Journal of Mathematical Extension
Vol. 16, No. 12, (2022) (4) 1-19
URL: https://doi.org/10.30495/JME.2022.2353
ISSN: 1735-8299
Original Research Paper

# Conformal Einstein PP-Wave as Quantum Solutions

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**Abstract.** We study conformal geometry on two essential classes of pp-wave spaces, i.e., Cahen-Wallach and two-symmetric spaces. This study leads to the general description of conformally Einstein metrics on the spaces under consideration. Having settled a model for the potential energy of the capacitor, we prove that the multiplying functions of conformal Einstein pp-wave spaces are solutions to the Schrödinger equation.

**AMS Subject Classification:** MSC 53C18; 53C50; 53B30
**Keywords and Phrases:** Conformal Einstein equation, Schrödinger equation, Lorentzian two-symmetric space, quantum solutions

## 1 Introduction

Quantum theory is an accurate theory for describing physical systems. In classical physics, forces play a significant role, but in quantum me-

Received: April 2022; Accepted: August 2022
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chanics, potentials do. We use the capacitor model through potential and the time-independent Schrödinger equation to describe some conformal Einstein pp-wave manifolds. The pp-waves metrics are interpreted as criteria for representing gravitational waves [19]. Another well-known feature of pp-wave metrics is that all the polynomial scalar invariants vanish [18]. A Lorentzian manifold (M, g) (of dimension n + 2) is called a pp-wave if locally there exist coordinates $(x, t^1, \dots, t^n, y)$, such that

$$g = 2dxdy + \sum_{i=1}^{n}(dt^i)^2 + H(dy)^2,$$

where H is a function of $t^1, \dots, t^n, y$. Cahen-Wallach spaces are pp-waves with $H = \sum_{i=1}^{n} \lambda_i(t_i)^2$, and from now on, we will use the term CW-space. In physics, CW-spaces are used as the background for theories of supergravity, because they have a large space of symmetries and may accept many generators of supersymmetry. Another important sub-class of pp-waves is the two-symmetric spaces. Generally, $k$-symmetric manifolds are great sub-classes of symmetric manifolds. A pseudo-Riemannian manifold $(M, g)$ having $R$ as its curvature tensor is entitled $k$-symmetric whenever

$$\nabla^k R = 0, \quad \nabla^{k-1} R \neq 0, \quad k \geq 1.$$

The Riemannian $k$-symmetric spaces, contrary to the pseudo-Riemannian ones, are locally symmetric [20]. According to studies, these spaces are one of the most critical subspaces of symmetric spaces. In recent years, the local classification of two-symmetric Lorentzian spaces has been presented in two independent and simultaneous works [2, 5]. In [2], the authors proved that a Lorentzian space $(M^{n+2}, g)$ is two-symmetric iff a locally coordinates $(x, t^1, \dots, t^n, y)$ exists, such that

$$g = 2dxdy + \sum_{i=1}^{n}(dt^i)^2 + (M_{ij}y + N_{ij})t^i t^j (dy)^2,$$

where $M_{ij}$ is a non-vanishing diagonal real matrix, $\lambda_1 \leq \cdots \leq \lambda_n$ are its diagonal components, and $N_{ij}$ is a real symmetric matrix. According to this result, for a Lorentzian two-symmetric space $(M^4, g)$, there are local coordinates $(x, y, u, v)$ such that

$$\begin{aligned} g =& 2dxdv + (dy)^2 + (du)^2 \\ &+ \Big(v(ay^2 + bu^2) + py^2 + 2qyu + su^2\Big)(dv)^2, \end{aligned} \tag{1}$$

for real constants $a, b, p, q$, and $s$ where $a^2 + b^2 \neq 0$. One of the essential concepts of both physical and geometrical is conformal geometry. We call $(G, g)$ and $(G, \tilde{g})$ *conformally equivalent* if we can find a smooth function $\varphi$ such that $\tilde{g} = \varphi^{-2} g$. Note that the category of metrics $\tilde{g}$ is called the conformal class of $g$. With these descriptions our main result is as follows:

We write down the conformally Einstein equation and show the general expression of the solutions in Theorem 3.2 and Theorem 4.2, showing that the space of conformal Einstein metrics is two-dimensional in the two situations under consideration (Cahen-Wallach and two-symmetric spaces).

Different geometric properties of a manifold could also be studied through its conformal class. For example, a manifold that is conformally equivalent to a flat (res. symmetric, Einstein) manifold is called *conformally flat* (rep. *conformally symmetric, conformally Einstein*) [8, 9, 11, 1]. It is clear that Weyl conformal tensor is invariant under a conformal transformation, but both the connection and the curvature tensor change under conformal transformation. While the Ricci tensor change under conformal transformations, it is natural to investigate the necessary and sufficient condition for a conformal manifold to be conformally Einstein. Recently, the second author studied the wave equation on Lorentzian conformally flat spaces [3] (or see [4]). Conformally Einstein manifolds play a significant role in mathematical physics. The conformally Einstein equation (5) presents a system of PDE by using local coordinates. The equation in dimension $n = 2$ is trivial, which means local conformal flatness in dimension $n = 3$. However, it helps to get excellent solutions in dimension $n = 4$. In 1920, Brinkmann investigated a locally conformal manifold to be an Einstein manifold. Later, in 1924, he found global properties about conformally Einstein spaces [7]. In dimension four, conformal Einstein of non-reductive homogeneous spaces was investigated in [10]. In [16, 17], the product of surfaces with non-zero scalar curvature for conformally Einstein metrics in four-dimensional is considered. Gover and Nurowski in [13] found some tensorial conditions for a space to be conformally Einstein, and it is still an interesting subject to study. The purpose of this paper is to investigate the conformal Einstein Cahen-Wallach and Lorentzian two-symmetric four spaces.

This paper is structured as follows. We introduce the notion of some essential preliminaries about the subject in Section 2. We study conformally Einstein two-symmetric Lorentzian spaces in the third section. Section 4 is assigned to the study of conformally Einstein CW-spaces. In Section 5, we study the Schrödinger equations and the multiplying functions of conformally Einstein CW and two-symmetric Lorentzian spaces.

## 2 Preliminaries

We recall some basic definitions and concepts that we will use in the text. Suppose that $(M^4, g)$ is a connected pseudo-Riemannian manifold. The curvature tensor is determined by using $R(U,V) = [\nabla_U, \nabla_V] - \nabla_{[U,V]}$. If we put $R(\partial_k, \partial_l)\partial_j = R^i_{jkl}\partial_i$, then we create the Ricci tensor $\varrho$ by contracting on the first and third indices of $R$. We show the scalar curvature tensor $\tau$ by contracting all of the coefficients of $R$. Also, we obtain Weyl tensor with the following set of equations:

$$
\begin{aligned}
W_{iklm} = R_{iklm} &+ \frac{1}{2}(R_{im}g_{kl} - R_{il}g_{km} + R_{kl}g_{im} - R_{km}g_{il}) \\
&+ \frac{1}{6}R(g_{il}g_{km} - g_{im}g_{kl}).
\end{aligned}
$$

The expression for the divergence of the Weyl tensor is

$$
\begin{aligned}
divW(X,Y,Z) = &-\frac{1}{2}((\nabla_X \varrho)(Y,Z) - (\nabla_Y \varrho)(X,Z)) \\
&+ \frac{1}{12}(X(\tau)g(Y,Z) - Y(\tau)g(X,Z)).
\end{aligned}
$$

Since any conformally Einstein space is Bach-flat, let us consider the Bach tensor too. If we suppose that $W$ is the Weyl conformal tensor on $(M, g)$, then the *Bach tensor* is assumed by

$$
\mathcal{B} = \mathrm{div}_1\mathrm{div}_4 W + \frac{n-3}{n-2}W[\varrho],
$$

where $\{e_i\}$, is a pseudo-orthonormal basis, $\varepsilon_i = g(e_i, e_i)$, and $n$ is the dimension of $M$. The tensor $W[\varrho]$ is the Ricci contraction of $W$ by

$$W[\varrho](U,V) = \sum_{i,j} \varepsilon_i \varepsilon_j W(e_i, U, V, e_j)\varrho(e_i, e_j).$$

The symmetric $(0,2)$-tensor field

$$\mathcal{S} = \varrho - \frac{\tau}{2(n-1)} g \tag{2}$$

is called *Schouten tensor*. The failure of the Schouten tensor to be Codazzi (i.e., its covariant derivative is totally symmetric) is measured by the $(0,3)$-Cotton tensor given by:

$$\mathcal{C}_{ijk} = (\nabla_i \mathcal{S})_{jk} - (\nabla_j \mathcal{S})_{ik}. \tag{3}$$

The divergence of the Weyl tensor and the Cotton tensor satisfy

$$\mathcal{C}_{ijk} = -\frac{n-2}{n-3} \nabla^l W_{ijkl}. \tag{4}$$

Finally, $(M, g)$ is called (locally) conformally Einstein if every point $p \in M$ has an open neighborhood $U$ and a positive smooth function $\varphi$ defined on $U$ such that $(U, \bar{g} = \varphi^{-2} g)$ is Einstein. A manifold is conformally Einstein iff the equation

$$(n-2)\mathrm{Hes}_\varphi + \varphi\varrho = \frac{1}{n}\{(n-2)\Delta\varphi + \varphi\tau\} g. \tag{5}$$

has a non-constant solution, where $\mathrm{Hes}_\varphi = \nabla d\varphi$ and $\varrho$ are the Hessian of $\varphi$ and the Ricci tensor of $g$, respectively. The conformal Einstein equation in dimensions $n = 2$ and $n = 3$ are trivial cases. Therefore, the first non-trivial conformal Einstein is dimension four.

# 3 Conformally Einstein Two-Symmetric Lorentzian Spaces

A necessary and sufficient condition was given in [6], where it was shown that a four-dimensional pp-wave is conformally Einstein if and only if the Weyl tensor is harmonic (i.e., $divW = 0$). Accordingly, it is sufficient to check the accuracy of this statement.

**Theorem 3.1.** *The Weyl tensor of four-dimensional two-symmetric Lorentzian spaces is harmonic.*

**Proof.** Suppose $(M^4, g)$ is a two-symmetric Lorentzian manifold in which the metric $g$ is given in $(x, y, u, v)$ by Equation (1). We apply $\{\partial_i = \frac{\partial}{\partial x^i} : i = 1...4\}$ for local basis of the tangent space. The non-vanishing elements of the connection can be obtained:

$$\begin{aligned}
\nabla_{\partial_2}\partial_4 &= (ayv + py + qu)\partial_1,\\
\nabla_{\partial_3}\partial_4 &= (buv + sv + qy)\partial_1,\\
\nabla_{\partial_4}\partial_4 &= \frac{ay^2 + bu^2}{2}\partial_1 - (ayv + py + qu)\partial_2\\
&\quad - (buv + qy + su)\partial_3.
\end{aligned} \tag{6}$$

The non-vanishing elements of $R$ are determined by relations:

$$\begin{aligned}
R(\partial_2, \partial_4) &= (ax^4 + p)\partial_1 dy + q\partial_1 du - (ax^4 + p)\partial_2 dv - q\partial_3 dv,\\
R(\partial_3, \partial_4) &= q\partial_1 dy + (bx^4 + s)\partial_1 du - q\partial_2 dv - (bx^4 + s)\partial_3 dv,
\end{aligned}$$

Two-symmetric Lorentzian spaces are not flat. The Ricci tensor matrix is as follows;

$$\varrho(\partial_i, \partial_j) = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -(a+b)v - (s+p) \end{pmatrix}. \tag{7}$$

From (7), it is not hard to see that $(M, g)$ is Einstein iff $(M, g)$ is Ricci flat iff $a = -b,\ s = -p$. The metric inverse of $g$ is obtained by

$$g^{ij} = \begin{pmatrix} -((ay^2 + bu^2)v + py^2 + 2qyu + su^2) & 0 & 0 & 1\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The scalar curvature $\tau = g^{ab}\varrho_{ab}$ vanishes. By (2), the components of the Schouten tensor are calculated as follows:

$$\mathcal{S}(\partial_4, \partial_4) = -(a+b)v - (s+p). \tag{8}$$

Also, from (8), the Cotton tensor considered by (3) is

$$\mathcal{C}_{ijk} = 0, \quad 1 \leq i, j, k \leq 4. \tag{9}$$

The non-vanishing elements of the Weyl tensor (up to symmetries) are obtained as

$$W_{2424} = -W_{3434} = \frac{(b-a)v + s - p}{2}, \qquad W_{2434} = -q. \tag{10}$$

From (10), $(M, g)$ is conformally flat iff $b = a,\ s = p$ and $q = 0$. Using (4), and (9), clearly $divW$ vanishes. $\square$

From (7), remember that the only non-zero value of Ricci tensor $\varrho$ was $\varrho(\partial_4, \partial_4) = -((a+b)v + (s+p))$. We set $\alpha' = a + b$ and $E' = -(s+p)$, so $\varrho(\partial_4, \partial_4) = -\alpha' v + E'$ is linear, and the multiplying functions are calculated as follows:

**Theorem 3.2.** *Let $(M^4, g)$ be a two-symmetric Lorentzian space. According to the value of $\alpha'$, we have two separate cases:*

*(1) If $\alpha' = 0$, then $(M^4, g)$ is locally conformally Einstein with the multiplying function*

$$\varphi(x, y, u, v) = c_1 sin(\sqrt{\frac{E'}{2}}v) + c_2 cos(\sqrt{\frac{E'}{2}}v).$$

*(2) If $\alpha' \neq 0$, then $(M^4, g)$ is locally conformally Einstein with the multiplying Airy wave function*

$$\begin{aligned}\varphi(x, y, u, v) = \quad & c_1 \mathrm{AiryAi}\left(\frac{1}{2^{1/3}} \frac{\alpha' v - E'}{(-\alpha')^{2/3}}\right) \\ & + c_2 \mathrm{AiryBi}\left(\frac{1}{2^{1/3}} \frac{\alpha' v - E'}{(-\alpha')^{2/3}}\right),\end{aligned}$$

*where $c_1, c_2$ are arbitrary real constants.*

**Proof.** We need to investigate Equation (5). Let $\varphi$ be a solution of the conformally Einstein equation (5). Then $\sigma = -2\ln(\varphi)$ is a function that satisfies $\mathfrak{C} = \mathcal{C} + W(.,.,., \nabla\sigma)$ which is a $(0,3)$-tensor field. By Proposition 4.1, in [15], conditions $\mathfrak{C} = 0$ and $\mathcal{B} = 0$ are necessary for any solution of (5). If $(M^4, g)$ is weakly-generic, conditions $\mathfrak{C} = 0$ and $\mathcal{B} = 0$ are also sufficient for any solution of (5). ($(M^4, g)$ is weakly-generic if we consider the Weyl tensor, as map $W : TM \to \otimes^3 TM$ then $W$ is injective.)

**Case(1)**. Let $(M^4, g)$ be a two-symmetric Lorentzian space and $\alpha' = 0$. First, we need to show that $(M, g)$ is not trivially conformally Einstein; namely, it is not conformal flat. Considering the condition $\alpha' = 0$, from (10), one sees $(M, g)$ is conformally flat iff $2bv + s - p = 0,\ q = 0$. Thus we suppose that $2bv + s - p \neq 0$ and $q \neq 0$. If we consider the Weyl tensor, as map $W : TM \to \otimes^3 TM$, then $ker(W) = \{(x, y, u, v) : y = u = v = 0\} \neq 0$. It shows that $(M^4, g)$ is not weakly-generic. Let $\varphi(x, y, u, v)$ be an arbitrary function on $M$ which is positive and $\sigma = -2\ln(\varphi)$. Then by performing a simple calculation, the gradient of $\sigma$ is calculated as follows:

$$\nabla\sigma = \frac{2}{\varphi}\Big\{\ (\varphi_4 - \big[bv(y^2 - u^2) + py^2 + 2qyu + su^2\big]\varphi_1)\partial_1 - \varphi_2\partial_2\ - \varphi_3\partial_3\ - \varphi_1\partial_4\ \Big\},$$

So, the only non-zero elements of the tensor $\mathfrak{C} = \mathcal{C} + W(.,.,.,\nabla\sigma)$ are as follows (note that $\mathfrak{C}_{ijk} = -\mathfrak{C}_{jik}$ for all $1 \leq i, j, k \leq 4$).

$$\begin{aligned}
\varphi\mathfrak{C}_{242} &= (-2bv - s + p)\varphi_1,\\
\varphi\mathfrak{C}_{243} &= 2q\varphi_1,\\
\varphi\mathfrak{C}_{424} &= 2q\varphi_3 + (-2bv - s + p)\varphi_2,\\
\varphi\mathfrak{C}_{342} &= 2q\varphi_1,\\
\varphi\mathfrak{C}_{343} &= (-2bv - s + p)\varphi_1,\\
\varphi\mathfrak{C}_{344} &= -2q\varphi_2 + (-2bv - s + p)\varphi_3.
\end{aligned} \tag{11}$$

Since for any conformally Einstein manifold, $\mathfrak{C}$ is necessarily 0, so for (11), we have

$$\mathfrak{C}_{ijk} = 0, \qquad 1 \leq i, j, k, \leq 4. \tag{12}$$

Since $(M, g)$ is not conformally flat $(-2bv - s + p)$ and $q$ are not zero, so $\varphi\mathfrak{C}_{243} = 2q\varphi_1 = 0$, results that $\varphi_1 = 0$, and $\varphi$ is independent on the coordinate $x$. Again, using (11) and (12) we obtain

$$0 = (-2bv - s + p)\varphi\mathfrak{C}_{424} - 2q\varphi\mathfrak{C}_{344} = ((-2bv - s + p)^2 + (2q)^2)\varphi_2,$$
$$0 = (-2bv - s + p)\varphi\mathfrak{C}_{344} + 2q\varphi\mathfrak{C}_{424} = ((-2bv - s + p)^2 + (2q)^2)\varphi_3,$$

Since $-2bv - s + p \neq 0$ and $q \neq 0$, so $\varphi_2 = \varphi_3 = 0$. Therefore $\varphi$ is not dependent on the coordinates $y$ and $u$ too. Thus, $\mathfrak{C} = 0$ shows that

$$\varphi(x, y, u, v) = \Phi(v),$$

where $\Phi$ is a smooth function. We set

$$\psi = 2\mathrm{Hes}_\varphi + \varphi\rho - \frac{1}{4}\{2\triangle\varphi + \varphi\tau\}g.$$

Obviously, $\psi_{ij} = -\psi_{ji}$ for all $i, j \in \{1, \cdots, 4\}$. From (6), a simple calculation shows that $\mathrm{Hes}_\varphi = \mathrm{Hes}_\Phi = \Phi_{44}$. On the other hand, $\triangle\varphi = \triangle\Phi = 0$. Thus, the components of the tensor $\psi$ are as follows:

$$\psi(4, 4) = 2\Phi_{44} + E'\Phi,$$

where $\Phi_i = \partial\Phi/\partial x_i$ denote the corresponding partial derivatives. Since $\psi$ has to be 0, so we have the differential equation

$$2\varphi_{44} + E'\varphi = 0.$$

We obtain the solution:

$$\varphi(x, y, u, v) = c_1 sin(\sqrt{\frac{E'}{2}}v) + c_2 cos(\sqrt{\frac{E'}{2}}v),$$

for some arbitrary real constants $c_1$ and $c_2$. For $c_1 = c_2 = 1$ and $E' = 2$ the function $\varphi(v)$ is plotted in Figure 1.

**Case(2)**. By (10), we already saw that the two-symmetric Lorentzian manifold $(M^4, g)$ was conformally flat iff $b = a,\ s = p,\ q = 0$. Thus $(M^4, g)$ is not trivially conformally Einstein. If we consider the Weyl tensor, as map $W : TM \to \otimes^3 TM$, then $ker(W) = \{(x, y, u, v) : y = u = v = 0\} \neq 0$. It shows that $(M^4, g)$ is not weakly-generic. Let $\varphi(x, y, u, v)$ be an arbitrary function on $M$ which is positive and $\sigma =$

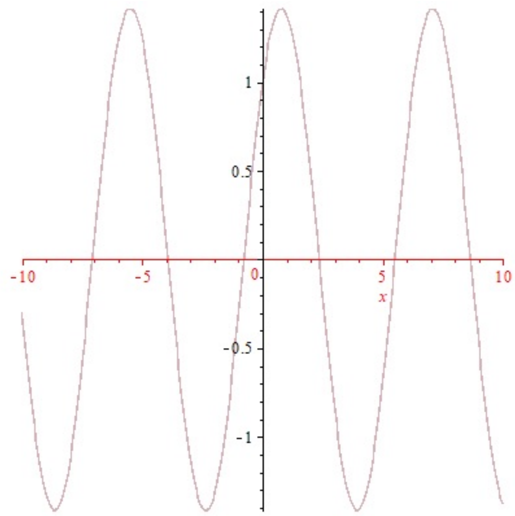


**Figure 1:** The plot of $\Phi^1(v)$.

$-2\ln(\varphi)$. Then by performing a preliminary calculation, the gradient of $\sigma$ is calculated as follows:

$$\nabla\sigma = \frac{2}{\varphi}\Big\{\ (\varphi_4 - \big[(ay^2+bu^2)v + py^2 \\ +2qyu + su^2\big]\varphi_1)\partial_1 - \varphi_2\partial_2\ - \varphi_3\partial_3\ - \varphi_1\partial_4\ \Big\},$$

Therefore, the only non-zero elements of the tensor $\mathfrak{C} = \mathcal{C} + W(.,.,.,\nabla\sigma)$ are as follows, where $\mathfrak{C}_{ijk} = -\mathfrak{C}_{jik}$ and $1 \leq i,j,k \leq 4$.

$$\begin{aligned}
\varphi\mathfrak{C}_{242} &= (-bv - s + av + p)\varphi_1,\\
\varphi\mathfrak{C}_{243} &= 2q\varphi_1,\\
\varphi\mathfrak{C}_{424} &= 2q\varphi_3 + (-bv - s + av + p)\varphi_2,\\
\varphi\mathfrak{C}_{342} &= 2q\varphi_1,\\
\varphi\mathfrak{C}_{343} &= (-bv - s + av + p)\varphi_1,\\
\varphi\mathfrak{C}_{344} &= -2q\varphi_2 + (-bv - s + av + p)\varphi_3.
\end{aligned} \tag{13}$$

By (12), $\mathfrak{C}_{ijk} = 0$ for all $1 \leq i,j,k, \leq 4$. Since $(M^4, g)$ is not conformally flat, $-bv - s + av + p$ and $q$ are not zero, so $\varphi\mathfrak{C}_{243} = 2q\varphi_1 = 0$, results that $\varphi_1 = 0$, and $\varphi$ is independent on the coordinate $x$. Again, using (13) and (12), we obtain

$$\begin{aligned} 0 &= (-bv - s + av + p)\varphi\mathfrak{C}_{424} - 2q\varphi\mathfrak{C}_{344} \\ &= ((-bv - s + av + p)^2 + (2q)^2)\varphi_2, \\ 0 &= (-bv - s + av + p)\varphi\mathfrak{C}_{344} + 2q\varphi\mathfrak{C}_{424} \\ &= ((-bv - s + av + p)^2 + (2q)^2)\varphi_3, \end{aligned}$$

Since $-bv - s + av + p \neq 0$ and $q \neq 0$, so $\varphi_2 = \varphi_3 = 0$. Therefore $\varphi$ is not dependent on the coordinates $y$ and $u$ too. Thus, $\mathfrak{C} = 0$ shows that

$$\varphi(x, y, u, v) = \Phi(v),$$

where $\Phi$ is a smooth function. We now calculate the Equation (5) for $\varphi$. Set

$$\psi = 2\mathrm{Hes}_\varphi + \varphi\rho - \frac{1}{4}\{2\triangle\varphi + \varphi\tau\}g.$$

Obviously, $\psi_{ij} = -\psi_{ji}$ for all $i, j \in \{1, \cdots, 4\}$. From (6), a simple calculation shows that $\mathrm{Hes}_\varphi = \mathrm{Hes}_\Phi = \Phi_{44}$. On the other hand, $\triangle\varphi = \triangle\Phi = 0$. Thus, the components of the tensor $\psi$ are as follows:

$$\psi(4, 4) = 2\Phi_{44} - (\alpha' v - E')\Phi,$$

where $\Phi_i = \partial\Phi/\partial x_i$ denote the corresponding partial derivatives. Since $\psi$ has to be 0, we get the following equation:

$$2\varphi_{44} - (\alpha' v - E')\varphi = 0, \tag{14}$$

known as the Airy equation or the Stokes equation. This type of equation can only be solved with Maple. Using the Maple command "desolve", Equation (14) can be solved in terms of the Airy wave functions AiryAi and AiryBi, as follows:

$$\varphi(x, y, u, v) = c_1 \mathrm{AiryAi}\left(\frac{1}{\sqrt[3]{2}}\frac{\alpha' v - E'}{(-\alpha')^{2/3}}\right) + c_2 \mathrm{AiryBi}\left(\frac{1}{\sqrt[3]{2}}\frac{\alpha' v - E'}{(-\alpha')^{2/3}}\right)$$

The expression $(-\alpha')^{2/3}$ in the above argument may be imaginary. We always deal with functions with real value. Therefore we have to select $\alpha'$ so that it allows us to find solutions with real value. If $\alpha'$ is chosen

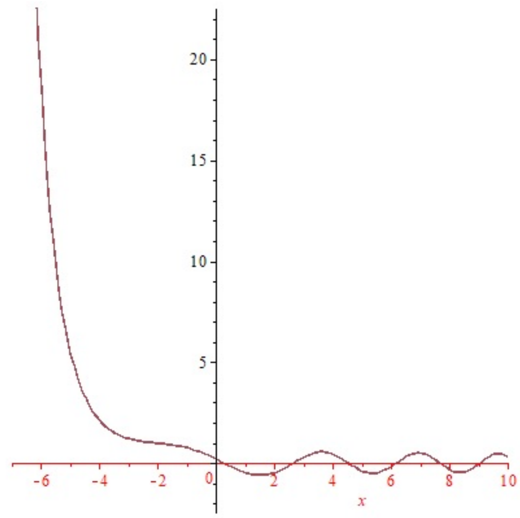


**Figure 2:** The plot of $\Phi^2(v)$.

as a unit, we can select the roots with the real value of $(-\alpha')^{2/3}$ and let the term become one. Thus by selecting the unit slope $\alpha' = a + b$, we can choose the real roots of $(-\alpha)^{2/3}$. For $c_1 = c_2 = 1$, $\alpha' = -1$ and $E' = -2$ the function $\Phi(v)$ is as follows:

$$\varphi(x, y, u, v) = \mathrm{AiryAi}\left(\frac{-v-2}{\sqrt[3]{2}}\right) + \mathrm{AiryBi}\left(\frac{-v-2}{\sqrt[3]{2}}\right),$$

which is plotted in Figure 2. $\square$

## 4 Conformally Einstein Cahen-Wallach Spaces

As we already mentioned, CW-spaces are pp-waves with $H = \sum_{i=1}^{n} \lambda_i (t_i)^2$. So for a CW-space $(M^4, g)$, there are local coordinates $(x, y, u, v)$ such that

$$g = 2dxdv + (dy)^2 + (du)^2 + \left(py^2 + su^2\right)(dv)^2, \tag{15}$$

for real constants $p$ and $s$. Indeed, CW-space is a special case of two-symmetric Lorentzian space. Comparing metrics (15) and (1), we see that for CW-spaces:

$$a = b = q = 0. \tag{16}$$

Thus, we have the following result immediately.

**Corollary 4.1.** *The Weyl tensor of four-dimensional CW-spaces is harmonic.*

In the previous section, Theorem 3.2, we examined conformally Einstein two-symmetric Lorentzian spaces in two separate states, $a = b$, and $a \neq b$, with multiplying functions (3.2) and (11), respectively. Therefore, we expect CW-spaces to be conformally Einstein with one of the multiplying functions (3.2) or (11). But, by (16), for CW-spaces we have $a = b = q = 0$. Thus, we need to obtain the multiplying function again for this special case of two-symmetric Lorentzian spaces (i.e., CW-spaces). Consider CW-space $(M^4, g)$ with local coordinates $(x, y, u, v)$ which the metric $g$ is given by Equation (15). From (7), when $a = b = q = 0$, the Ricci tensor is;

$$\varrho(\partial_i, \partial_j) = -(s+p)(dv)^2 \tag{17}$$

From (17), it is not hard to see that $(M^4, g)$ is Einstein iff $(M^4, g)$ is Ricci flat iff $s = -p$. The non-vanishing elements of the Weyl tensor (up to symmetries) are obtained as

$$W_{2424} = -W_{3434} = \frac{s-p}{2}. \tag{18}$$

From (10), $(M, g)$ is conformally flat iff $s = p$. By (17), the only non-zero value of Ricci tensor $\varrho$ was $\varrho(\partial_4, \partial_4) = -(s+p)$. We set $E' = -(s+p)$. The multiplying functions are calculated as follows:

**Theorem 4.2.** *Let $(M^4, g)$ be a CW-space. Then $(M^4, g)$ is locally conformally Einstein with the multiplying function*

$$\varphi(x, y, u, v) = c_1 sin(\sqrt{\frac{E'}{2}}v) + c_2 cos(\sqrt{\frac{E'}{2}}v).$$

*where $c_1, c_2$ are arbitrary real constants.*

**Proof.** Now we have to investigate the necessary condition $\mathfrak{C} = 0$ for any solution of (5). First, we suppose that $(M^4, g)$ is not conformally flat (i.e., $s \neq p$). $(M^4, g)$ is not weakly-generic (if we consider the Weyl tensor, as map $W : TM \to \otimes^3 TM$, then $ker(W) = \{(x, y, u, v) : y = u = v = 0\} \neq \phi$). Thus we have to investigate Equation (5) itself too. Let $\varphi(x, y, u, v)$ be an arbitrary function on $M$ which is positive and $\sigma = -2\ln(\varphi)$. Then by performing a simple calculation, the gradient of $\sigma$ is calculated as follows:

$$\nabla\sigma = \frac{2}{\varphi}\{ (-\varphi_4 + s^2\varphi_1 + p^2\varphi_1)\partial_1 \\ -\varphi_2\partial_2 - \varphi_3\partial_3 - \varphi_1\partial_4 \},$$

So, the only elements of the tensor $\mathfrak{C} = \mathcal{C} + W(.,.,.,\nabla\sigma)$ that do not become zero are as follows. $\mathfrak{C}_{ijk} = -\mathfrak{C}_{jik}$ where $1 \leq i, j, k \leq 4$.

$$\begin{aligned} \varphi\mathfrak{C}_{422} &= -(p+s)\varphi_1, \\ \varphi\mathfrak{C}_{244} &= -(p+s)\varphi_2, \\ \varphi\mathfrak{C}_{343} &= -(p+s)\varphi_1, \\ \varphi\mathfrak{C}_{434} &= -(p+s)\varphi_3. \end{aligned} \tag{19}$$

Since for any conformally Einstein manifold, $\mathfrak{C}$ is necessarily 0, so for (19), we have

$$\mathfrak{C}_{ijk} = 0, \qquad 1 \leq i, j, k, \leq 4.$$

Because we assumed that $(M^4, g)$ is not conformally flat, from Equation (18), we see that $-(p+s)$ can not be zero. So, (19) results that $\varphi_1 = 0$, and $\varphi$ is not dependent on the coordinate $x, y$, and $u$. Thus, $\mathfrak{C} = 0$ results that

$$\varphi(x, y, u, v) = \Phi(v),$$

where $\Phi$ is a smooth function. We set

$$\psi = 2\mathrm{Hes}_\varphi + \varphi\rho - \frac{1}{4}\{2\triangle\varphi + \varphi\tau\}g.$$

Obviously, $\psi_{ij} = -\psi_{ji}$ for all $i, j \in \{1, \cdots, 4\}$. A simple calculation shows that $\mathrm{Hes}_\varphi = \mathrm{Hes}_\Phi = \Phi_{44}$. On the other hand, $\triangle\varphi = \triangle\Phi = 0$. Thus, the components of the tensor $\psi$ are as follows:

$$\psi(4, 4) = 2\Phi_{44} - (p+s)\Phi,$$

where $\Phi_i = \partial\Phi/\partial x_i$ denote the corresponding partial derivatives. Since $\psi$ has to be 0, so

$$2\varphi_{44} - (p+s)\varphi = 0.$$

which can be solved as follows:

$$\varphi(v) = c_1 sin(\sqrt{\frac{E'}{2}}v) + c_2 cos(\sqrt{\frac{E'}{2}}v),$$

for some constants $c_1$ and $c_2$. This proves the theorem. □

# 5 Schrödinger Equation

The Schrödinger equation forms the basis of the quantum description, which governs non-relativistic wave mechanics. In matters of quantum mechanics, we do not work with forces. We work with potentials. We have to model the setting as a potential well instead of a force diagram.

For this purpose, we settle a model for capacitor potential energy. Following [12, 14], consider an electric field generated by a capacitor (parallel plate capacitor). Ignoring edge factors, this field is monotone across the capacitor. We apply the time-independent Schrödinger equation for time-independent potential as follows:

$$-\frac{h^2}{2m}\frac{d^2\Phi(x)}{dx^2} = (E - U(x))\Phi(x), \tag{20}$$

where $E$ is the total energy of the system and $\dfrac{h^2}{2m}$ is constant. The expression $U(x)$ indicates the particle's potential energy. Thus the expression $(E - U(x))$ is assumed as the particle's kinetic energy. An expression in the Schrödinger equation that controls the treatment of the wave function is the expression $U(x)$. According to the two available cases in Theorem 3.2 for the conformal Einstein Lorentzian two-symmetric manifolds, we divide the model into two areas relying on the potential available in these areas. Zero and linear potential areas are considered, namely $U(x) = 0$ and $U(x) = \alpha x$, where the slope $\alpha$ is a non-zero constant.

**Corollary 5.1.** *The Schrödinger equation* (20) *in each area is as follows:*

(1) *Let $U(x) = 0$, i.e., there is no external influence, so the Schrödinger equation (20) turns into*
$$\frac{d^2\Phi(x)}{dx^2} = -k^2\Phi(x), \tag{21}$$
*where* $k = \sqrt{\dfrac{2mE}{h^2}}$.

(2) *Let $U(x) = \alpha x$, i.e., in the central area of the capacitor, the potential is linear, so the Schrödinger equation (20) turns into*
$$\frac{d^2\Phi(x)}{dx^2} = k^2(\alpha x - E)\Phi(x), \tag{22}$$
*where* $k = \sqrt{\dfrac{2m}{h^2}}$.

We now want to prove the main result:

**Theorem 5.2.** *The multiplying functions (3.2) and (11) in Theorem 3.2 are solutions to the Schrödinger equations (21) and (22) in Corollary 5.1, respectively.*

**Proof. Case(1).** By Theorem 3.2, the multiplying function (3.2) is as follows:
$$\varphi(v) = c_1 sin(\sqrt{\frac{E'}{2}}v) + c_2 cos(\sqrt{\frac{E'}{2}}v).$$
If we set $E' = \dfrac{4mE}{h^2}$, then a preliminary calculation shows that $\varphi$ satisfies (21), which proves the theorem for Case(1).
**Case(2).** By Theorem 3.2, the multiplying function (11) was
$$\varphi(v) = c_1 \text{AiryAi}\left(\frac{1}{2^{1/3}}\frac{\alpha' v - E'}{(-\alpha')^{2/3}}\right) + c_2 \text{AiryBi}\left(\frac{1}{2^{1/3}}\frac{\alpha' v - E'}{(-\alpha')^{2/3}}\right),$$
where AiryAi and AiryBi are wave functions. They are linked to the famous Bessel functions, in the form of Ai and Bi. A$i(x)$ for $xin\mathbb{R}$ is determined by the improper Riemann integral:
$$\text{Ai(x)} = \frac{1}{\pi}\lim_{b\to\infty}\int_0^b cos(\frac{t^3}{3} + xt)dt,$$

that converges by Dirichlet's exam, and the Airy function B$i(x)$ is determined by:

$$\mathrm{B}i(x)=\frac{1}{\pi}\lim_{b\to\infty}\int_0^b[-exp(\frac{t^3}{3}+xt)+sin(\frac{t^3}{3}+xt)]dt,$$

which they are linearly independent. Putting $m=\frac{h^2}{4}$, $\alpha=\alpha'$, $E=E'$ and $v$ instead of $x$, Equation (22), becomes

$$\frac{d^2\Phi(v)}{dv^2}=\frac{1}{2}(\alpha' v-E')\Phi(v). \tag{23}$$

Substituting function (11) into the Schrodinger equation (23), and by evaluating the second derivative gives

$$\frac{d\varphi(v)}{dv}=-(\frac{\alpha'}{2})^{1/3}$$
$$\left(c_1\mathrm{AiryAi}\left(1,\frac{1}{2^{1/3}}\frac{\alpha' v-E'}{(-\alpha')^{2/3}}\right)+c_2\mathrm{AiryBi}\left(1,\frac{1}{2^{1/3}}\frac{\alpha' v-E'}{(-\alpha')^{2/3}}\right)\right),$$
$$\frac{d^2\varphi(v)}{dv^2}=$$
$$\frac{\alpha' v-E'}{2}\left(c_1\mathrm{AiryAi}\left(\frac{1}{2^{1/3}}\frac{\alpha' v-E'}{(-\alpha')^{2/3}}\right)+c_2\mathrm{AiryBi}\left(\frac{1}{2^{1/3}}\frac{\alpha' v-E'}{(-\alpha')^{2/3}}\right)\right),$$

which is equal to the other side of Schrodinger equation (23), namely $\frac{1}{2}(\alpha' v-E')\Phi(v)$. This proves the theorem for Case (2). $\square$

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