GH^1(A,M) = \{0\}$, whenever $H^1(A,M) = \{0\}$.

We recall that the dual space $M^*$ of $M$ is a Banach $A$– module by regarding
the module structure as follows

\[(a.f)(x) = f(xa), (f.a)(x) = f(ax).\]

The Banach algebra \(A\) is said to be \textit{generalized amenable} (resp. \textit{weakly generalized amenable}) if every generalized derivation \(\delta : A \to M^*\) (resp. \(\delta : A \to A^*\)) is inner; i.e. \(GH^1(A, M^*) = \{0\}\) (resp. \(GH^1(A, A^*) = \{0\}\)). The notion of an amenable Banach algebra was introduced by Johnson in [8]. Bade et al [2], later defined the concept of weak amenability for commutative Banach algebras. Let \(I\) be a closed two sided ideal of \(A\). Then \(A\) is said to be \(I-\text{weakly generalized amenable}\) if every generalized derivation \(\delta : A \to I^*\) is inner. Further, we call \(A\) \textit{ideally generalized amenable} if \(GH^1(A, I^*) = \{0\}\), for every closed two sided ideal \(I\) of \(A\). The notion of ideal amenability was first appeared in the framework of Gorji and Yazdanpanah in [6]. The reader is referred to books [4, 11] for more information on this subject.

Let \(A_1, A_2\) be unital Banach algebras and \(X\) be a unital \(A_1 - A_2\)-module in the sense that \(1_{A_1} x 1_{A_2} = x\), for every \(x \in X\). In this paper, we deal with the module generalized derivations from the triangular Banach algebra of the form \(T := \begin{pmatrix} A_1 & X \\ 0 & A_2 \end{pmatrix}\) into the associated triangular \(T-\text{bimodule} T^*\) of the form \(T^* := \begin{pmatrix} A_1^* & X^* \\ 0 & A_2^* \end{pmatrix}\). Such algebras were introduced by Forrest and Marcoux in [5]. Applying several results concerning the relations between module generalized derivations from \(A_i\) into the dual space \(A_i^*\) (for \(i = 1, 2\)) and such derivations from \(T\) into \(T^*\), we show that the so-called generalized first cohomology group from \(T\) to \(T^*\) is isomorphic to the directed sum of the generalized first cohomology group from \(A_1\) to \(A_1^*\) and the generalized first cohomology group from \(A_2\) to \(A_2^*\). Also, we establish a one to one corresponding between weak (resp. ideal) generalized amenability of \(T\) and those amenability of \(A_i\) (\(i = 1, 2\)).

2. Module Derivations on Triangular Banach Algebras

Let \(T := \{ \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} : a \in A_1, x \in X, b \in A_2 \}\). Then \(T\) equipped with the usual \(2 \times 2\) matrix addition and formal multiplication with the norm

\[
\| \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \| := \| a \| + \| x \| + \| b \|
\]
is a Banach algebra which is called the triangular Banach algebra associated to $X$. We define $T^*$ as \( \left\{ \left( \begin{array}{cc} f & h \\ 0 & g \end{array} \right) \right\} ; f \in A_1^*, h \in X^*, g \in A_2^* \} \) and
\[
\left( \begin{array}{cc} f & h \\ 0 & g \end{array} \right) \left( \begin{array}{cc} a & x \\ 0 & b \end{array} \right) := \left( \begin{array}{cc} af + xh & bh \\ 0 & bg \end{array} \right).
\]
Then $T^*$ is a triangular $T-$ bimodule with respect to the following module structure
\[
\left( \begin{array}{cc} f & h \\ 0 & g \end{array} \right) \left( \begin{array}{cc} a & x \\ 0 & b \end{array} \right) := \left( \begin{array}{cc} fa & ha \\ 0 & hx + gb \end{array} \right).
\]
The following results show some interesting relations between module derivations from $A_i$ to $A_i^*$ (for $i = 1, 2$) and those from $T$ to $T^*$. Let $d_i : A_i \to A_i^*$ be a bounded module derivation and $\delta_i : A_i \to A_i^*$ be a bounded module $d_i-$ derivation, for $i = 1, 2$. Define $\Delta_1, \Delta_2 : T \to T^*$ by
\[
\Delta_1\left( \begin{array}{cc} a & x \\ 0 & b \end{array} \right) := \left( \begin{array}{cc} \delta_1(a) & 0 \\ 0 & 0 \end{array} \right) \quad \text{and} \quad \Delta_2\left( \begin{array}{cc} a & x \\ 0 & b \end{array} \right) := \left( \begin{array}{cc} 0 & 0 \\ 0 & \delta_2(b) \end{array} \right).
\]

**Proposition 2.1.** $\Delta_i$ is a bounded $D_i-$ derivation (for $i = 1, 2$), where
\[
\begin{aligned}
D_1\left( \begin{array}{cc} a & x \\ 0 & b \end{array} \right) &:= \left( \begin{array}{cc} d_1(a) & 0 \\ 0 & 0 \end{array} \right) \quad \text{and} \quad D_2\left( \begin{array}{cc} a & x \\ 0 & b \end{array} \right) := \left( \begin{array}{cc} 0 & 0 \\ 0 & d_2(b) \end{array} \right).
\end{aligned}
\]
Moreover $\Delta_i$ (resp. $D_i$) is inner if and only if so is $\delta_i$ (resp. $d_i$).

**Proof.** By simple calculations, it can be observed that $D_1$ is a derivation and $\Delta_1$ is a $D_1-$ derivation. Also
\[
\| \left( \begin{array}{cc} \delta_1(a) & 0 \\ 0 & 0 \end{array} \right) \| = \| \delta_1(a) \| \leq \| \delta_1 \| \left\{ \| a \| + \| x \| + \| b \| \right\}.
\]
Hence $\Delta_1$ (and similarly $D_1$) is bounded.

Suppose that $\Delta_1$ is inner. Then there exist a bounded linear module map $\Psi : T \to T^*$ and $\left( \begin{array}{cc} f' & h' \\ 0 & g' \end{array} \right) \in T^*$ such that
\[
\left( \begin{array}{cc} \delta_1(a) & 0 \\ 0 & 0 \end{array} \right) = \Delta_1\left( \begin{array}{cc} a & 0 \\ 0 & 0 \end{array} \right) = \Psi\left( \begin{array}{cc} a & 0 \\ 0 & 0 \end{array} \right) - \left( \begin{array}{cc} a & 0 \\ 0 & 0 \end{array} \right) \left( \begin{array}{cc} f' & h' \\ 0 & g' \end{array} \right) = \Psi\left( \begin{array}{cc} a & 0 \\ 0 & 0 \end{array} \right) - \left( \begin{array}{cc} af' & 0 \\ 0 & 0 \end{array} \right).
Since $\Psi\left(\begin{array}{cc} a & 0 \\ 0 & 0 \end{array}\right) \in T^*$, so there exists $\left(\begin{array}{cc} f_1 & h_1 \\ 0 & g_1 \end{array}\right) \in T^*$ for which

$\Psi\left(\begin{array}{cc} a & 0 \\ 0 & 0 \end{array}\right) = \left(\begin{array}{cc} f_1 & h_1 \\ 0 & g_1 \end{array}\right)$. Define $\psi_1 : A_1 \rightarrow A_1^*$ by $\psi_1(a) := f_1$.

It follows from boundedness and linearity of $\Psi$ that, $\psi_1$ is a bounded linear operator. Also, there exist $h_3 \in X^*$ and $g_3 \in A_1^*$ such that

$\Psi\left(\begin{array}{cc} ab & 0 \\ 0 & 0 \end{array}\right) = \left(\begin{array}{cc} \psi_1(ab) & h_3 \\ 0 & g_3 \end{array}\right)$ and

$\left(\begin{array}{cc} \psi_1(ab) & h_3 \\ 0 & g_3 \end{array}\right) = \Psi\left(\begin{array}{cc} ab & 0 \\ 0 & 0 \end{array}\right) = \Psi\left(\begin{array}{cc} a & 0 \\ 0 & 0 \end{array}\right) \left(\begin{array}{cc} b & 0 \\ 0 & 0 \end{array}\right) = \Psi\left(\begin{array}{cc} a & 0 \\ 0 & 0 \end{array}\right) \left(\begin{array}{cc} h_1 & 0 \\ 0 & 0 \end{array}\right) = \left(\begin{array}{cc} \psi_1(a) & h_1 \\ 0 & g_1 \end{array}\right) \left(\begin{array}{cc} b & 0 \\ 0 & 0 \end{array}\right) = \left(\begin{array}{cc} \psi_1(ab) & h_1b \\ 0 & g_1 \end{array}\right)$.

Thus $\psi_1$ is a module map and

$\left(\begin{array}{cc} \delta_1(a) & 0 \\ 0 & 0 \end{array}\right) = \left(\begin{array}{cc} \psi_1(a) & h_1 \\ 0 & g_1 \end{array}\right) - \left(\begin{array}{cc} af' & 0 \\ 0 & 0 \end{array}\right) = \left(\begin{array}{cc} \psi_1(a) - af' & h_1 \\ 0 & g_1 \end{array}\right)$. 

Hence $\delta_1(a) = \psi_1(a) - af'$, $g_1 = 0$ and $h_1 a = 0$ for all $a \in A_1$. Therefore $\delta_1$ is an inner generalized derivation and $h_1 = 0 = g_1$.

Conversely, if $\delta_1(a) : A_1 \rightarrow A_1^*$ is an inner generalized derivation, then there exist $\psi_1 \in \mathcal{B}(A_1, A_1^*)$ and $f' \in A_1^*$ such that $\delta_1(a) = \psi_1(a) - af'$. Then

$\Delta_1\left(\begin{array}{cc} a & x \\ 0 & b \end{array}\right) = \left(\begin{array}{cc} \delta_1(a) & 0 \\ 0 & 0 \end{array}\right) = \left(\begin{array}{cc} \psi_1(a) - af' & 0 \\ 0 & 0 \end{array}\right) = \left(\begin{array}{cc} \psi_1(a) & 0 \\ 0 & 0 \end{array}\right) - \left(\begin{array}{cc} a & x \\ 0 & b \end{array}\right) \left(\begin{array}{cc} f' & 0 \\ 0 & 0 \end{array}\right)$.

Define $\Psi : T \rightarrow T^*$ by $\Psi\left(\begin{array}{cc} a & x \\ 0 & b \end{array}\right) = \left(\begin{array}{cc} \psi_1(a) & 0 \\ 0 & 0 \end{array}\right)$. It follows from boundedness and linearity of $\psi_1$ that, $\Psi$ is a bounded linear module map and consequently, $\Delta_1$ is an inner generalized derivation. □
Theorem 2.2. Let $T^*$ be the triangular bimodule \( \begin{pmatrix} A_1^* & X^* \\ 0 & A_2^* \end{pmatrix} \) associated to the triangular Banach algebra $T$. Assume that $D : T \to T^*$ be a bounded derivation and $\Delta : T \to T^*$ be a bounded $D$-derivation. Then for $i = 1, 2$, there exist a continuous derivation $d_i : A_i \to A_i^*$, a continuous $d_i$-derivation $\delta_i : A_i \to A_i^*$, and $h_0, h'_0 \in X^*$ such that
\[
\Delta\left( \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \right) = \begin{pmatrix}
\delta_1(a) - xh_0 & h'_0 a - bh_0 \\
0 & h'_0 x + \delta_2(b)
\end{pmatrix}.
\]

Proof. First we show that there exist an element $h_0 \in X^*$, a continuous derivation $d_1 : A_1 \to A_1^*$, and a continuous derivation $d_2 : A_2 \to A_2^*$ such that
\[
D\left( \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \right) = \begin{pmatrix}
d_1(a) - xh_0 & h_0 a - bh_0 \\
0 & h_0 x + d_2(b)
\end{pmatrix}.
\]

For this aim using some ideas of [5], we can verify that

(i) There exists $h_0 \in X^*$ such that $D\left( \begin{pmatrix} 1_A & 0 \\ 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & h_0 \\ 0 & 0 \end{pmatrix}$.

(ii) There exists a bounded derivation $d_1 : A_1 \to A_1^*$ such that $D\left( \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} d_1(a) & h_0 a \\ 0 & 0 \end{pmatrix}$.

(iii) $D\left( \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} -xh_0 & 0 \\ 0 & h_0 x \end{pmatrix}$.

(iv) There exist a bounded derivation $d_2 : A_2 \to A_2^*$ such that $D\left( \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \right) = \begin{pmatrix} 0 & -bh_0 \\ 0 & d_2(b) \end{pmatrix}$.

Now a similar calculation shows that

(i') There exist $f \in A_1^*$, $h'_0 \in X^*$ such that $\Delta\left( \begin{pmatrix} 1_A & 0 \\ 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} f & h'_0 \\ 0 & 0 \end{pmatrix}$.

(ii') There exists a bounded $d_1$-derivation $\delta_1 : A_1 \to A_1^*$ such that $\Delta\left( \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} \delta_1(a) & h'_0 a \\ 0 & 0 \end{pmatrix}$.

(iii') $\Delta\left( \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} -xh_0 & 0 \\ 0 & h'_0 x \end{pmatrix}$.
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(iv') There exist a bounded \( d_2 \) - derivation \( \delta_2 : A_2 \to A_2^* \) such that \( \Delta(\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}) = \begin{pmatrix} 0 & -bh_0 \\ 0 & \delta_2(b) \end{pmatrix} \).

and finally \( \Delta(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}) = \begin{pmatrix} \delta_1(a) - xb_0 & h_0 \alpha - bh_0 \\ 0 & h_0 x + \delta_2(b) \end{pmatrix} \).

For this aim following the parts (i) and (ii), we only prove the parts (i') and (ii'). The other parts are similar.

(i') There exist \( f \in A_1^* \), \( h \in X^* \), and \( g \in A_2^* \) such that \( \Delta(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} f & h \\ 0 & g \end{pmatrix} \).

On the other hand
\[
\begin{pmatrix} f & h \\ 0 & g \end{pmatrix} = \Delta(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}) = \Delta\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right] = \Delta(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Delta(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}) \\
= \begin{pmatrix} f & h \\ 0 & g \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} h_0 \\ 0 \end{pmatrix} + \begin{pmatrix} f & h \\ 0 & g \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\]

It follows that \( g = 0 \). Taking \( h'_0 := h \), completes the proof.

(ii') There exist \( f_1 \in A_1^* \), \( h_1 \in X^* \), and \( g_1 \in A_2^* \) such that \( \Delta(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} f_1 & h_1 \\ 0 & g_1 \end{pmatrix} \). On the other hand
\[
\begin{pmatrix} f_1 & h_1 \\ 0 & g_1 \end{pmatrix} = \Delta(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}) = \Delta\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\right] = \Delta(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}) \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Delta(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}) \\
= \begin{pmatrix} f & h'_0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} d_4(\alpha) & h_0 \alpha \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} f & h'_0 \alpha \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a + d_4(\alpha) & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\]
It follows that $g_1 = 0$, $h_1 = h'_0a$, and $f_1 = fa + d_1(a)$.
Take $\delta_1(a) := f_1$. We show that $\delta_1$ is a $d_1-$ derivation. Trivially $\delta_1$ is linear. Moreover
\[
\begin{pmatrix}
\delta_1(a_1a_2) & h'_0a_1a_2 \\
0 & 0
\end{pmatrix} = \Delta \begin{pmatrix}
a_1a_2 & 0 \\
0 & 0
\end{pmatrix}
\]
\[
= \Delta \begin{pmatrix}
a_1 & 0 \\
0 & 0
\end{pmatrix} \begin{pmatrix}
a_2 & 0 \\
0 & 0
\end{pmatrix} + \begin{pmatrix}
a_1a_2 & 0 \\
0 & 0
\end{pmatrix} \begin{pmatrix}
d_1(a_2) & h_0a_2 \\
0 & 0
\end{pmatrix}
\]
\[
= \begin{pmatrix}
\delta_1(a_1)a_2 & h'_0a_1a_2 \\
0 & 0
\end{pmatrix} + \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}
\]
\[
= \begin{pmatrix}
\delta_1(a_1)a_2 + a_1d_1(a_2) & h'_0a_1a_2 \\
0 & 0
\end{pmatrix}.
\]

Therefore $\delta_1$ is a $d_1-$ derivation. Further since $\Delta$ is bounded, so
\[
\| \delta_1(a) \| \leq \| \delta_1(a) \| + \| h'_0a \|
\]
\[
= \left\| \begin{pmatrix}
\delta_1(a) & h'_0a
\end{pmatrix}
\right\|
\]
\[
= \| \Delta \begin{pmatrix}
a & 0 \\
0 & 0
\end{pmatrix} \| 
\]
\[
\leq \| \Delta \| \| a \|.
\]

It follows that $\delta_1$ is bounded and $\| \delta_1 \| \leq \| \Delta \|$.

**Theorem 2.3.** Let $A_1$, $A_2$ be unital Banach algebras, $X$ be an $A_1-A_2-$module and let $T^*$ be the triangular bimodule $\begin{pmatrix}$
A_1^* & X^* \\
0 & A_2^*
\end{pmatrix}$ associated to the triangular Banach algebra $T$. Then
\[
GH^1(T, T^*) \cong GH^1(A_1, A_1^*) \oplus GH^1(A_2, A_2^*).
\]

**Proof.** Define $\pi : GZ^1(A_1, A_1^*) \oplus GZ^1(A_2, A_2^*) \to GH^1(T, T^*)$ by $\pi(\delta_1, \delta_2) := [\Delta^5_{a_1}]$, where
$\Delta^5_{a_1} := \Delta_1 + \Delta_2$ (as we defined in Proposition 2.1) and $[\Delta^5_{a_1}]$ represents the equivalent class of $\Delta^5_{a_1}$ in $GH^1(T, T^*)$. Clearly $\pi$ is linear. We are going to show that $\pi$ is surjective. For, let $\Delta$ be a bounded $D-$ derivation from $T$ to $T^*$. Let $\delta_1, \delta_2, h_0$ and $h'_0$ be as in Theorem 2.2. Then trivially
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\[
(\triangle - \triangle_{\delta_2}^\delta)(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}) = \begin{pmatrix} \delta_1(a) - xh_0 & h_0' a - bh_0 \\ 0 & \delta_2(h) + h_0' x \end{pmatrix} - \begin{pmatrix} \delta_1(a) & 0 \\ 0 & \delta_2(b) \end{pmatrix} \\
= \begin{pmatrix} -xh_0 & h_0' a - bh_0 \\ 0 & h_0' x \end{pmatrix} \\
= \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} - \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} - \begin{pmatrix} 0 & h_0 \\ 0 & 0 \end{pmatrix}.
\]

Taking \(\Psi(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}) := \begin{pmatrix} a & x' \\ 0 & b' \end{pmatrix}\), we conclude that

\[
\triangle - \triangle_{\delta_2}^\delta \in GN^1(T, T^*).
\]

This implies that \([\triangle] = [\triangle_{\delta_2}^\delta]\) and \(\pi\) is surjective. Therefore

\[
GH^1(T, T^*) \cong GZ^1(A_1, A_1^*) \oplus GZ^1(A_2, A_2^*)/\ker \pi.
\]

It is enough to show that \(\ker \pi = GN^1(A_1, A_1^*) \oplus GN^1(A_2, A_2^*)\). For this aim, note that if

\((\delta_1, \delta_2) \in \ker \pi\), then \(\triangle_{\delta_2}^\delta\) is an inner generalized derivation. So there exist

\(\Psi \in B(T, T^*)\) and \(\begin{pmatrix} f' & h' \\ 0 & g' \end{pmatrix}\) \(\in T^*\) such that

\[
\begin{pmatrix} \delta_1(a) & 0 \\ 0 & \delta_2(b) \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \\
= \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} - \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} - \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} - \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} f' & h' \\ 0 & g' \end{pmatrix}.
\]

Since \(\Psi(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}) \in T^*\), so there exists \(\begin{pmatrix} f_1 & h_1 \\ 0 & g_1 \end{pmatrix}\) \(\in T^*\) for which \(\Psi(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}) = \begin{pmatrix} f_1 & h_1 \\ 0 & g_1 \end{pmatrix}\). Define \(\psi_1 : A_i \to A_i^*\) \((i = 1, 2)\), by \(\psi_1(a) := f_1\) and \(\psi_2(b) := g_1\).

Following the arguments as stated in the proof of Proposition 2.1, it can be obtained that for \(i = 1, 2\), \(\psi_i : A_i \to A_i^*\) is a bounded linear module map and hence \(\delta_1(a) = \psi_1(a) - af'\) and \(\delta_2(b) = \psi_2(b) - bg'\) for all \(a \in A_1, b \in A_2\). So \(\delta_1\) and \(\delta_2\) are the inner generalized derivations. Hence

\((\delta_1, \delta_2) \in GN^1(A_1, A_1^*) \oplus GN^1(A_2, A_2^*)\).
Conversely, if \((\delta_1, \delta_2) \in GN^1(A_1, A_1^*) \oplus GN^1(A_2, A_2^*)\), then \(\delta_1\) and \(\delta_2\) are inner.
By Proposition 2.1, \(\Delta_i\) is an inner \(D_i\)-derivation, for \(i = 1, 2\). Hence \(\Delta_1 + \Delta_2 = \Delta_\delta\) is an inner generalized derivation. Therefore

\[
\ker \pi = HG_{d_1}N^1(A_1, A_1^*) \oplus HG_{d_2}N^1(A_2, A_2^*).
\]

Now we have

\[
GH^1(T, T^*) = \frac{GZ^1(A_1, A_1^*) \oplus GZ^1(A_2, A_2^*)}{GZ^1(A_1, A_1^*)} \oplus GZ^1(A_2, A_2^*) \\
\cong \frac{GZ^1(A_1, A_1^*)}{GZ^1(A_1, A_1^*)} \oplus \frac{GZ^1(A_2, A_2^*)}{GZ^1(A_2, A_2^*)} \\
= GH^1(A_1, A_1^*) \oplus GH^1(A_2, A_2^*). \quad \square
\]

In the rest of this paper, we investigate ideal generalized amenability of triangular Banach algebras. For this aim, first we characterize the form of each closed two sided ideal of triangular Banach algebras as following:

**Proposition 2.4.** Let \(A_1\), \(A_2\) be unital Banach algebras, \(X\) be a unital \(A_1 - A_2\)-module and let \(T\) be the unital triangular Banach algebra associated to \(X\). Then, \(T\) is a closed two sided ideal of \(T\) if and only if there exist a closed two sided ideal \(I_1\) of \(A_1\), a closed two sided ideal \(I_2\) of \(A_2\) and a closed \(A_1 - A_2\)-submodule \(Y\) of \(X\) in such a way that \(T = \begin{pmatrix} I_1 & Y \\ 0 & I_2 \end{pmatrix}\) and \(I_1X \cup XI_2 \subseteq Y\).

**Proof.** Suppose that \(T\) is a closed two sided ideal in \(T\). Define \(I_1\), \(I_2\) and \(Y\) as follows:

\[
I_1 := \{a \in A_1; \text{there exist } b \in A_2 \text{ and } x \in X \text{ with } \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \in T\},
\]

\[
I_2 := \{b \in A_2; \text{there exist } a \in A_1 \text{ and } x \in X \text{ with } \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \in T\} \text{ and }
\]

\[
Y := \{x \in X; \text{there exist } a \in A_1 \text{ and } b \in A_2 \text{ with } \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \in T\}.
\]

It is easy to check that \(I_1\) is a closed two sided ideal in \(A_1\), \(I_2\) is a closed two sided ideal in \(A_2\) and \(Y\) is a closed \(A_1 - A_2\)-submodule of \(X\). Also if \(a \in I_1\) and \(x_1 \in X\), then there exist \(b \in A_2\) and \(x \in X\) with \(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \in T\) and consequently, \(\begin{pmatrix} 0 & ax_1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \begin{pmatrix} 0 & x_1 \\ 0 & 0 \end{pmatrix} \in T\) which means \(ax_1 \in Y\). Similarly, one can show that \(XI_2 \subseteq Y\). To prove the closedness of \(Y\), let \(\{x_n\}\) be a sequence of \(Y\) fulfilling \(x_n \rightarrow x\), for some \(x \in X\). Hence, for each \(n \in \mathbb{N}\) there exist \(a_n \in A_1\) and \(b_n \in A_2\) such that \(\begin{pmatrix} a_n & x_n \\ 0 & b_n \end{pmatrix} \in T\).
On the other hand, for each \( \epsilon > 0 \) there exist \( N \in \mathbb{N} \) such that for each \( n \geq N \),
\[
\| x_n - x \| < \epsilon.
\]
It follows that
\[
\| \left( \begin{array}{cc} a_n & x_n \\ 0 & b_n \end{array} \right) - \left( \begin{array}{cc} a_n & x \\ 0 & b_n \end{array} \right) \| = \| x_n - x \| < \epsilon.
\]
Applying the closedness of \( \mathcal{I} \), we conclude that \( x \in Y \) which completes the proof of the closedness of \( Y \). The converse is similar.

\[\text{Remark 2.5. Let } A_1, A_2 \text{ be unital Banach algebras, } X \text{ be a unital } A_1 - A_2 - \text{module and let}
\]
\[
\mathcal{I} = \left( \begin{array}{cc} I_1 & Y \\ 0 & I_2 \end{array} \right)
\]
be a closed ideal of the unital triangular Banach algebra \( T \) associated to \( X \). Define \( \mathcal{I}^* \) as
\[
\{ \left( \begin{array}{cc} f & h \\ 0 & g \end{array} \right); f \in I_1^*, h \in Y^*, g \in I_2^* \} \text{ and } \left( \begin{array}{cc} f & h \\ 0 & g \end{array} \right) \left( \begin{array}{cc} a & x \\ 0 & b \end{array} \right) := f(a) + h(x) + g(b).
\]
It is trivial that \( \mathcal{I}^* \) is a triangular \( \mathcal{I} - \) bimodule with respect to the module structure as stated at the first part of Section 2. Let \( d_i : A_i \to I_i^* \) be a bounded derivation and \( \delta_i : A_i \to I_i^* \) be a bounded \( d_i - \) derivation, for \( i = 1, 2 \). Define \( \Delta_1, \Delta_2 : T \to \mathcal{I}^* \) by
\[
\Delta_1 \left( \begin{array}{cc} a & x \\ 0 & b \end{array} \right) := \left( \begin{array}{cc} \delta_1(a) & 0 \\ 0 & 0 \end{array} \right) \text{ and } \Delta_2 \left( \begin{array}{cc} a & x \\ 0 & b \end{array} \right) := \left( \begin{array}{cc} 0 & 0 \\ 0 & \delta_2(b) \end{array} \right).
\]
(i) Following exactly the method has been used in Proposition 2.1, shows that \( \Delta_i \) is a bounded \( D_i - \) derivation (for \( i = 1, 2 \)), where
\[
D_1 \left( \begin{array}{cc} a & x \\ 0 & b \end{array} \right) := \left( \begin{array}{cc} d_1(a) & 0 \\ 0 & 0 \end{array} \right) \text{ and } D_2 \left( \begin{array}{cc} a & x \\ 0 & b \end{array} \right) := \left( \begin{array}{cc} 0 & 0 \\ 0 & d_2(b) \end{array} \right).
\]
Moreover \( \Delta_i \) (resp. \( D_i \)) is inner if and only if so is \( \delta_i \) (resp. \( d_i \)).
(ii) Assume that \( D : T \to \mathcal{I}^* \) is a bounded derivation and \( \Delta : T \to \mathcal{I}^* \) be a bounded \( D_i - \) derivation. Then similar to the proof of Theorem 2.2, it can be shown that for \( i = 1, 2 \) there exist a bounded derivation \( d_i : A_i \to I_i^* \), a bounded \( d_i - \) derivation \( \delta_i : A_i \to I_i^* \), and \( h_0, h_0^* \in X^* \) such that
\[
\Delta \left( \begin{array}{cc} a & x \\ 0 & b \end{array} \right) = \left( \begin{array}{cc} \delta_1(a) - xh_0 & h_0^*a - bh_0 \\ 0 & h_0^*x + \delta_2(b) \end{array} \right).
\]
(iii) As an immediate consequence of the parts (i) and (ii), it is easily seen that
\[ GH^1(T, I^* +) \cong GH^1(A_1, I_1^*) \oplus GH^1(A_2, I_2^*). \]

**Corollary 2.6.** $T$ is weakly (resp. ideally) generalized amenable if and only if $A_i$ is weakly (resp. ideally) generalized amenable, for $i = 1, 2$.

In [7] (resp. [6]), it has been proved that every $C^*$-algebra is weakly (resp. ideally) amenable. Therefore $H^1(A, A^*) = \{0\}$ (resp. $H^1(A, I^*) = \{0\}$, for every closed two sided ideal $I$ of $A$) and it follows from the characterization of generalized inner derivation that $GH^1(A, A^*) = \{0\}$ (resp. $GH^1(A, I^*) = \{0\}$, for every closed two sided ideal $I$ of $A$) which means every $C^*$-algebra is weakly (resp. ideally) generalized amenable.

**Corollary 2.7.** For each $C^*$-algebra $A$, the triangular Banach algebras
\[
\begin{pmatrix}
A & 0 \\
A & A \\
0 & A
\end{pmatrix}
\]
and
\[
\begin{pmatrix}
A & 0 \\
0 & A
\end{pmatrix}
\]
are weakly (resp. ideally) generalized amenable.

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**References**


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