On Elliptic Curves Via Heron Triangles and Diophantine Triples

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Abstract. In this article, we construct families of elliptic curves arising from the Heron triangles and Diophantine triples with the Mordell-Weil torsion subgroup of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. These families have ranks at least 2 and 3, respectively, and contain particular examples with rank equal to 7.

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1. Introduction

Triangles with integral sides and area have been considered by Indian mathematician Brahmagupta (598-668 A.D.). In general, the sides and area are related by a formula first proved by Greek mathematician Heron of Alexandria (c. 10 A.D-c. 75 A.D.) as

\[ S = \sqrt{P(P-a)(P-b)(P-c)}, \]

where \( P = (a + b + c)/2 \) is the semi perimeter.

Triangles with rational sides and area are known as the Heron triangles (for more information and fundamental results on Heron triangles, see [7, 8, 11]).

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Goins and Maddox have studied Heron triangles by considering the elliptic curve
\[ E^{(n)}_{\tau} : y^2 = x(x - n\tau)(x + n\tau^{-1}) \]
as a generalization of the congruent number problem (see [8]). In the same paper, they also have found 4 curves of rank 3 with torsion subgroup \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \). Campbell and Goins ([2]) by analyzing the elliptic curve
\[ E_t : y^2 = x^3 + (t^2 + 2)x^2 + x \]
defined over the rational function field \( \mathbb{Q}(t) \) described connections between the problem of finding Heron triangles with a given area possessing at least one side of a particular length and rational Diophantine quadruples and quintuples. They also have studied the relation between these problems and elliptic curves with torsion subgroup \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z} \), and found a new elliptic curve with this torsion having rank 3 and an infinite family of elliptic curves with torsion subgroup \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z} \) and rank at least 1. Having constructed a family of Diophantine triples such that the correspondent elliptic curve over \( \mathbb{Q} \) has torsion subgroup \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) and rank 5, Aguirre et al. [1] have obtained two examples of elliptic curves over \( \mathbb{Q} \) with torsion subgroup \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) and rank equal to 11. Dujella and Peral in a joint work [6] have created subfamilies of elliptic curves coming from the Heron triangles of ranks at least 3, 4, and 5. They also have given examples of elliptic curves over \( \mathbb{Q} \) with rank equal to 9 and 10.

This paper is organized as follows. In Section 2, a family of elliptic curves arising from Heron triangles introduced by Fine [7] is considered and shown that the family has torsion subgroup \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \), and rank at least 2, and a subfamily of rank \( \geq 3 \). In Theorem 2.6, a subfamily of \( Y^2 = (aX + 1)(bX + 1)(cX + 1) \) of rank \( \geq 2 \) is given. This is a generalization of Dujella’s work done in [4]. Therein, Dujella extended the Diophantine triple \( (a, b, c) = (k - 1, k + 1, 4k) \) to a quadruple by studying \( Y^2 = (aX + 1)(bX + 1)(cX + 1) \), and proved that this elliptic curve has generic rank 1 over \( \mathbb{Q} \). In Section 3, some examples of elliptic curves with rank 7 are given.
2. Main Results

Let $S$ be the area of the triangle $(a, b, c)$, i.e., $S = \sqrt{P(P-a)(P-b)(P-c)}$, where $P = (a+b+c)/2$. This formula, due to Heron, ensures us to have an elliptic curve $v^2 = u(u-a)(u-b)(u-c)$ with non-torsion point $(u, v) = (P, S)$. The curve therefore is birationally equivalent to $y^2 = (x + ab)(x + bc)(x + ac)$, with corresponding (non-torsion) point $(x, y) = (-abcP^{-1}, abcSP^{-2})$, and is equivalent to $Y^2 = (aX + 1)(bX + 1)(cX + 1)$, with corresponding point $(X, Y) = (-P^{-1}, SP^{-2})$. In the sequel, we are going to treat with special families coming from these two kinds of elliptic curves.

Consider the elliptic curve $E_k : y^2 = (x + a(k)b(k))(x + b(k)c(k))(x + a(k)c(k))$ associated to the Fine triple:

$$\begin{cases}
  a(k) = 10k^2 - 8k + 8, \\
  b(k) = k(k^2 - 4k + 20), \\
  c(k) = (k + 2)(k^2 - 4),
\end{cases} \tag{1}$$

arising from a Heron triangle which has rational area $4k(k^2 - 4)^2$ (see [7]). (Note that multiplication of sides in (1) by $(2(k^2 - 4))^{-1}$ implies that the resulting triangle to have area $k$.) One can easily check that $E_k$ has three rational points of order two:

$$\begin{cases}
  T_1 = (-k(10k^2 - 8k + 8)(k^2 - 4k + 20), 0), \\
  T_2 = (-k(k + 2)(k^2 - 4k + 20)(k^2 - 4), 0), \\
  T_3 = -(k + 2)(10k^2 - 8k + 8)(k^2 - 4), 0).
\end{cases}$$

As the change of coordinates $(x, y) \rightarrow (x - a(k)b(k), y)$ does not affect the group structure of $E_k(\mathbb{Q})$, we may consider $E_k$ in the form $y^2 = x^3 + Ax^2 + Bx$, in which

$$\begin{align*}
  A &= k^6 - 12k^5 + 116k^4 - 480k^3 + 304k^2 - 448k - 64, \\
  B &= 4k(5k^2 - 4k + 4)(k^2 - 4k + 20)(3k^2 - 12k - 4) \\
  &\times (k^3 - 8k^2 + 4k - 16). \tag{2}
\end{align*}$$

**Theorem 2.1.** Let $a(k)$, $b(k)$ and $c(k)$ be defined as (1), where $k$ is an arbitrary rational number different from 0, -2, and 2. Then the elliptic
curve

\[ E : y^2 = (x + a(k)b(k))(x + b(k)c(k))(x + a(k)c(k)) \]

defined over \( Q(k) \) has torsion subgroup \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \).

**Proof.** The points \( O \) (the point at infinity), \( T_1 = (-a(k)b(k), 0), T_2 = (-b(k)c(k), 0) \), and \( T_3 = (-a(k)c(k), 0) \) form a subgroup of the torsion group \( E(\mathbb{Q}(k))_{\text{tors}} \) isomorphic to \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \). By Mazur’s theorem and a theorem of Silverman (see [13], Theorem 11.4, p.271), it suffices to check that there exists no point \( E(\mathbb{Q}(k)) \) of order four, six or eight. If there exists a point \( T \) on \( E(\mathbb{Q}(k)) \) such that \( 2T \in \{ T_1, T_2, T_3 \} \), then by Proposition (see [9], 4.1, p.37), implies that all of the expressions

\[
-a(k)b(k) + a(k)b(k) = 0, \\
-a(k)b(k) + b(k)c(k) = k(k^2 - 4k + 20)(k^3 - 8k^2 + 4k - 16), \\
-a(k)b(k) + a(k)c(k) = 4(5k^2 - 4k + 4)(3k^2 - 12k - 4),
\]

must be perfect squares. But, it is easily seen that for \( k = 1 \) none of the above expressions are perfect squares. Similarly, if \( 2T = T_2 \) and \( 2T = T_3 \), then all of the expressions

\[
-b(k)c(k) + a(k)b(k) = -k(k^2 - 4k + 20)(k^3 - 8k^2 + 4k - 16), \\
-b(k)c(k) + b(k)c(k) = 0, \\
-b(k)c(k) + a(k)c(k) = -(k^2 - 12k + 4)(k - 2)^2(k + 2)^2,
\]

as well as

\[
-a(k)c(k) + a(k)b(k) = -4(5k^2 - 4k + 4)(3k^2 - 12k - 4), \\
-a(k)c(k) + b(k)c(k) = (k^2 - 12k + 4)(k - 2)^2(k + 2)^2, \\
-a(k)c(k) + a(k)c(k) = 0,
\]

must be perfect squares. But, it is easily seen that for \( k = 1 \) none of the above expressions are perfect squares. This contradiction shows that \( T \notin \{ T_1, T_2, T_3 \} \). Thus, by [10] it is to prove that there exists no point \( T \).
such that $3T \in \{T_1, T_2, T_3\}$. If there exists a point $T = (x, y)$ on $E(\mathbb{Q}(k))$ such that $3T = T_1$, $T \neq T_1$, then from $2T = -T + T_1$, the equation

$$x^4 - 6h_1(k)x^2 - 4h_1(k)h_2(k)x - 3h_2(k)^2 = 0,$$

is obtained in which

$$h_1(k) = -12k^5 - 480k^3 + 116k^4 + 304k^2 - 448k - 64 + k^6,$$

$$h_2(k) = 4k(5k^2 - 4k + 4)(k^2 - 4k + 20)(3k^2 - 12k - 4) \times (k^3 - 8k^2 + 4k - 16).$$

It can be easily seen that for $k = 1$, the equation (3), namely

$$x^4 + 3498x^2 + 195841360x - 21157921200 = 0$$

has no rational solution. Similarly it can be checked that there does not exist any point $T$ on $E(\mathbb{Q}(k))$ such that $3T = T_2$, $T \neq T_2$, and $3T = T_3$, $T \neq T_3$. Therefore, $E(\mathbb{Q}(k))_{\text{tors}} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. □

**Theorem 2.2.** With the terminology in Theorem 2.1, $\text{rank } E(\mathbb{Q}(k)) \geq 2$.

**Proof.** Evidently the non torsion points

$$\mathcal{P}_1 = (-a(k)b(k)c(k)P^{-1}(k), a(k)b(k)c(k)S(k)P(k)^{-2}),$$

$$\mathcal{P}_2 = (0, a(k)b(k)c(k)),$$

lie on $E(\mathbb{Q}(k))$, where $P(k)$ and $S(k)$ are respectively the associated semi perimeter and area to $(a(k), b(k), c(k))$.

For $k = 1$, the elliptic curve $E(\mathbb{Q}(k))$ turns into

$$E_1: y^2 = x^3 - \frac{73}{36}x^2 - \frac{85}{4}x + \frac{7225}{144},$$

with

$$\mathcal{P}_1 = \left(\frac{85}{18}, \frac{85}{27}\right), \quad \mathcal{P}_2 = \left(0, \frac{85}{12}\right).$$

The Néron-Tate height matrix [14, p. 230] associated to these points is of non vanishing determinant $\approx 2.30842249514247$ (carried out with SAGE [12]) showing that the points are linearly independent. Therefore
the rank of $E$ over $\mathbb{Q}(k)$ is $\geq 2$, and hence, by the specialization theorem of Silverman [13], the rank $E_k(\mathbb{Q}) \geq 2$, for all but finitely many rational numbers $k$. □

**Proposition 2.3.** For each $2 \leq r \leq 7$, there exists some $k$ such that $E_k$ defined in Theorem 2.1 has torsion subgroup $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and rank $r$.

**Proof.** The first part is readily obtained from Theorem 2.1. For the second part, it suffices to note that for the values of $k = 4,7,19,11,98/625$, and $88/31$, the corresponding ranks with using of Mwrank [3], are $2,3,4,5,6$, and $7$, respectively. □

Now, we are ready to show our main result:

**Theorem 2.4.** There exists a subfamily of $E_k$ of rank $\geq 3$ over $\mathbb{Q}(m)$.

**Proof.** Obviously the non torsion points

\[ \mathcal{P}_1 = (4(5k^2 - 4k + 4)(k^2 - 4k + 20), 8(5k^2 - 4k + 4)(k^2 - 4k + 20)(k - 2)^3), \]

\[ \mathcal{P}_2 = (2(5k^2 - 4k + 4)k(k^2 - 4k + 20), 2k(k - 2)(5k^2 - 4k + 4)(k^2 - 4k + 20)(k + 2)^3), \]

lie on the curve $E_k : y^2 = x^3 + Ax^2 + Bx$. In order to find a subfamily of rank $\geq 3$, we proceed as following. Let $B_1 = 2k(3k^2 - 13k - 4)(k^2 - 4k + 20)$, and for some rational numbers $M, N, e$, $\mathcal{P}_3 = (B_1 M^2 / e^2, B_1 M N / e^3)$ be on $E_k$. This implies the quartic equation $B_1 M^4 + A M^2 e^2 + B_2 e^4 = N^2$. Taking $M = e = 1$, we get $(k - 6)(k + 2)^5 = N^2$, hence, $(k - 6)(k + 2) = z^2$, where $z = N / (k + 2)^2$. Using the rational solution $(k, z) = (6, 0)$, the parametric solution is then $(k, z) = (2(3m^2 + 1) / (m^2 - 1), 8m / (m^2 - 1))$, where $m \in \mathbb{Q} \setminus \{\pm 1\}$. Therefore, $N = 2^{12}m^5 / (m^2 - 1)^3$ and $\mathcal{P}_3$ turns into

\[ \mathcal{P}_3 = (B_1, B_1 N) = \left( \frac{2^{12}(3m^2 + 1)(m^4 + 4m^2 + 1)(m^4 + 1)}{(m^2 - 1)^5}, \frac{2^{21}(3m^2 + 1)(m^4 + 4m^2 + 1)(m^4 + 1)m^5}{(m^2 - 1)^8} \right). \]
Thus $E_k$ turns into $E_m : y^2 = x^3 + Ax^2 + Bx$ with

$$A = \frac{2^{13}(m^{12} + 14m^{10} - 5m^8 + 4m^6 + 11m^4 + 6m^2 + 1)}{(m^2 - 1)^6},$$

$$B = -\frac{2^{24}(m^6 - 5m^4 - 3m^2 - 1)M_1M_3}{(-1 + m^2)^{10}},$$

and three non torsion points

$$P_1 = \left( \frac{2^{12}M_1}{(m^2 - 1)^4}, \frac{2^{19}M_1(1 + m^2)^3}{(m^2 - 1)^7} \right),$$

$$P_2 = \left( \frac{2^{12}(m^4 + 1)M_2}{(-1 + m^2)^5}, \frac{2^{20}(m^4 + 1)m^4(m^2 + 1)M_2}{(m^2 - 1)^8} \right),$$

$$P_3 = \left( \frac{2^{12}(m^4 + 1)M_3}{(m^2 - 1)^5}, \frac{2^{21}(m^4 + 1)m^5M_3}{(m^2 - 1)^8} \right),$$

where

$$M_1 = (m^4 + 1)(5m^4 + 4m^2 + 1),$$

$$M_2 = (3m^2 + 1)(5m^4 + 4m^2 + 1),$$

$$M_3 = (3m^2 + 1)(m^4 + 4m^2 + 1).$$

Regarding the specialization theorem, since for $m = 1/2$, the Néron-Tate height matrix associated to these points has non vanishing determinant $\approx 11.9727247292862$, then $E_m$ as a subfamily of $E_k$ is of rank $\geq 3$ over $\mathbb{Q}(m)$. □

We say that ([5]) the Diophantine triple $(a, b, c)$ has the property $D(n)$, for any non zero integer $n$, whenever there exist rational $r, s, t$ such that

$$ab + n = r^2, \quad ac + n = s^2, \quad bc + n = t^2.$$

**Theorem 2.5.** Let $(a, b, c) = (k - 1, k + 1, 4k)$ with property $D(1)$. Then there exists a subfamily of $C : Y^2 = (aX + 1)(bX + 1)(cX + 1)$ over $\mathbb{Q}$ with rank $\geq 2$.

**Proof.** Consider the triple $(a, b, c) = (k - 1, k + 1, 4k)$ with property $D(1)$. The curve $C_k : Y^2 = ((k - 1)X + 1)((k + 1)X + 1)(4kX + 1), k \in \mathbb{Q}$, has non
torsion point $P_1 = (0,1)$. [The triple $(k - 1, k + 1, 4k)$ has the property $D(1)$, but does not form any triangle (note $a(k) + b(k) < c(k)$).] In order to find a subfamily of $C_k$ of rank $\geq 2$, let $P_2 = (-P^{-1}, P^{-2}S)$, be on the curve, where $P = 3k$ and $S = k\sqrt{3(4k^2 - 1)}$. This implies to have some rational $u$ such that $-3(4k^2 - 1) = u^2$. Using the rational solution $(k, u) = (1/2, 0)$, the parametric solution for $k$ is then $k = \frac{m^2 - 12}{2(m^2 + 12)}$, where $m \in \mathbb{Q}$. Henceforth, $C_k$ turns into

$$C_m : Y^2 = \left(\frac{-m^2 - 36}{2(m^2 + 12)}X + 1\right)\left(\frac{3(m^2 + 4)}{2(m^2 + 12)}X + 1\right)\left(\frac{2(m^2 - 12)}{m^2 + 12}X + 1\right),$$

with two non torsion points

$$P_1 = (0,1),$$

$$P_2 = \left(-\frac{2(m^2 + 12)}{3(m^2 - 12)}, \frac{8m}{3(m^2 - 12)}\right).$$

The associated height matrix to these points at $m = 1$ has non vanishing determinant $\approx 2.87442404831027$ showing that these points are linearly independent, hence rank $C_m(\mathbb{Q}) \geq 2$ for all but finitely many $m$'s. The elliptic curve $y^2 = (ax + 1)(bx + 1)(cx + 1)$ with the point $(x, y)$ is isomorphic to $y^2 = x^3 + (ab + ac + bc)x^2 + abc(a + b + c)x + a^2b^2c^2$, with the corresponding point $(abcd, abc)$. □

**Remark 2.5.** We should mention that in [4], two subfamilies of $C_k$ from Theorem 2.5 with rank $\geq 2$ and one subfamily with rank $\geq 3$ were constructed. However they considered the problem for the integer values of $k$ between 1 and 1000, while in our case the values of $k = \frac{m^2 - 12}{2(m^2 + 12)}$, are rational numbers less than 1.

### 3. Specialization of High Rank

In this stage we want to find curves having large ranks possible. The main idea here is that a curve is more likely to have large rank if $|E(\mathbb{F}_p)|$ is relatively large for many primes $p$. We will use the following realization
of this idea. For a prime $p$ we put $a_p = a_p(E) = p + 1 - |E(\mathbb{F}_p)|$ and

$$SN(E, N) = \sum_{p \leq N, p \text{ prime}} \left(1 - \frac{p - 1}{|E(\mathbb{F}_p)|}\right) \log(p) = \sum_{p \leq N, p \text{ prime}} \left(\frac{-a_p + 2}{p + 1 - a_p}\right) \log(p).$$

This summation is defined as Mestre-Nagao sum. In order to give examples of high rank for $E_k : y^2 = x^3 + Ax^2 + Bx$ with $A$ and $B$ in the equation (2.2), we observe $k = p/q$, with $\gcd(p, q) = 1$, $|p|, |q| < 1000$, and Mestre-Nagao sums $SN(1000, E_k) > 20$, $SN(10000, E_k) > 30$, and $SN(100000, E_k) > 40$. Among these sieved $k$’s, it is considered the ones with high Selmer-rank. Then, rank computations are carried out with \texttt{Mwrank}. This process shows that for $k = \frac{39}{259}, \frac{67}{93}, \frac{88}{37}, \frac{98}{37}, \frac{263}{666}, \frac{280}{919}, \frac{593}{150}, \frac{596}{19}, \frac{609}{76}, \frac{845}{33}$, rank $E_k(\mathbb{Q}) = 7$.

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