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Original Research Paper

A New Carlson–Type Sharp Inequality

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Abstract. In this paper, we establish a new Carlson–type integral inequality with the best constant factor. The equivalent Beurling–Kjellberg type inequality and discrete form are considered.

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1 Introduction

Let f be a positive function, then the following inequality holds

$$\int_0^\infty f(x)dx \leq \sqrt{\pi}\left(\int_0^\infty f^2(x)dx\right)^{\frac{1}{4}}\left(\int_0^\infty x^2 f^2(x)dx\right)^{\frac{1}{4}}, \tag{1}$$

provided that the integrals on the right-hand side are convergent. The constant $\sqrt{\pi}$ is sharp and the equality holds for $f(x) = \frac{1}{x^2+1}$. Inequality

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(1) is called Carlson's inequality which was discovered in [5]. The corresponding discrete form for some positive sequence of numbers $(a_n)_{n\geq 1}$ is given as

$$\sum_{n=1}^{\infty} a_n < \sqrt{\pi} \left(\sum_{n=1}^{\infty} a_n^2 \right)^{\frac{1}{4}} \left(\sum_{n=1}^{\infty} n^2 a_n^2 \right)^{\frac{1}{4}}, \tag{2}$$

provided that the series on the right are convergent. The constant $\sqrt{\pi}$ is the best possible, in the sense that it cannot be decreased anymore.

Carlson's inequality plays an important role in several applications in mathematics, such as interpolation theory, see the book [10]. About Carlson's inequalities and their extensions we refer the reader to the book [10] and its references. Also, the reader may refer to the following book and papers [1]-[13].

In this paper, we establish a new Carlson–type integral inequality involving the missing term $\int_0^\infty x f^2(x)dx$ with the best constant factor. The equivalent Beurling–Kjellberg type inequality and discrete form are considered.

2 Integral Case

Recall that the Beta function $B(u, v)$ is defined for two positive parameters by

$$B(u, v) = \int_0^\infty \frac{t^{u-1}}{(t+1)^{u+\nu}} dt,$$

and it satisfies the following relation

$$B(s, t+1) = \frac{t}{s+t} B(s, t). \tag{3}$$

Our first result refers to a Carlson–type integral inequality.

Theorem 2.1. *Suppose that* $p > 1, q \neq 0, \frac{1}{p} + \frac{1}{q} = 1$ *and that* f *is a Lebesgue measurable nonnegative function such that* $0 < \int_0^\infty x^p f^p(x)dx <$

∞ and $0 < \int_0^\infty x^{p-2} f^p(x)dx < \infty$ then the following inequality holds

$$\left(\int_0^\infty f(x)dx\right)^p \leq C\left(\left(\int_0^\infty x^p f^p(x)dx \int_0^\infty x^{p-2} f^p(x)dx\right)^{\frac{1}{2}} + \int_0^\infty x^{p-1} f^p(x)dx\right), \tag{4}$$

where $C = 2B^{\frac{p}{q}}(q-1, q-1)$. Inequality (4) is sharp.

Proof. Let $a, b > 0$, for abbreviation we set $L := \int_0^\infty x^p f^p(x)dx$, $S := \int_0^\infty x^{p-2} f^p(x)dx$ and $T := \int_0^\infty x^{p-1} f^p(x)dx$. Since both L and S are finite, then by the Schwarz inequality ($T \leq (LS)^{\frac{1}{2}}$) T is also finite. Using Hölder inequality, we have

$$\begin{aligned}
\left(\int_0^\infty f(x)dx\right)^p &= \left(\int_0^\infty \frac{x^{\frac{2}{p}-1}}{(ax+b)^{\frac{2}{p}}} x^{1-\frac{2}{p}} (ax+b)^{\frac{2}{p}} f(x)dx\right)^p \\
&\leq \left(\int_0^\infty \frac{x^{q-2}}{(ax+b)^{2(q-1)}}dx\right)^{\frac{p}{q}} \int_0^\infty x^{p-2}(ax+b)^2 f^p(x)dx \\
&= \frac{1}{ab} B^{\frac{p}{q}}(q-1, q-1)\left[a^2 L + b^2 S + 2abT\right] \\
&= B^{\frac{p}{q}}(q-1, q-1)\left[\frac{a}{b}L + \frac{b}{a}S + 2T\right].
\end{aligned}$$

Let $t = \frac{a}{b}$, and $g(t) = tL + \frac{1}{t}S$, then g attains its minimum at $t = \frac{\sqrt{S}}{\sqrt{L}} = \frac{a}{b}$, thus we get (4). It remains to show that the inequality is sharp. To do this we consider the function $f(x) = \frac{x^{q-2}}{(x+1)^{2(q-1)}}$, then we find

$$\int_0^\infty f(x)dx = \int_0^\infty \frac{x^{q-2}}{(x+1)^{2(q-1)}}dx = B(q-1, q-1), \tag{5}$$

$$S = \int_0^\infty x^{p-2} f^p(x)dx = \int_0^\infty \frac{x^{q-2}}{(x+1)^{2q}}dx = B(q-1, q+1), \tag{6}$$

$$T = \int_0^\infty x^{p-1} f^p(x)dx = \int_0^\infty \frac{x^{q-1}}{(x+1)^{2q}}dx = B(q, q), \tag{7}$$

and

$$L=\int_0^\infty x^p f^p(x)dx=\int_0^\infty \frac{x^q}{(x+1)^{2q}}dx=B(q+1,q-1). \tag{8}$$

Using formula (3) for the Beta function, we find

$$B(q+1,q-1)=\frac{q}{2q-1}B(q-1,q)=\frac{q}{2\,(2q-1)}B(q-1,q-1), \tag{9}$$

and

$$B(q,q)=\frac{q-1}{2q-1}B(q-1,q)=\frac{q-1}{2\,(2q-1)}B(q-1,q-1). \tag{10}$$

By substituting (9) in (6) and in (8) and then substituting (10) in (7), we find that the right-hand side of (4) is

$$\begin{aligned}
&2B^{\frac{p}{q}}(q-1,q-1)\left[\frac{q}{2\,(2q-1)}B(q-1,q-1)+\frac{q-1}{2\,(2q-1)}B(q-1,q-1)\right]\\
&\quad=2B^{\frac{p}{q}}(q-1,q-1)B(q-1,q-1)\frac{q+q-1}{2\,(2q-1)}\\
&\quad=B^p(q-1,q-1).
\end{aligned} \tag{11}$$

The left-hand side of (4) is

$$\left(\int_0^\infty f(x)dx\right)^p=B^p(q-1,q-1). \tag{12}$$

From (11) and (12) we conclude the equality in (4). □

Remark 2.2. If we let $p=q=2$ in (4) we get

$$\begin{aligned}
&\left(\int_0^\infty f(x)dx\right)^2\\
&\quad\le 2\left[\left(\int_0^\infty x^2f^2(x)dx\int_0^\infty f^2(x)dx\right)^{\frac{1}{2}}+\int_0^\infty xf^2(x)dx\right].
\end{aligned} \tag{13}$$

Carlson's inequality follows from the inequality (13). Although this does not give the sharp constant $\sqrt{\pi}$. By the Schwarz inequality we find

$$\int_0^\infty xf^2(x)dx\le\left(\int_0^\infty x^2f^2(x)dx\int_0^\infty f^2(x)dx\right)^{\frac{1}{2}}.$$

Hence, inequality (13) becomes

$$\left(\int_0^\infty f(x)dx\right)^2 \leq 4\left(\int_0^\infty x^2 f^2(x)dx \int_0^\infty f^2(x)dx\right)^{\frac{1}{2}},$$

or

$$\int_0^\infty f(x)dx \leq 2\left(\int_0^\infty x^2 f^2(x)dx\right)^{\frac{1}{4}}\left(\int_0^\infty f^2(x)dx\right)^{\frac{1}{4}}.$$

The last inequality is Carlson's inequality with the constant $\sqrt{\pi}$ replaced by 2. But generally speaking, inequality (1) and inequality (13) are not comparable. To see this, consider two particular cases when $f_1(x) = \frac{1}{(x+1)^3}$ and $f_2(x) = \frac{1}{(x+1)^{7/4}}$. In the first setting, inequality (1) yields the estimate

$$\int_0^\infty f_1(x)dx \leq \sqrt{\pi}\left(\int_0^\infty f_1^2(x)dx\right)^{\frac{1}{4}}\left(\int_0^\infty x^2 f_1^2(x)dx\right)^{\frac{1}{4}} \approx 0.506468$$

while (13) implies the inequality

$$\int_0^\infty f_1(x)dx$$
$$\leq \sqrt{2}\left[\left(\int_0^\infty x^2 f_1^2(x)dx \int_0^\infty f_1^2(x)dx\right)^{\frac{1}{2}} + \int_0^\infty x f_1^2(x)dx\right]^{\frac{1}{2}} \approx 0.513127.$$

Clearly, the inequality (1) yields better estimate. In the second setting, inequality (1) yields the estimate

$$\int_0^\infty f_2(x)dx \leq \sqrt{\pi}\left(\int_0^\infty f_2^2(x)dx\right)^{\frac{1}{4}}\left(\int_0^\infty x^2 f_2^2(x)dx\right)^{\frac{1}{4}} \approx 1.43251$$

while (13) implies the inequality

$$\int_0^\infty f_2(x)dx$$
$$\leq \sqrt{2}\left[\left(\int_0^\infty x^2 f_2^2(x)dx \int_0^\infty f_2^2(x)dx\right)^{\frac{1}{2}} + \int_0^\infty x f_2^2(x)dx\right]^{\frac{1}{2}} \approx 1.35637.$$

In this case, the inequality (13) is more accurate.

Remark 2.3. Using same techniques as in [3, Remark 2.4], we prove that the inequality (4) is equivalent to the following Beurling–Kjellberg type inequality:

$$\begin{aligned}&\left(\int_{-\infty}^{\infty} f(x)dx\right)^p \\ &\quad\le 2^{p-1}C\left(\left(\int_{-\infty}^{\infty}|x|^p f^p(x)dx\int_{-\infty}^{\infty}|x|^{p-2}f^p(x)dx\right)^{\frac{1}{2}}\right. \\ &\qquad\left.+\int_{-\infty}^{\infty}|x|^{p-1}f^p(x)dx\right),\end{aligned} \tag{14}$$

where C is as defined in Theorem 2.1.

Assume that (4) holds. By using inequality (4), Hölder inequality and Schwarz inequality, we have

$$\begin{aligned}&\left(\int_{-\infty}^{\infty} f(x)dx\right)^p \\ &=\left(\int_{-\infty}^{0} f(x)dx+\int_{0}^{\infty} f(x)dx\right)^p \\ &\le C\left[\left(\left(\int_{-\infty}^{0}|x|^p f^p(x)dx\int_{-\infty}^{0}|x|^{p-2}f^p(x)dx\right)^{\frac{1}{2}}+\int_{-\infty}^{0}|x|^{p-1}f^p(x)dx\right)^{\frac{1}{p}}\right. \\ &\quad\left.+\left(\left(\int_{0}^{\infty}|x|^p f^p(x)dx\int_{0}^{\infty}|x|^{p-2}f^p(x)dx\right)^{\frac{1}{2}}+\int_{0}^{\infty}|x|^{p-1}f^p(x)dx\right)^{\frac{1}{p}}\right]^p \\ &\le 2^{p-1}C\left[\left(\int_{-\infty}^{0}|x|^p f^p(x)dx\int_{-\infty}^{0}|x|^{p-2}f^p(x)dx\right)^{\frac{1}{2}}\right. \\ &\quad\left.+\left(\int_{0}^{\infty}|x|^p f^p(x)dx\int_{0}^{\infty}|x|^{p-2}f^p(x)dx\right)^{\frac{1}{2}}+\int_{-\infty}^{\infty}|x|^{p-1}f^p(x)dx\right]^p \\ &\le 2^{p-1}C\left(\left(\int_{-\infty}^{\infty}|x|^p f^p(x)dx\int_{-\infty}^{\infty}|x|^{p-2}f^p(x)dx\right)^{\frac{1}{2}}\right. \\ &\quad\left.+\int_{-\infty}^{\infty}|x|^{p-1}f^p(x)dx\right),\end{aligned}$$

which is (14). In the other direction, let $g(x) = f(x)$ if $x \geq 0$ and $g(x) = f(-x)$ if $x < 0$. Then

$$\begin{aligned}
&\left(2\int_0^\infty f(x)dx\right)^p \\
&= \left(\int_{-\infty}^\infty g(x)dx\right)^p \\
&\leq 2^{p-1}C\left(\left(\int_{-\infty}^\infty |x|^p g^p(x)dx \int_{-\infty}^\infty |x|^{p-2} g^p(x)dx\right)^{\frac{1}{2}}\right. \\
&\quad \left. + \int_{-\infty}^\infty |x|^{p-1} g^p(x)dx\right) \\
&\leq 2^p C\left(\left(\int_0^\infty x^p f^p(x)dx \int_0^\infty x^{p-2} f^p(x)dx\right)^{\frac{1}{2}} + \int_0^\infty x^{p-1} f^p(x)dx\right),
\end{aligned}$$

that is, we get (4). Therefore, inequalities (4) and (14) are equivalent.

3 Discrete Case

Now, our main goal is to establish discrete analogue of Carlson–type inequality derived in the previous section.

Theorem 3.1. *Suppose that $p > 1, q \neq 0, \frac{1}{p}+\frac{1}{q} = 1$ and that $(a_n)_{n\geq 1}$ is a sequence of positive numbers such that $\sum_{n=1}^\infty n^p a_n^p < \infty$ and $\sum_{n=1}^\infty n^{p-2} a_n^p < \infty$, then the following inequality holds*

$$\left(\sum_{n=1}^\infty a_n\right)^p < C\left(\left(\sum_{n=1}^\infty n^p a_n^p \sum_{n=1}^\infty n^{p-2} a_n^p\right)^{\frac{1}{2}} + \sum_{n=1}^\infty n^{p-1} a_n^p\right), \tag{15}$$

where the constant $C = 2B^{\frac{p}{q}}(q-1, q-1)$ is the best possible.

Proof. Let $a, b > 0$, for abbreviation we set $\ell := \sum_{n=1}^\infty n^p a_n^p$, $s := \sum_{n=1}^\infty n^{p-2} a_n^p$ and $t := \sum_{n=1}^\infty n^{p-1} a_n^p$. Since both ℓ and s are finite, then by Schwarz inequality $(t \leq (\ell s)^{\frac{1}{2}})$ t is also finite. Using Hölder inequality, we have

$$\begin{aligned}\sum_{n=1}^{\infty} a_n &= \sum_{n=1}^{\infty} \frac{n^{\frac{2}{p}-1}}{(an+b)^{\frac{2}{p}}} n^{1-\frac{2}{p}} (an+b)^{\frac{2}{p}} a_n \\ &\leq \left(\sum_{n=1}^{\infty} \frac{n^{q-2}}{(an+b)^{2(q-1)}} \right)^{\frac{1}{q}} \left(\sum_{n=1}^{\infty} n^{p-2} (an+b)^2 a_n^p \right)^{\frac{1}{p}} \\ &< \left(\int_0^{\infty} \frac{x^{q-2}}{(ax+b)^{2(q-1)}} dx \right)^{\frac{1}{q}} \\ &\quad \times \left(a^2 \sum_{n=1}^{\infty} n^p a_n^p + 2ab \sum_{n=1}^{\infty} n^{p-1} a_n^p + b^2 \sum_{n=1}^{\infty} n^{p-2} a_n^p \right)^{\frac{1}{p}}.\end{aligned}$$

Hence, we find

$$\begin{aligned}\left(\sum_{n=1}^{\infty} a_n \right)^p &< B^{\frac{p}{q}}(q-1, q-1) \left(\frac{a}{b}\ell + \frac{b}{a}s + 2t \right) \\ &= 2B^{\frac{p}{q}}(q-1, q-1) \left(\sqrt{\ell s} + t \right).\end{aligned}$$

Similarly, as in the proof of Theorem 2.1, if we set $\frac{\sqrt{s}}{\sqrt{\ell}} = \frac{a}{b}$ we get (15). It remains to show that the constant C is the best possible, to do that we assume that there exists a positive constant $D < C$ such that inequality (15) is valid if we replace C by D. For $0 < \lambda < 1$, we set $\tilde{a}_n = \frac{\lambda n^{q-2}}{(n+\lambda)^{2q-2}}$, then we find

$$\begin{aligned}\sum_{n=1}^{\infty} \tilde{a}_n &> \int_1^{\infty} \frac{\lambda x^{q-2}}{(x+\lambda)^{2q-2}} dx \\ &= \int_0^{\infty} \frac{\lambda x^{q-2}}{(x+\lambda)^{2q-2}} dx - \int_0^1 \frac{\lambda x^{q-2}}{(x+\lambda)^{2q-2}} dx \\ &= \lambda^{2-q} B(q-1, q-1) - \lambda O(1). \qquad (16)\end{aligned}$$

Moreover, we have

$$\begin{aligned}\sum_{n=1}^{\infty} n^{p-2}\tilde{a}_n^p &= \lambda^p \sum_{n=1}^{\infty} \frac{n^{q-2}}{(n+\lambda)^{2q}} \\ &< \lambda^p \int_0^{\infty} \frac{x^{q-2}}{(x+\lambda)^{2q}} dx \\ &= \lambda^{p-q-1} B(q-1, q+1), \end{aligned} \tag{17}$$

$$\begin{aligned}\sum_{n=1}^{\infty} n^{p}\tilde{a}_n^p &= \lambda^p \sum_{n=1}^{\infty} \frac{n^{q}}{(n+\lambda)^{2q}} \\ &< \lambda^p \int_0^{\infty} \frac{x^{q}}{(x+\lambda)^{2q}} x \\ &= \lambda^{p-q+1} B(q-1, q+1), \end{aligned} \tag{18}$$

and

$$\begin{aligned}\sum_{n=1}^{\infty} n^{p-1}\tilde{a}_n^p &= \lambda^p \sum_{n=1}^{\infty} \frac{n^{q-1}}{(n+\lambda)^{2q}} \\ &< \lambda^p \int_0^{\infty} \frac{x^{q-1}}{(x+\lambda)^{2q}} dx \\ &= \lambda^{p-q} B(q, q). \end{aligned} \tag{19}$$

Substituting relations (9) in (17) and (18) and then (10) in (19) we find

$$\begin{aligned}&\big(\lambda^{2-q} B(q-1, q-1) - \lambda O(1)\big)^p \\ &\quad < D\left(\left(\frac{B(q-1,q+1)}{\lambda^{q-p+1}} \frac{B(q-1,q+1)}{\lambda^{q-p-1}}\right)^{\frac{1}{2}} + \frac{B(q,q)}{\lambda^{q-p}}\right) \\ &\quad = \frac{D}{2\lambda^{q-p}} B(q-1, q-1).\end{aligned}$$

Multiplying the last inequality by λ^{q-p}, we have

$$\big(B(q-1, q-1) - \lambda^{q-1} O(1)\big)^p < \frac{D}{2} B(q-1, q-1). \tag{20}$$

If we let $\lambda \to 0^+$ in (20) we get

$$D \geq 2B^{p-1}(q-1, q-1) = C$$

which contradicts our assumption that $D < C$. The theorem is proved. □

Remark 3.2. If we set $p = q = 2$ in (15) we get

$$\left(\sum_{n=1}^{\infty} a_n\right)^2 < 2\left(\left(\sum_{n=1}^{\infty} n^2 a_n^2 \sum_{n=1}^{\infty} a_n^2\right)^{\frac{1}{2}} + \sum_{n=1}^{\infty} n a_n^2\right). \tag{21}$$

Discrete Carlson's inequality follows from the inequality (21). Although this does not give the sharp constant $\sqrt{\pi}$. By the Schwarz inequality for series we get

$$\sum_{n=1}^{\infty} n a_n^2 \leq \left(\sum_{n=1}^{\infty} n^2 a_n^2 \sum_{n=1}^{\infty} a_n^2\right)^{\frac{1}{2}}.$$

Thus, from the last inequality (21) becomes

$$\sum_{n=1}^{\infty} a_n < 2\left(\sum_{n=1}^{\infty} n^2 a_n^2 \sum_{n=1}^{\infty} a_n^2\right)^{\frac{1}{2}}.$$

The last inequality is Carlson's inequality (2) with the constant π replaced by 4. We also note that, in a manner similar to Remark 2.2 with $\hat{a}_n = 1/n^2$ and $\tilde{a}_n = 1/n^{\frac{31}{20}}$, inequality (2) and inequality (21) are not comparable.

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